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INFINITELY MANY SOLUTIONS OF SYSTEMS OF KIRCHHOFF-TYPE EQUATIONS WITH GENERAL POTENTIALS

GUOFENG CHE AND HAIBO CHEN

ABSTRACT. This paper is concerned with the following systems of Kirchhoff-type equations:

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u \\ \qquad +V(x)u=F_u(x,u,v) \qquad x\in\mathbb{R}^N, \\ -\left(c+d\displaystyle\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x\right)\Delta v \\ \qquad +V(x)v=F_v(x,u,v) \qquad x\in\mathbb{R}^N, \\ u(x)\to 0, \qquad v(x)\to 0 \qquad \text{as } |x|\to\infty. \end{cases}$$

Under some more relaxed assumptions on $V(x)$ and $F(x,u,v)$, we prove the existence of infinitely many negative-energy solutions for the above system via the genus properties in critical point theory. Some recent results from the literature are greatly improved and extended.

1. Introduction.

In this paper, we consider the following systems of Kirchhoff-type equations:

(1.1)
$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=F_u(x,u,v) & x\in\mathbb{R}^N, \\ -\left(c+d\displaystyle\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x\right)\Delta v+V(x)v=F_v(x,u,v) & x\in\mathbb{R}^N, \\ u(x)\longrightarrow 0, \qquad v(x)\longrightarrow 0 & \text{as } |x|\to\infty, \end{cases}$$

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where $a,\ c > 0$, $b,\ d \geq 0$, $V(x)$ and $F(x,u,v)$ satisfy the following hypotheses:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) = a > 0$;

(f_1) $F \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$, and there exist $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 2$, $1 < \beta_1 < \beta_2 < \cdots < \beta_m < 2$, $c_i \in L^{2/(2-\alpha_i)}(\mathbb{R}^N, \mathbb{R}^+)$ and $d_i \in L^{2/(2-\beta_i)}(\mathbb{R}^N, \mathbb{R}^+)$ such that

$$|F_u(x,u,v)| \leq \sum_{i=1}^{m} \alpha_i c_i(x)|(u,v)|^{\alpha_i - 1},$$

and

$$|F_v(x,u,v)| \leq \sum_{i=1}^{m} \beta_i d_i(x)|(u,v)|^{\beta_i - 1},$$

for any $(x,u) \in \mathbb{R}^N \times \mathbb{R}^2$, where $|(u,v)| = (u^2+v^2)^{1/2}$;

(f_2) there exist a bounded open set $J \subset \mathbb{R}^N$ and three constants $a_1,\ a_2 > 0$ and $a_3 \in (1,2)$ such that

$$F(x,u,v) \geq a_2|(u,v)|^{a_3}$$
$$\text{for all } (x,u,v) \in J \times [-a_1,a_1] \times [-a_1,a_1];$$

(f_3) $F(x,u,v) = F(x,-u,-v)$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

Kirchhoff-type problems are related to the stationary analog of the equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\triangle u = f(x,u), \quad \text{in } \Omega, \tag{1.2}$$

where u denotes the displacement, $f(x,u)$ the external force and b the initial tension, while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in **[10]** to describe the transversal oscillations of a stretched string, in particular, taking into account the subsequent change in string length caused by oscillations. For more details on the physical and mathematical background of problem (1.2), the reader is referred to **[1, 10, 12, 13]**, and the references therein.

Kirchhoff-type problems are often referred to as being nonlocal due to the presence of the integral over the entire Ω, which provokes some mathematical difficulties and also makes the study of such a class of problem particularly interesting. There has been much research on the existence of nontrivial solutions by using variational methods, for example, see [**1, 3, 5, 7, 8, 9, 11, 15, 17, 18, 19, 20, 21, 24, 25, 26, 27, 28, 29**], and the references therein. In [**5**], by using Nehari manifolds and the fibering map, Chen, Kuo and Wu established the existence of multiple positive solutions for Kirchhoff type equations that involve sign-changing weight functions. Jin and Wu [**9**] obtained three existence results of infinitely many radial solutions for a class of Kirchhoff-type problems by using the Fountain theorem. In [**24**], Wu obtained four new existence results for nontrivial solutions and a sequence of high energy solutions for Schrödinger-Kirchhoff type equations by using a Symmetric mountain pass theorem.

Recently, Wu [**25**] obtained five new critical point theorems on the product spaces and three existence theorems for the sequence of high energy solutions for problem (1.1). They assumed that the potential $V(x)$ satisfies (V_1) and

(V_2) for any $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

Later, under conditions (V_1) and (V_2), Zhou, Wu and Wu [**29**] presented a new proof technique to prove the existence of high energy solutions for problem (1.1) under some assumptions that are weaker than those in [**25**], which unify and sharply improve [**25,** Theorems 3.1–3.3], as well as some results in other literature, such as [**24,** Theorems 1–4]. We emphasize hypotheses (V_1) and (V_2), which appeared in Bartsch and Wang [**2**], were used to guarantee the compact embedding of the working space (see [**30,** Lemma 3.4]). Evidently, if assumptions (V_1) and (V_2) are replaced by (V_1), then the compactness of the embedding fails, and the situation becomes more complicated. More recently, the authors in [**4, 6, 11, 15, 18, 22**] dealt with this case.

Motivated by the above facts, the aim of this paper is to study the existence of nontrivial solutions and infinitely many negative-energy solutions for problem (1.1) under some more general conditions on $V(x)$ via variational methods. To the best of our knowledge, there has been little work concerning this case up until now.

Now, we state our main results.

Theorem 1.1. *Assume that conditions* (V_1), (f_1) *and* (f_2) *hold. Then, problem* (1.1) *possesses at least one nontrivial solution.*

Theorem 1.2. *Assume that conditions* (V_1) *and* (f_1)–(f_3) *hold. Then, problem* (1.1) *possesses infinitely many nontrivial solutions.*

By Theorems 1.1 and 1.2, we have the following corollaries.

