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FAMILIES OF CALABI-YAU ELLIPTIC FIBRATIONS IN $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$

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ABSTRACT. Let B be a smooth projective surface, and let $\mathcal{L}$ be an ample line bundle on B. The aim of this paper is to study the families of elliptic Calabi-Yau threefolds sitting in the bundle $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ as anticanonical divisors. We will show that the number of such families is finite.

Introduction. While the theory of elliptic surfaces is a well-settled and consolidated subject, in the case of elliptic threefolds, there are still many interesting and open questions. Not only are the theoretical aspects of the theory important, but, in addition, the research of the families of examples plays a central role: one of the main motivations is the close connection with the theory of strings (and, in particular, F-theory, see e.g., [**21**]), which is a physical subject whose main object of study is, in fact, elliptic fibrations on Calabi-Yau manifolds. To give two examples, in [**7**], the E_6 and E_7 families of elliptic Calabi-Yau threefolds are defined, and, in [**6**], the authors defined the D_5 family.

In this paper, we focus on a manner of constructing elliptic fibrations on the Calabi-Yau threefolds, working in the field $\mathbb{C}$ of complex numbers.

A simple way to produce Calabi-Yau varieties is to consider smooth anticanonical subvarieties of some reasonable ambient space; in fact, by adjunction, these varieties will automatically be Calabi-Yau. Giving different shades to the word 'reasonable,' different classes of ambient spaces are considered in attempting to describe its anticanonical subvarieties. In particular, the class of toric Fano Gorenstein fourfolds has been deeply studied for the following reasons:

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(1) since any anticanonical divisor of a Fano variety is ample, we are sure to find effective divisors in the anticanonical system;
(2) Gorenstein varieties may be singular, but, in this case, they have nice resolutions of the singularities and then the anticanonical subvarieties of the resolution can be studied;
(3) toric varieties are simple since most of the problems that may have to be solved can be translated into a combinatorial problem, which is simpler to deal with.

To each toric Fano Gorenstein fourfold is associated a reflexive four-dimensional polyhedron, and viceversa, so the first attempt to describe the Calabi-Yau subvarieties in these ambient spaces is to classify all of the reflexive four-dimensional polyhedra. Such a classification is known, and there are 473,800,776 four-dimensional reflexive polyhedra (see e.g., [**15, 16**]). Among these, in [**2**], the 102,581 elliptic fibrations over $\mathbb{P}^2$ are identified.

The elliptic fibrations we describe in this paper are anticanonical hypersurfaces in a projective bundle Z over a surface B of the form

$$Z = \mathbb{P}(\mathcal{L}^a \otimes \mathcal{L}^b \otimes \mathcal{O}_B)$$

for $\mathcal{L}$ an ample line bundle on B. Observe that, even in the case where the base B is toric, e.g., $B = \mathbb{P}^2$, the ambient bundle is typically not Fano.

The aim of this paper is to show that, once B and $\mathcal{L}$ are fixed, then the bundle $Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ can house Calabi-Yau elliptic fibrations only for a finite number of choices of (a, b):

Main theorem (Theorem 2.1)**.** *Let B be a smooth projective surface, and let $\mathcal{L}$ be an ample line bundle on B. Consider the projective bundle $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, with $a \geq b \geq 0$. Then, only for a finite number of pairs (a, b), the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ is a Calabi-Yau elliptic fibration over B.*

As will be seen in subsections 2.2 and 2.5, we may fail to find a Calabi-Yau elliptic fibration for the following reasons: the fibration has no sections or its total space is singular.

The outline of this paper is as follows. In Section 1, we recall the definitions of elliptic fibration and of Calabi-Yau variety. In Section 2, we state the finiteness result (Theorem 2.1) and prove it (subsections 2.5.1–2.5.4). Finally, in Section 3, we give some concrete examples: we will find explicit bounds on the number of different families when the base B is a del Pezzo surface (and, in particular, for $B = \mathbb{P}^2$), and when the basis B is a Hirzebruch surface $\mathbb{F}_e$.