Corollary 1.3. *Assume that conditions* (V_1), (V_2) *hold, and* $F(x,u,v)$ *satisfies the following conditions*:

(f_4) $F(x,u,v) = b(x)G(u,v)$, *where* $G \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $b \in C^1(\mathbb{R}^N,\mathbb{R})\bigcap L^{2/(2-\gamma_1)}(\mathbb{R}^N,\mathbb{R})$, *for the constant* $\gamma_1 \in (1,2)$, *and some* $x_0 > 0$ *such that* $b(x_0) > 0$;

(f_5) *there exist constants* M, $m > 0$ *and* $\gamma_0 \in (1,2)$ *such that*

$$m|(u,v)|^{\gamma_0} \le G(u,v) \le M|(u,v)|^{\gamma_1} \quad \text{for all } (u,v) \in \mathbb{R}\times\mathbb{R}.$$

Then, problem (1.1) *possesses at least one nontrivial solution.*

Corollary 1.4. *Assume that conditions* (V_1), (V_2), (f_4), (f_5) *and* $G(-u,-v) = G(u,v)$ *hold for any* $(u,v) \in \mathbb{R}\times\mathbb{R}$. *Then, problem* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.5. It is not difficult to find the functions $V(x)$ and $F(x,u,v)$ satisfying all of the conditions of Theorem 1.2. For example, let

$$V(x) = 1 + \sin^2 x_1$$

and

$$F(x,u,v) = \frac{\sin^2 x_1}{1+e^{|x|}}|(u,v)|^{7/6} + \frac{\cos^2 x_1}{1+e^{|x|}}|(u,v)|^{4/3},$$

where $x = \{x_1, x_2, \ldots, x_{\mathbb{N}}\}$. Then,

$$F_u(x,u,v) = \frac{7\sin^2 x_1}{6(1+e^{|x|})}|(u,v)|^{-5/6}u + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|(u,v)|^{-2/3}u,$$

$$F_v(x,u,v) = \frac{7\sin^2 x_1}{6(1+e^{|x|})}|(u,v)|^{-5/6}v + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|(u,v)|^{-2/3}v,$$

$$|F_u(x,u,v)| \leq \frac{7\sin^2 x_1}{6(1+e^{|x|})}|(u,v)|^{1/6} + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|(u,v)|^{1/3},$$
$$|F_v(x,u,v)| = \frac{7\sin^2 x_1}{6(1+e^{|x|})}|(u,v)|^{1/6} + \frac{4\cos^2 x_1}{3(1+e^{|x|})}|(u,v)|^{1/3},$$

and

$$F_u(x,u,v) \geq \frac{\cos^2 1}{1+e}|(u,v)|^{4/3},$$

for all $(x,u,v) \in J \times [-1,1] \times [-1,1]$. This also shows that (f_2) holds, where

$$\frac{7}{6} = \alpha_1 = \beta_1 < \beta_2 = \alpha_2 = \frac{4}{3},$$
$$c_1(x) = d_1(x) = \frac{\sin^2 x_1}{1+e^{|x|}},$$
$$c_2(x) = d_2(x) = \frac{\cos^2 x_1}{1+e^{|x|}},$$

and

$$a_1 = 1, \qquad a_2 = \frac{\cos^2 1}{1+e}, \qquad a_3 = \frac{4}{3}, \qquad J = B(0,1).$$

Notation 1.6. Throughout this paper, we shall denote by $\|\cdot\|_r$ the L^r-norm and C various positive generic constants, which may vary from line to line.

$$2^* = \frac{2N}{N-2} \quad \text{for } N \geq 3 \quad \text{and} \quad 2^* = \infty,\ N = 1,2,$$

is the critical Sobolev exponent. Also, if we take a subsequence of a sequence $\{(u_n,v_n)\}$, we shall denote it again by $\{(u_n,v_n)\}$.

The remainder of this paper is as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

2. Variational setting and preliminaries.

Let

$$H^1(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\big\},$$

with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right)^{1/2}.$$

Let

$$X := \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty \right\},$$

with the inner product and norm

$$\langle u, v \rangle_X = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv)\, dx, \quad \|u\|_X = \langle u, u \rangle_X^{1/2}.$$

As is standard, for $1 \le p < +\infty$, we let

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{1/p}, \quad u \in L^p(\mathbb{R}^N),$$

and

$$\|u\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Then, $E = X \times X$ is a Hilbert space with the following inner product:

$$\langle (u, v), (\varphi, \psi) \rangle = \langle u, \varphi \rangle_X + \langle v, \psi \rangle_X, \quad (u, v),\ (\varphi, \psi) \in X \times X,$$

and the norm

$$||(u, v)||^2 = \langle (u, v), (u, v) \rangle = \|u\|_X^2 + \|v\|_X^2, \quad (u, v) \in X \times X.$$

Lemma 2.1. *Suppose that condition (V_1) holds. Then, the embedding*

$$E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$$

is continuous for $2 \le r \le 2^*$ and

$$E \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^N) \times L^r_{\text{loc}}(\mathbb{R}^N)$$

is compact for $2 \le r < 2^*$.

Proof. By [**30,** Lemma 3.4], we know that, under the assumption (V_1), the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2^*]$, and $X \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^N)$ is compact for $r \in [2, 2^*)$, i.e., there exist constants $C_r > 0$ such that

$$\|u\|_r \le C_r \|u\|_X, \quad \text{for all } u \in X,$$

and, for any bounded sequence $\{u_n\} \subset X$, there exists a subsequence of $\{u_n\}$ such that $u_n \rightharpoonup u_0$ in X,

$$u_n \to u_0 \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^N), \quad r \in [2, 2^*).$$

Therefore, for any $(u, v) \in E$, there exists a $C > 0$ such that

$$||(u,v)||_r^r \le C(||u||_r^r + ||v||_r^r) \le C(||u||_X^r + ||v||_X^r) \le C||(u,v)||^r,$$

that is, $||(u,v)||_r \le C||(u,v)||$, i.e.,

$$E \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$$

is continuous for $2 \le r \le 2^*$.

On the other hand, suppose that $\{(u_n, v_n)\} \subset E$ are bounded, i.e., $\{u_n\}$ and $\{v_n\}$ are bounded in X. Then, there exist subsequences $\{u_n\}$ and $\{v_n\}$ such that

$$u_n \longrightarrow u_0, \qquad v_n \longrightarrow v_0 \text{ in } L^r_{\text{loc}}(\mathbb{R}^N), \quad r \in [2, 2^*).$$

Therefore,

$$0 \le ||(u_n, v_n) - (u_0, v_0)||_r^r \le C(||u_n - u_0||_r^r + ||v_n - v_0||_r^r) \longrightarrow 0,$$

as $n \to \infty$, that is,

$$(u_n, v_n) \longrightarrow (u_0, v_0), \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^N) \times L^r_{\text{loc}}(\mathbb{R}^N), \quad r \in [2, 2^*),$$

i.e.,

$$E \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^N) \times L^r_{\text{loc}}(\mathbb{R}^N)$$

is compact for $r \in [2, 2^*)$. The proof is complete. $\square$

Lemma 2.2. *Assume that* (V_1) *and* (f_1) *hold. Then, the functional* $I : E \to \mathbb{R}$, *defined by*

$$\begin{aligned} I(u,v) = {} & \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\bigg(\int_{\mathbb{R}^N} |\nabla u|^2 dx\bigg)^2 \\ & + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx + \frac{c}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx \\ & + \frac{d}{4}\bigg(\int_{\mathbb{R}^N} |\nabla v|^2 dx\bigg)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) v^2 dx \\ & - \int_{\mathbb{R}^N} F(x,u,v)\, dx, \end{aligned} \tag{2.1}$$

is well defined and of class $C^1(E,\mathbb{R})$*, and*

$$
\begin{aligned}
\langle I'(u,v),(\varphi,\psi)\rangle = &\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla\varphi\,dx\\
&+\int_{\mathbb{R}^N}V(x)u\varphi dx+\int_{\mathbb{R}^N}V(x)v\psi\,dx\\
&+\left(c+d\int_{\mathbb{R}^N}|\nabla v|^2dx\right)\int_{\mathbb{R}^N}\nabla v\nabla\psi\,dx\\
&-\int_{\mathbb{R}^N}F_u(x,u,v)\varphi\,dx-\int_{\mathbb{R}^N}F_v(x,u,v)\psi\,dx.
\end{aligned}\tag{2.2}
$$

Moreover, the critical points of I *in* E *are solutions to problem* (1.1)*.*

Proof. Set

$$\Phi(u,v)=\int_{\mathbb{R}^N}F(x,u,v)\,dx.$$

Then, by the definition of I, it suffices to show that $\Phi(u,v)\in C^1(E,\mathbb{R})$ and

$$\langle\Phi'(u,v),(\varphi,\psi)\rangle=\int_{\mathbb{R}^N}F_u(x,u,v)\varphi\,dx+\int_{\mathbb{R}^N}F_v(x,u,v)\psi\,dx.\tag{2.3}$$

First, we prove the existence of the Gateaux derivative of Φ. From (f_1), we have

$$
\begin{aligned}
|F(x,u,v)| &= |F(x,u,v)-F(x,0,0)|\\
&\le\int_0^1|F_u(x,tu,tv)||u|\,dt+\int_0^1|F_v(x,tu,tv)||v|\,dt\\
&\le\sum_{i=1}^m c_i(x)|(u,v)|^{\alpha_i}+\sum_{i=1}^m d_i(x)|(u,v)|^{\beta_i}.
\end{aligned}\tag{2.4}
$$

Then, for any $(u,v)\in E$, it follows from (V_1), (2.4) and the Hölder inequality that

(2.5)

$$
\begin{aligned}
\int_{\mathbb{R}^N}|F(x,u,v)|\,dx &\le \int_{\mathbb{R}^N}\left[\sum_{i=1}^m c_i(x)|(u,v)|^{\alpha_i}+\sum_{i=1}^m d_i(x)|(u,v)|^{\beta_i}\right]dx\\
&\le\sum_{i=1}^m a^{-\alpha_i/2}\left(\int_{\mathbb{R}^N}|c_i(x)|^{2/(2-\alpha_i)}dx\right)^{(2-\alpha_i)/2}
\end{aligned}
$$

$$\times\left(\int_{\mathbb{R}^N} V(x)|(u,v)|^2dx\right)^{\alpha_i/2}$$
$$+\sum_{i=1}^m a^{-\beta_i/2}\left(\int_{\mathbb{R}^N}|d_i(x)|^{2/(2-\beta_i)}dx\right)^{(2-\beta_i)/2}$$
$$\times\left(\int_{\mathbb{R}^N} V(x)|(u,v)|^2dx\right)^{\beta_i/2}$$
$$\leq\sum_{i=1}^m a^{-\alpha_i/2}||c_i||_{2/(2-\alpha_i)}||(u,v)||^{\alpha_i}$$
$$+\sum_{i=1}^m a^{-\beta_i/2}||d_i||_{2/(2-\beta_i)}||(u,v)||^{\beta_i},$$

which implies that I, defined by (2.1) is well defined on E.