1. Elliptic fibrations and Calabi-Yau manifolds. In this section, we recall the definition and main properties of elliptic fibrations (subsection 1.1) and Calabi-Yau manifolds (subsection 1.2). Throughout this paper, all the varieties are defined over $\mathbb{C}$.

1.1. Elliptic fibrations. Elliptic fibrations are the geometric realization of elliptic curves over the function field of a variety. Their study has been encouraged by physics, and, in particular, string theory: to each elliptic fibration correspond a physical scenario, and the fibration itself determines the number of elementary particles, their charges and masses (see, e.g., **[21]**).

Definition 1.1. We say that $\pi : X \to B$ is an *elliptic fibration* over B if:

(i) X and B are projective varieties of dimension n and $n - 1$, respectively, with X smooth;
(ii) π is a surjective morphism with connected fibres;
(iii) the generic fibre of π is a smooth connected curve of genus 1;
(iv) a section $\sigma : B \to X$ of π is given.

When $\pi : X \to B$ satisfies only the first three requirements above, we say that it is a *genus one fibration*.

We will denote the fibre over the point $P \in B$ with X_P.

Remark 1.2. Let $\pi : X \to B$ be an elliptic fibration, with section σ. Then, each smooth fibre X_P is an elliptic curve, where we choose as the origin the point $\sigma(P)$.

A *morphism* between two elliptic fibrations

$$\pi : X \longrightarrow B \quad \text{and} \quad \pi' : X' \longrightarrow B$$

is a morphism of varieties over B, i.e., a morphism

$$f : X \longrightarrow X'$$

such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes.

Not every fibre of π needs to be smooth: the *discriminant locus* of the fibration is the subset of B over which the fibres are singular

$$\Delta = \{P \in B \mid X_P \text{ is singular}\} \subseteq B.$$

A *rational section* of π is a rational map $s : B \dashrightarrow X$ such that $\pi \circ s = \mathrm{id}$ over the domain of s. The *Mordell-Weil group* of the fibration is

$$\mathrm{MW}(X) = \{s : B \dashrightarrow X \mid s \text{ is a rational section}\},$$

where the group law is given by addition fibrewise. Observe that, although the elements of the Mordell-Weil group are rational sections, we require its zero element to be a section.

1.1.1. The Weierstrass model of an elliptic fibration. The main reason for requiring that an elliptic fibration admits a section is that we can use the presence of this section to define the *Weierstrass model* of the fibration (see [**19,** Theorem 2.1]).

Let $\pi : X \to B$ be an elliptic fibration. By a slight abuse of notation, we still call the image of the distinguished section $S = \sigma(B)$ the distinguished section of X. Denote by i the inclusion $i : S \hookrightarrow X$. Then, we define the *fundamental line bundle* of the fibration as the line bundle on B

$$\mathcal{F} = \left(\pi_* i_* \mathcal{N}_{S|X}\right)^{-1},$$

and the *Weierstrass model* of X is then the image of the birational morphism

$$f : X \longrightarrow \mathbb{P}(\pi_* \mathcal{O}_X(3S)) = \mathbb{P}(\mathcal{F}^{\otimes 2} \oplus \mathcal{F}^{\otimes 3} \oplus \mathcal{O}_B)$$

defined by the canonical morphism $\pi^*\pi_*\mathcal{O}_X(3S) \to \mathcal{O}_X(3S)$. For the surjectivity of this map, we refer to [**19,** Proof of Theorem 2.1].

Remark 1.3. Let $p : W \to B$ be the Weierstrass model of $\pi : X \to B$. Then, W is defined in $\mathbb{P}(\mathcal{F}^{\otimes 2} \oplus \mathcal{F}^{\otimes 3} \oplus \mathcal{O}_B)$ by a Weierstrass equation

$$W : y^2 z = x^3 + \alpha_{102} x z^2 + \alpha_{003} z^3, \tag{1.1}$$

where $\alpha_{102} \in H^0(B, \mathcal{F}^{\otimes 4})$, $\alpha_{003} \in H^0(B, \mathcal{F}^{\otimes 6})$.