For any function
$$\theta:\mathbb{R}^N\longrightarrow(0,1),$$

by (f_1) and the Hölder inequality, we have

(2.6)
$$\begin{aligned}
&\int_{\mathbb{R}^N}\max_{t\in[0,1]}|F_u(x,u(x)+t\theta(x)\varphi(x),v(x)+t\theta(x)\psi(x))\varphi(x)|\,dx\\
&\quad=\int_{\mathbb{R}^N}\max_{t\in[0,1]}|F_u(x,u(x)+t\theta(x)\varphi(x),v(x)+t\theta(x)\psi(x))||\varphi(x)|\,dx\\
&\quad\leq\sum_{i=1}^m\alpha_i\int_{\mathbb{R}^N}(c_i(x)|(u(x)+t\theta(x)\varphi(x),v(x)+t\theta(x)\psi(x))|^{\alpha_i-1})|\varphi(x)|\,dx\\
&\quad\leq C\bigg[\sum_{i=1}^m\int_{\mathbb{R}^N}(c_i(x)(|u(x)|^{\alpha_i-1}+|\varphi(x)|^{\alpha_i-1}))|\varphi(x)|\,dx\\
&\qquad\qquad+\sum_{i=1}^m\int_{\mathbb{R}^N}(c_i(x)(|v(x)|^{\alpha_i-1}+|\psi(x)|^{\alpha_i-1}))|\varphi(x)|dx\bigg]\\
&\quad\leq C\bigg[\sum_{i=1}^m a^{-\alpha_i/2}\left(\int_{\mathbb{R}^N}|c_i(x)|^{2/(2-\alpha_i)}dx\right)^{(2-\alpha_i)/2}\\
&\quad\times\left(\int_{\mathbb{R}^N}V(x)|u(x)|^2dx\right)^{(\alpha_i-1)/2}\left(\int_{\mathbb{R}^N}V(x)|\varphi(x)|^2dx\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{m} a^{-\alpha_i/2}\Big(\int_{\mathbb{R}^N} |c_i(x)|^{2/(2-\alpha_i)}dx\Big)^{(2-\alpha_i)/2}\\
&\times \Big(\int_{\mathbb{R}^N} V(x)|\varphi(x)|^2dx\Big)^{\alpha_i/2}\\
&\quad + \sum_{i=1}^{m} a^{-\alpha_i/2}\Big(\int_{\mathbb{R}^N} |c_i(x)|^{2/(2-\alpha_i)}dx\Big)^{(2-\alpha_i)/2}\\
&\times \Big(\int_{\mathbb{R}^N} V(x)|v(x)|^2dx\Big)^{(\alpha_i-1)/2}\Big(\int_{\mathbb{R}^N} V(x)|\varphi(x)|^2dx\Big)^{1/2}\\
&\quad + \sum_{i=1}^{m} a^{-\alpha_i/2}\Big(\int_{\mathbb{R}^N} |c_i(x)|^{2/(2-\alpha_i)}dx\Big)^{(2-\alpha_i)/2}\\
&\times \Big(\int_{\mathbb{R}^N} V(x)|\psi(x)|^2dx\Big)^{(\alpha_i-1)/2}\Big(\int_{\mathbb{R}^N} V(x)|\varphi(x)|^2dx\Big)^{1/2}\Bigg]\\
&\leq C\sum_{i=1}^{m} ||c_i||_{2/(2-\alpha_i)}\big(||u||^{\alpha_i-1}+||\varphi||^{\alpha_i-1}+||v||^{\alpha_i-1}+||\psi||^{\alpha_i-1}\big)||\varphi||\\
&< +\infty.
\end{aligned}
$$

Similarly, we have

$$
\int_{\mathbb{R}^N} \max_{t\in[0,1]} |F_v(x,u(x)+t\theta(x)\varphi(x),v(x)+t\theta(x)\psi(x))\psi(x)|\,dx < +\infty. \tag{2.7}
$$

Then, by (2.1), (2.6), (2.7) and Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
\langle \Phi'(u,v),(\varphi,\psi)\rangle &= \lim_{t\to 0^+}\frac{\Phi(u+t\varphi,v+t\psi)-\Phi(u,v)}{t}\\
&= \lim_{t\to 0^+}\Big[\int_{\mathbb{R}^N} F_u(x,u+t\theta\varphi,v+t\theta\psi)\varphi\,dx\\
&\qquad + \int_{\mathbb{R}^N} F_v(x,u+t\theta\varphi,v+t\theta\psi)\psi\,dx\Big]\\
&= \int_{\mathbb{R}^N} F_u(x,u,v)\varphi\,dx + \int_{\mathbb{R}^N} F_v(x,u,v)\psi\,dx,
\end{aligned} \tag{2.8}
$$

which implies that (2.3) holds.

Now, we show that $\Phi(u,v) \in C^1(E,\mathbb{R})$. Let $(u_n,v_n) \to (u,v)$ in E. Then, $(u_n,v_n) \to (u,v)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, and

$$\lim_{n\to\infty}(u_n,v_n) = (u,v) \text{ almost everywhere } x \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.9}$$

Now, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |F_u(x,u_n,v_n) - F_u(x,u,v)|^2 dx = 0. \tag{2.10}$$

Otherwise, there exist a constant $\varepsilon_0 > 0$ and a sequence $\{(u_{ni},v_{ni})\}$ such that

$$\int_{\mathbb{R}^N} |F_u(x,u_n,v_n) - F_u(x,u,v)|^2 dx \geq \varepsilon_0, \quad \text{for all } i \in \mathbb{N}. \tag{2.11}$$

In fact, since $(u_n,v_n) \to (u,v)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, passing to a subsequence, if necessary, it can be assumed that

$$\sum_{i=1}^{\infty} ||(u_{ni},v_{ni}) - (u,v)||_2^2 < +\infty.$$

Set

$$\omega(x) = \left(\sum_{i=1}^{\infty} ||(u_{ni},v_{ni}) - (u,v)||_2^2\right)^{1/2}.$$