Remark 1.4. The discriminant locus Δ of a Weierstrass fibration $p : W \to B$ is not only a subset of B, but also a subvariety (actually, a subscheme) of B. It is defined in terms of the coefficients of the Weierstrass equation (1.1) by

$$\Delta : 4\alpha_{102}^3 + 27\alpha_{003}^2 = 0.$$

1.2. Calabi-Yau manifolds. Calabi-Yau manifolds are the higher-dimensional analogues of elliptic curves and $K3$ surfaces. The mathematical models of F-theory are all examples of Calabi-Yau manifolds: this property is needed on the total space of an elliptic fibration in order to get a physically consistent model (see, e.g., [**17, 18**]).

Definition 1.5. A *Calabi-Yau manifold* is a smooth compact Kähler variety X with

(a) trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$;
(b) $h^{0,q}(X) = 0$ for $q = 1, \dots \dim X - 1$, where $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$.

Example 1.6. If X is a Calabi-Yau variety of dimension 1, then X is a smooth Riemann surface of genus 1. In the case of dimension 2, the Calabi-Yau surfaces are the $K3$. In dimension 3, the Fermat quintic in $\mathbb{P}^4$, and, in fact, any smooth quintic, is a classical example of Calabi-Yau variety (see, for instance, [**4, 12**]). Other Calabi-Yau threefolds, which are complete intersections in projective spaces, are the complete intersection of two hypersurfaces of degree 3 in $\mathbb{P}^5$, of a hyperquadric and a hypersurface of degree 4 in $\mathbb{P}^5$, of two hyperquadrics and a hypercubic in $\mathbb{P}^6$, or the complete intersection of four hyperquadrics in $\mathbb{P}^7$. For other examples of Calabi-Yau manifolds, see e.g., [**14**].

2. A finiteness result.

2.1. Notation and general setting. In this section, we will fix the notation to be used throughout the remainder of the paper. Let B be a smooth projective surface, and let $\pi : X \to B$ be an elliptic threefold over B. As we observed in subsection 1.1.1, the Weierstrass model of π sits in a projective bundle of the form $\mathbb{P}(\mathcal{F}^{\otimes 2} \oplus \mathcal{F}^{\otimes 3} \oplus \mathcal{O}_B)$ for a suitable line bundle $\mathcal{F}$ on B. Now, we want to investigate all of the elliptic fibrations that can be embedded in similar ambient spaces.

This is the general framework with which we will be working. Let B be a smooth projective surface and $\mathcal{L}$ an ample line bundle on B. Let

$$p : Z \longrightarrow B$$

be the projective bundle of *lines* associate to the rank two vector bundle

$$\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B,$$

i.e.,

$$Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B).$$

Let $X \in |-K_Z|$ be an anticanonical subvariety, and let $\pi : X \to B$ be the restriction to X of the structure map p of Z.

2.2. Statement of the problem. The aim of the paper is to give an answer to the next question.

Main question. *For how many* (*and for which*) *pairs* (a, b) *is it true that, for the generic anticanonical subvariety* X *of* $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, *the map* π *defines a Calabi-Yau elliptic fibration over* B*?*

At first sight, the answer seems to be "almost for all pairs," for the following reasons:

(1) anticanonical subvarieties are Calabi-Yau by adjunction;
(2) since the generic fibre of π is a plane cubic curve (cf., (2.2)), we have always a genus 1 fibration.

Nevertheless, we are wrong. In fact, the map π can have no sections, or the total space X of the fibration can be singular. This last case can occur for two reasons:

(1) the generic $X \in |-K_Z|$ is reducible (see subsection 2.5.4);
(2) there is a section of π passing through a singular point of a fibre.

In the second case, if the singularities of X admit a small resolution, we can obtain a Calabi-Yau elliptic fibration; however, then, the resolved fibration would live in another ambient space, so we exclude them from this paper.