Then, $\omega \in L^2(\mathbb{R}^N)$. Evidently,

(2.12)

$$\begin{aligned}
&|F_u(x,u_{ni},v_{ni}) - F_u(x,u,v)|^2 \\
&\quad\leq 2|F_u(x,u_{ni},v_{ni})|^2 + 2|F_u(x,u,v)|^2 \\
&\quad\leq 2^{m-1}\sum_{i=1}^{m} \alpha_i^2|c_i(x)|^2\left[|(u_{ni},v_{ni})|^{2(\alpha_i-1)} + |(u,v)|^{2(\alpha_i-1)}\right] \\
&\quad\leq 2^{m-1}\sum_{i=1}^{m}(2^{\alpha_i-1}+1)\alpha_i^2|c_i(x)|^2\left[|(u_{ni},v_{ni}) - (u,v)|^{2(\alpha_i-1)}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. + |(u,v)|^{2(\alpha_i-1)}\right] \\
&\quad\leq 2^{m-1}\sum_{i=1}^{m}(2^{\alpha_i-1}+1)\alpha_i^2|c_i(x)|^2\left[|\omega(x)|^{2(\alpha_i-1)} + |(u,v)|^{2(\alpha_i-1)}\right] \\
&\quad:= h(x), \quad \text{for all } i \in \mathbb{N},\ x \in \mathbb{R}^N,
\end{aligned}$$

and

$$
\begin{aligned}
(2.13)\quad \int_{\mathbb{R}^N} h(x)\,dx &= 2^{m-1}\sum_{i=1}^{m}(2^{\alpha_i-1}+1)\alpha_i^2 \\
&\quad \cdot \int_{\mathbb{R}^N} |c_i(x)|^2\left[|\omega(x)|^{2(\alpha_i-1)}+|(u,v)|^{2(\alpha_i-1)}\right]dx \\
&\le 2^{m-1}\sum_{i=1}^{m}(2^{\alpha_i-1}+1)\alpha_i^2||c_i||_{2/(2-\alpha_i)}^2 \\
&\quad \cdot \left(||\omega||_2^{2(\alpha_i-1)}+||(u,v)||_2^{2(\alpha_i-1)}\right) < +\infty.
\end{aligned}
$$

It follows from (2.12), (2.13) and Lebesgue's dominated convergence theorem that (2.10) holds.

Analogously, we obtain

$$
(2.14)\qquad \lim_{n\to\infty}\int_{\mathbb{R}^N}|F_v(x,u_n,v_n)-F_v(x,u,v)|^2dx = 0.
$$

Then, by (2.2), (2.10) and (2.14), we have

$$
\begin{aligned}
&|\langle \Phi'(u_n,v_n)-\Phi'(u,v),(\varphi,\psi)\rangle| \\
&\quad \le \int_{\mathbb{R}^N}|F_u(x,u_n,v_n)-F_u(x,u,v)||\varphi|\,dx \\
&\qquad + \int_{\mathbb{R}^N}|F_v(x,u_n,v_n)-F_v(x,u,v)||\psi|\,dx \\
&\quad \le a^{-1/2}\left[\left(\int_{\mathbb{R}^N}|F_u(x,u_n,v_n)-F_u(x,u,v)|^2dx\right)^{1/2}||\varphi||\right. \\
&\qquad \left. + \left(\int_{\mathbb{R}^N}|F_v(x,u_n,v_n)-F_v(x,u,v)|^2dx\right)^{1/2}||\psi||\right] \\
&\quad \longrightarrow 0, \quad \text{as } n\to\infty,
\end{aligned}
$$

which implies that $\Phi\in C^1(E,\mathbb{R})$. Moreover, by a standard argument, it is easy to verify that the critical points of I in E are solutions of problem (1.1), see **[23]**. The proof is complete. □

Theorem 2.3 ([**14**]). *Let E be a real Banach space, and let $I\in C^1(E,\mathbb{R})$ satisfy the* (PS) *condition. If I is bounded from below, then $c=\inf_E I$ is a critical value of I.*

In order to find the multiplicity of nontrivial critical points of I, we will use the "genus" properties; thus, we recall the following definitions and results, see [**16**].

Let E be a Banach space, $c \in \mathbb{R}$ and $I \in C^1(E,\mathbb{R})$. Set

$$\Sigma = \big\{A \subset E\backslash\{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\big\},$$

$$K_c = \{u \in E : I(u) = c, I'(u) = 0\}, \qquad I^c = \{u \in E : I(u) \leq c\}.$$

Definition 2.4. For $A \in \Sigma$, we say that the genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^N \setminus \{0\})$ and n is the smallest integer with this property.

Theorem 2.5. *Let E be an even C^1 functional on E which satisfies the* (PS) *condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\}, \qquad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u).$$

(i) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I.*

(ii) *If there exists an $r \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R}$ and $c \neq I(0)$, then $\gamma(K_c) \geq r+1$.*

3. Proofs of main results. In this section, we will prove Theorems 1.1 and 1.2. In order to complete the proof, we need the following lemma.

Lemma 3.1. *Assume that* (V_1) *and* (f_1) *hold. Then, I is bounded from below and satisfies the* (PS) *condition.*

Proof. From Lemma 2.1, (f_1), the Sobolev embedding theorem and the Hölder inequality, we have

(3.1)
$$\begin{aligned} I(u,v) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\bigg(\int_{\mathbb{R}^N} |\nabla u|^2 dx\bigg)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx \\ + \frac{c}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{d}{4}\bigg(\int_{\mathbb{R}^N} |\nabla v|^2 dx\bigg)^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2dx - \int_{\mathbb{R}^N} F(x,u,v)\,dx\\
\geq{}& \frac{1}{2}\min\{a,c,1\}||(u,v)||^2 - \int_{\mathbb{R}^N} F(x,u,v)dx\\
\geq{}& \frac{1}{2}\min\{a,c,1\}||(u,v)||^2 - \sum_{i=1}^{m}\int_{\mathbb{R}^N} c_i(x)|(u,v)|^{\alpha_i}dx\\
&-\sum_{i=1}^{m}\int_{\mathbb{R}^N} d_i(x)|(u,v)|^{\beta_i}dx\\
\geq{}& \frac{1}{2}\min\{a,c,1\}||(u,v)||^2\\
&-\sum_{i=1}^{m} a^{-\alpha_i/2}\bigg(\int_{\mathbb{R}^N}|c_i(x)|^{2/(2-\alpha_i)}dx\bigg)^{(2-\alpha_i)/2}\\
&\times\bigg(\int_{\mathbb{R}^N} V(x)|(u,v)|^2dx\bigg)^{\alpha_i/2}\\
&-\sum_{i=1}^{m} a^{-\beta_i/2}\bigg(\int_{\mathbb{R}^N}|d_i(x)|^{2/(2-\beta_i)}dx\bigg)^{(2-\beta_i)/2}\\
&\times\bigg(\int_{\mathbb{R}^N} V(x)|(u,v)|^2dx\bigg)^{\beta_i/2}\\
\geq{}& \frac{1}{2}\min\{a,c,1\}||(u,v)||^2 - \sum_{i=1}^{m} a^{-\alpha_i/2}||c_i||_{2/(2-\alpha_i)}||(u,v)||^{\alpha_i}\\
&-\sum_{i=1}^{m} a^{-\beta_i/2}||d_i||_{2/(2-\beta_i)}||(u,v)||^{\beta_i},
\end{aligned}
$$