Theorem 2.1. *Let B be a smooth projective surface, and let $\mathcal{L}$ be an ample line bundle on B. Consider the projective bundle $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, with $a \geq b \geq 0$. Then, only for a finite number of pairs (a, b), the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ is a Calabi-Yau elliptic fibration over B.*

Remark 2.2. Our theorem can be considered as a reflex of the more general statement [**11,** Theorem 0.1] that there are only a finite number of deformation families of Calabi-Yau elliptic threefolds over rational surfaces with the property that any Calabi-Yau elliptic threefold over a rational surface is birational to one elliptic fibration in these families (see, also, [**5,** Theorem 1.1] for an analogue statement for Calabi-Yau elliptic fourfolds and fivefolds).

Remark 2.3. The theorem states only the finiteness, but its proof also provides a sort of algorithm to detect a finite superset of the set of pairs satisfying the main question.

Remark 2.4. Consider the projective bundle

$$\mathbb{P}(\mathcal{L}^\alpha \oplus \mathcal{L}^\beta \oplus \mathcal{O}_B),$$

with

$$(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}.$$

After tensoring $\mathcal{L}^\alpha \oplus \mathcal{L}^\beta \oplus \mathcal{O}_B$ with $\mathcal{L}^{-m}$, where $m = \min\{\alpha, \beta, 0\}$, and a permutation of the addends, a new vector bundle is obtained, of the form

$$\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B \quad \text{with } a \geq b \geq 0,$$

and such that

$$\mathbb{P}(\mathcal{L}^\alpha \oplus \mathcal{L}^\beta \oplus \mathcal{O}_B) \simeq \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B).$$

Thus, the bound on the possible (a, b)s in the hypothesis of Theorem 2.1 is not restrictive.

Before proving Theorem 2.1, in subsection 2.3, we will take a short digression on the projective bundle Z and its anticanonical subvarieties.

2.3. Calabi-Yau's in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$. We are interested in studying the anticanonical subvarieties of

$$Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B).$$

In this section, we want first to compute the Chern classes of Z and then find out what an equation looks like for an anticanonical subvariety.

2.3.1. The ambient bundle. The bundle projection $p : Z \to B$ gives the relative tangent bundle exact sequence

$$0 \longrightarrow \mathcal{T}_{Z|B} \longrightarrow \mathcal{T}_Z \longrightarrow p^*\mathcal{T}_B \longrightarrow 0,$$

from which we see that

$$c(Z) = c(\mathcal{T}_{Z|B})p^*c(B).$$

To compute the total Chern class of the relative tangent bundle, we exploit the fact that it fits into an Euler-type exact sequence (see **[9,** page 435, B.5.8]):

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow p^*E \otimes \mathcal{O}_Z(1) \longrightarrow \mathcal{T}_{Z|B} \longrightarrow 0,$$

where $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B$.

An explicit computation leads to the following results

$$\begin{aligned}
c_1(Z) &= p^*c_1(B) + (a+b)p^*L + 3\xi,\\
c_2(Z) &= abp^*L^2 + (a+b)p^*Lc_1(B) + 2(a+b)p^*L\xi\\
&\qquad + 3p^*c_1(B)\xi + p^*c_2(B) + 3\xi^2,\\
c_3(Z) &= 2(a+b)p^*c_1(B)L\xi + 3p^*c_1(B)\xi^2 + 3p^*c_2(B)\xi,\\
c_4(Z) &= 3p^*c_2(B)\xi^2,
\end{aligned} \tag{2.1}$$

where $L = c_1(\mathcal{L})$ and $\xi = c_1(\mathcal{O}_Z(1))$.

2.3.2. Equations for anticanonical subvarieties. Consider the projective bundle

$$Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B),$$

and let x, y and z denote sections on Z whose vanishing gives the subvariety of Z corresponding to the embeddings

$$\mathcal{L}^b \oplus \mathcal{O}_B \hookrightarrow E, \qquad \mathcal{L}^a \oplus \mathcal{O}_B \hookrightarrow E, \qquad \mathcal{L}^a \oplus \mathcal{L}^b \hookrightarrow E,$$

respectively. Then,

$$\begin{aligned} x &\in H^0(Z, p^*\mathcal{L}^a \otimes \mathcal{O}_Z(1)) \\ y &\in H^0(Z, p^*\mathcal{L}^b \otimes \mathcal{O}_Z(1)) \\ z &\in H^0(Z, \mathcal{O}_Z(1)), \end{aligned}$$

and we can use $(x : y : z)$ as global homogeneous coordinates in Z over B.