which implies that $I(u,v)\to+\infty$, as $\|(u,v)\|\to\infty$, since $a,\ c>0$, α_i, $\beta_i\in(1,2)$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS) condition. Assume that $\{(u_n,v_n)\}$ is a (PS) sequence of I such that $I(u_n,v_n)$ is bounded and $||I'(u_n,v_n)||\to 0$, as $n\to\infty$. Then, it follows from (3.1) that there exists a constant $C>0$ such that

$$
||(u_n,v_n)||_2 \leq a^{-1/2}||(u_n,v_n)|| \leq C, \quad n\in\mathbb{N}. \tag{3.2}
$$

Then by Lemma 2.1, there exists a $(u,v) \in E$ such that

$$(u_n, v_n) \rightharpoonup (u,v) \quad \text{in } E,$$

$$(u_n, v_n) \longrightarrow (u,v) \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N) \times L^s_{\text{loc}}(\mathbb{R}^N),\ s \in [2, 2^*), \tag{3.3}$$

$$(u_n, v_n) \longrightarrow (u,v) \text{ almost everywhere } \mathbb{R}^N.$$

On the other hand, for any given $\varepsilon > 0$, by (f_1), we can choose $R_\varepsilon > 0$ such that

$$\left(\int_{|x| > R_\varepsilon} |c_i(x)|^{2/(2-\alpha_i)} dx \right)^{(2-\alpha_i)/2} < \varepsilon, \quad i = 1, 2, \ldots, m. \tag{3.4}$$

It follows from (3.3) that there exists an $n_0 > 0$ such that

$$\int_{|x| \le R_\varepsilon} |(u_n, v_n) - (u,v)|^2 dx < \varepsilon^2, \quad \text{for } n \ge n_0. \tag{3.5}$$

Therefore, by (f_1), (3.2), (3.5) and the Hölder inequality, for any $n \ge n_0$, we have

(3.6)

$$\begin{aligned}
&\int_{|x| \le R_\varepsilon} \big|F_u(x, u_n, v_n) - F_u(x,u,v)\big| \big|(u_n, v_n) - (u,v)\big|\, dx \\
&\quad \le \left(\int_{|x| \le R_\varepsilon} |F_u(x, u_n, v_n) - F_u(x,u,v)|^2 dx \right)^{1/2} \\
&\quad \times \left(\int_{|x| \le R_\varepsilon} |(u_n, v_n) - (u,v)|^2 dx \right)^{1/2} \\
&\quad \le \varepsilon \left[\int_{|x| \le R_\varepsilon} 2(|F_u(x, u_n, v_n)|^2 + |F_u(x,u,v)|^2)\, dx \right]^{1/2} \\
&\quad \le 2\varepsilon \left[\sum_{i=1}^m \alpha_i^2 \int_{|x| \le R_\varepsilon} |c_i(x)|^2 (|(u_n, v_n)|^{2(\alpha_i - 1)} + |(u,v)|^{2(\alpha_i - 1)}\, dx \right]^{1/2} \\
&\quad \le 2\varepsilon \left[\sum_{i=1}^m \alpha_i^2 ||c_i||^2_{2/(2-\alpha_i)} \big(||(u_n, v_n)||_2^{2(\alpha_i - 1)} + ||(u,v)||_2^{2(\alpha_i - 1)} \big) \right]^{1/2} \\
&\quad \le 2\varepsilon \left[\sum_{i=1}^m \alpha_i^2 ||c_i||^2_{2/(2-\alpha_i)} \big(C^{2(\alpha_i - 1)} + ||(u,v)||_2^{2(\alpha_i - 1)} \big) \right]^{1/2}.
\end{aligned}$$

For $n \in \mathbb{N}$, it follows from (f_1), (3.2), (3.4) and the Hölder inequality that

$$
\begin{aligned}
&\int_{|x|>R_\varepsilon} \big|F_u(x,u_n,v_n) - F_u(x,u,v)\big|\big|(u_n,v_n)-(u,v)\big|\,dx \\
&\quad\le \sum_{i=1}^{m} \alpha_i \int_{|x|>R_\varepsilon} |c_i(x)|\big(|(u_n,v_n)|^{\alpha_i-1} + |(u,v)|^{\alpha_i-1}\big) \\
&\quad\times \big(|(u_n,v_n)| + |(u,v)|\big)\,dx \\
&\le 2\sum_{i=1}^{m} \alpha_i \int_{|x|>R_\varepsilon} |c_i(x)|\big(|(u_n,v_n)|^{\alpha_i} + |(u,v)|^{\alpha_i}\big)\,dx \\
&\quad\le 2\sum_{i=1}^{m} \alpha_i \Big(\int_{|x|>R_\varepsilon} |c_i(x)|^{2/(2-\alpha_i)} dx\Big)^{(2-\alpha_i)/2} \\
&\quad\times \big(||(u_n,v_n)||_2^{\alpha_i} + ||(u,v)||_2^{\alpha_i}\big) \\
&\quad\le 2\sum_{i=1}^{m} \alpha_i \Big(\int_{|x|>R_\varepsilon} |c_i(x)|^{2/(2-\alpha_i)} dx\Big)^{(2-\alpha_i)/2} \big(C^{\alpha_i} + ||(u,v)||_2^{\alpha_i}\big) \\
&\quad\le 2\varepsilon \sum_{i=1}^{m} \alpha_i \big(C^{\alpha_i} + ||(u,v)||_2^{\alpha_i}\big).
\end{aligned} \tag{3.7}
$$

Since ε is arbitrary, combining (3.6) and (3.7), we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} \big(F_u(x,u_n,v_n) - F_u(x,u,v)\big)\big((u_n,v_n)-(u,v)\big)\,dx = 0. \tag{3.8}
$$

Arguing in the same manner, we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} \big(F_v(x,u_n,v_n) - F_v(x,u,v)\big)\big((u_n,v_n)-(u,v)\big)\,dx = 0. \tag{3.9}
$$

Then, by (2.2), (3.8), (3.9) and the weak convergence of $\{(u_n,v_n)\}$, we obtain

(3.10)

$$
\begin{aligned}
o_n(1) &= \langle I'(u_n,v_n) - I'(u,v), (u_n-u, v_n-v)\rangle \\
&= \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} |\nabla(u_n-u)|^2 dx \\
&\quad + \int_{\mathbb{R}^N} V(x)|u_n-u|^2 dx + \int_{\mathbb{R}^N} V(x)|v_n-v|^2 dx \\
&\quad - \int_{\mathbb{R}^N} \big[F_u(x,u_n,v_n) - F_u(x,u,v)\big](u_n-u)\,dx \\
&\quad + \left(c + d\int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)\int_{\mathbb{R}^N} |\nabla(v_n-v)|^2 dx \\
&\quad - \int_{\mathbb{R}^N} \big[F_v(x,u_n,v_n) - F_v(x,u,v)\big](v_n-v)\,dx \\
&\quad - b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla(u_n-u)\,dx \\
&\quad - d\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla v \nabla(v_n-v)\,dx \\
&\geq \min\{a,c,1\}||(u_n-u, v_n-v)||^2 \\
&\quad - b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla(u_n-u)\,dx \\
&\quad - d\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla v \nabla(v_n-v)\,dx \\
&\quad - \int_{\mathbb{R}^N} \big[F_u(x,u_n,v_n) - F_u(x,u,v\big](u_n-u)\,dx \\
&\quad - \int_{\mathbb{R}^N} \big[F_v(x,u_n,v_n) - F_v(x,u,v)\big](v_n-v)\,dx.
\end{aligned}
$$

On the other hand, the boundedness of $\{u_n\}$ and $\{v_n\}$ imply

$$
b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla(u_n-u)\,dx \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.11}
$$

$$
d\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla v \nabla(v_n-v)\,dx \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.12}
$$

Then, by (3.8)–(3.12), we have $(u_n, v_n) \to (u, v)$ in E. Therefore, I satisfies the (PS) condition. The proof is complete. $\square$

Proof of Theorem 1.1. From Lemmas 2.2 and 3.1, the conditions of Theorem 2.1 are satisfied. Thus, $c = \inf_E I(u, v)$ is a critical value of I, that is, there exists a critical point $(u^*, v^*) \in E$ such that $I(u^*, v^*) = c$.

Now, we show that $(u^*, v^*) \neq (0, 0)$. Let

$$(u, v) \in \left(W_0^{1,2}(J) \bigcap X\right) \times \left(W_0^{1,2}(J) \bigcap X\right) \setminus \{(0, 0)\},$$

$||u||_\infty \leq 1$ and $||v||_\infty \leq 1$. Then, by (2.1) and (f_2), we have

$$\begin{aligned} I(tu, tv) &= \frac{at^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{bt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 \\ &\quad + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(x, tu, tv)\, dx \\ &\quad + \frac{ct^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{dt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^2 \\ &\quad + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x) v^2 dx \\ &= \frac{at^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{bt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 \\ &\quad + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_J F(x, tu, tv)\, dx \\ &\quad + \frac{ct^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{dt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^2 \\ &\quad + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x) v^2 dx \\ &\leq \frac{t^2}{2} \max\{a, c, 1\} ||(u, v)||^2 + \frac{bt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 \\ &\quad + \frac{dt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^2 - a_2 t^{a_3} \int_J |(u, v)|^{a_3} dx, \end{aligned} \tag{3.13}$$

where $0 < t < a_1$, a_1 is given in (f_2). Since $1 < a_3 < 2$, it follows from (3.13) that $I(tu, tv) < 0$ for $t > 0$ small enough. Therefore, $I(u^*, v^*) = c < 0$, that is, (u^*, v^*) is a nontrivial critical point of I, and thus, (u^*, v^*) is a nontrivial solution to problem (1.1). The proof is complete. $\square$

Proof of Theorem 1.2. From Lemmas 2.2 and 3.1, $I \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the (PS) condition. It follows from (2.1) and (f_3) that I is even and $I(0,0) = 0$. In order to apply Theorem 2.2, we now show that, for any $n \in \mathbb{N}$, there exists an $\varepsilon > 0$ such that

$$\gamma(I^{-\varepsilon}) \geq n. \tag{3.14}$$

For any $n \in \mathbb{N}$, we take n disjoint open sets J_i such that

$$\bigcup_{i=1}^{n} J_i \subset J.$$

For $i = 1, 2, \ldots, n$, let

$$(u_i, v_i) \in \left(W_0^{1,2}(J_i) \bigcap X\right) \times \left(W_0^{1,2}(J_i) \bigcap X\right) \setminus \{(0,0)\},$$

$||u_i||_\infty \leq \infty$, $||v_i||_\infty \leq \infty$, $||(u_i, v_i)|| = 1$,

$$E_n = \operatorname{span}\big\{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\big\},$$

and

$$S_n = \big\{(u,v) \in E_n : ||(u,v)|| = 1\big\}.$$

Then, for any $(u,v) \in E_n$, there exist $\lambda_i \in \mathbb{R}, i = 1, 2, \ldots, n$, such that

$$(u(x), v(x)) = \sum_{i=1}^{n} \lambda_i (u_i(x), v_i(x)), \quad x \in \mathbb{R}^N. \tag{3.15}$$

Then, we obtain

$$||(u,v)||_{a_3} = \left(\int_{\mathbb{R}^N} |(u,v)|^{a_3} dx\right)^{1/a_3} = \left(\sum_{i=1}^{n} |\lambda_i|^{a_3} \int_{J_i} |(u,v)|^{a_3} dx\right)^{1/a_3}, \tag{3.16}$$

and

$$\begin{aligned} \|(u,v)\|^2 &= \|u\|_X^2 + \|v\|_X^2 \\ &= \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)\, dx \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&+\int_{R^N}(|\Delta u|^2+|\nabla u|^2+V(x)|u|^2)\,dx\\
&=\sum_{i=1}^n\lambda_i^2\int_{J_i}(|\Delta u_i|^2+|\nabla u_i|^2+V(x)|u_i|^2)\,dx\\
&\quad+\sum_{i=1}^n\lambda_i^2\int_{J_i}(|\Delta u_i|^2+|\nabla u_i|^2+V(x)|u_i|^2)\,dx\\
&=\sum_{i=1}^n\lambda_i^2\int_{\mathbb{R}^N}(|\Delta u_i|^2+|\nabla u_i|^2+V(x)|u_i|^2)\,dx\\
&\quad+\sum_{i=1}^n\lambda_i^2\int_{\mathbb{R}^N}(|\Delta u_i|^2+|\nabla u_i|^2+V(x)|u_i|^2)\,dx\\
&=\sum_{i=1}^n\lambda_i^2||(u_i,v_i)||^2=\sum_{i=1}^n\lambda_i^2.
\end{aligned}
$$

Since all norms are equivalent in a finite-dimensional normed space, so there exists a $c_0>0$ such that

$$c_0||(u,v)||\le||(u,v)||_{a_3},\quad \text{for any } (u,v)\in E_n. \tag{3.18}$$

Then, from (2.1), (f_2), (3.15)–(3.18) and the Sobolev embedding inequality, for $(u,v)\in S_n$, we have

(3.19)

$$
\begin{aligned}
I(tu,tv)&=\frac{at^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2\\
&\quad+\frac{t^2}{2}\int_{\mathbb{R}^N}V(x)u^2dx+\frac{ct^2}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx\\
&\quad+\frac{dt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2+\frac{t^2}{2}\int_{\mathbb{R}^N}V(x)v^2dx\\
&\quad-\int_{\mathbb{R}^N}F(x,tu,tv)\,dx\\
&=\frac{at^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{bt^4}{4}(\int_{\mathbb{R}^N}|\nabla u|^2dx)^2\\
&\quad+\frac{t^2}{2}\int_{\mathbb{R}^N}V(x)u^2dx+\frac{ct^2}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{dt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2+\frac{t^2}{2}\int_{\mathbb{R}^N}V(x)v^2dx\\
&-\sum_{i=1}^{n}\int_{J_i}F(x,tu,tv)\,dx\\
\leq&\frac{t^2}{2}\max\{a,c,1\}||(u,v)||^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2\\
&+\frac{dt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2-a_2t^{a_3}\sum_{i=1}^{n}|\lambda_i|^{a_3}\int_{J_i}|(u_i,v_i)|^{a_3}dx\\
\leq&\frac{t^2}{2}\max\{a,c,1\}||(u,v)||^2+\frac{(b+d)t^4}{4}||(u,v)||^4-a_2(c_0t)^{a_3}||(u,v)||^{a_3}\\
=&\frac{t^2}{2}\max\{a,c,1\}+\frac{(b+d)t^4}{4}-a_2(c_0t)^{a_3},
\end{aligned}
$$

where $0 < t \leq a_1$ and $1 < a_3 < 2$. Then, it follows from (3.19) that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$I(\delta u,\delta v) < -\varepsilon, \quad \text{for any } (u,v)\in S_n. \tag{3.20}$$

Let

$$
\begin{aligned}
S_n^\delta &= \big\{(\delta u,\delta v):(u,v)\in S_n\big\},\\
\Omega &= \left\{(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbb{R}^N:\sum_{i=1}^{n}\lambda_i^2<\delta^2\right\}.
\end{aligned}
$$

It follows from (3.20) that

$$I(u,v) < -\varepsilon, \quad \text{for } (u,v)\in S_n^\delta,$$

which, together, with the fact that $I\in C^1(E,\mathbb{R})$ and is even, implies that

$$S_n^\delta \subset I^{-\varepsilon}\in\Sigma. \tag{3.21}$$

On the other hand, from (3.15) and (3.17), there exists an odd homeomorphism mapping $\phi\in C(S_n^\delta,\partial\Omega)$. By some properties of the genus (see [**15,** Propositions 7.5, 7.7, 3^0]), we have

$$\gamma(I^{-\varepsilon}) \geq \gamma(S_n^\delta) = n. \tag{3.22}$$

Thus, the proof of (3.2) holds. Set

$$c_n = \inf_{A\in\Sigma_n} \sup_{(u,v)\in A} I(u,v). \tag{3.23}$$

It follows from (3.22) and the fact that I is bounded from below on E that $-\infty < c_n \leq -\varepsilon < 0$, that is to say, for any $n \in \mathbb{N}$, c_n is a real negative number. From Theorem 2.5, I has infinitely many nontrivial critical points; therefore, problem (1.1) possesses infinitely many nontrivial solutions. The proof is complete. □

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School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, P.R. China
Email address: cheguofeng222@163.com

School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, P.R. China
Email address: math_chb@163.com