Since $c_1(Z) = p^*c_1(B) + (a+b)p^*L + 3\xi$, by (2.1), an equation F defining an anticanonical hypersurface must be cubic in $(x : y : z)$, of the form

$$F = \sum_{i+j+k=3} \alpha_{ijk} x^i y^j z^k, \tag{2.2}$$

and the coefficient α_{ijk} of the monomial $x^i y^j z^k$ must be a section of a suitable line bundle, according to Table 1.

Table 1: Cubic monomials and the weight of their coefficients.

Monomial	Weight of the coefficient
x^3	$c_1(B) - 2aL + bL$
x^2y	$c_1(B) - aL$
xy^2	$c_1(B) - bL$
y^3	$c_1(B) + aL - 2bL$
x^2z	$c_1(B) - aL + bL$
xyz	$c_1(B)$
y^2z	$c_1(B) + aL - bL$
xz^2	$c_1(B) + bL$
yz^2	$c_1(B) + aL$
z^3	$c_1(B) + aL + bL$

2.3.3. Chern classes of anticanonical subvarieties. We want to compute the Chern classes of a smooth $X \in |-K_Z|$. We have

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

and the normal bundle sequence of X in Z

$$0 \longrightarrow \mathcal{T}_X \longrightarrow i^*\mathcal{T}_Z \longrightarrow \mathcal{N}_{X|Z} \longrightarrow 0,$$

which gives the following relation between the total Chern classes

$$i^*c(Z) = c(X)c(\mathcal{N}_{X|Z}) = c(X)i^*(1 - K_Z).$$

Since we know $c(Z)$ from subsection 2.3.1, and $1 - K_Z$ is a unit in the Chow ring of Z, we deduce the following formulae for the Chern classes of X:

$$\begin{aligned} c_1(X) &= 0, \\ c_2(X) &= 3\xi^2_{|X} + \pi^*(2(a+b)L + 3c_1(B))\xi_{|X} \\ &\qquad + \pi^*((a+b)Lc_1(B) + abL^2 + c_2(B)), \\ c_3(X) &= -9\pi^*c_1(B)\xi^2_{|X} - \pi^*(2(a^2 - ab + b^2)L^2 \\ &\qquad + 6(a+b)Lc_1(B) + 3c_1(B)^2)\xi_{|X}. \end{aligned} \tag{2.3}$$

Remark 2.5. In particular, we have a formula for the Euler-Poincaré characteristic of our varieties:

$$\chi_{\mathrm{top}}(X) = \deg c_3(X) = -6(a^2 - ab + b^2)L^2 - 18c_1(B)^2.$$

2.4. Hypersurfaces in Calabi-Yau threefolds. In this section, we recall a result which will be crucial in the proof of our Main theorem 2.1 (see subsection 2.5.2). Using Proposition 2.6, we will, in fact, reduce our general problem to a simpler one, concerning only the base surface of the elliptic fibration.

Assume that X is any threefold with $c_1(X) = 0$ and that $i : S \hookrightarrow X$ is the inclusion of a smooth surface. The techniques used in subsections

2.3.1 and 2.3.3 can be used to acquire more information on how S is embedded in X.

From the normal bundle sequence

$$0 \longrightarrow T_S \longrightarrow i^*T_X \longrightarrow \mathcal{N}_{S|X} \longrightarrow 0,$$

we obtain that

$$i^*c(X) = c(S)c(\mathcal{N}_{S|X}). \tag{2.4}$$

To compute $i_*c(\mathcal{N}_{S|X})$, we can argue in two ways:

- By the self-intersection formula, $c(\mathcal{N}_{S|X}) = i^*(1 + [S])$, where $[S]$ is the class of S in the Chow ring of X. Thus,

$$i_*c(\mathcal{N}_{S|X}) = i_*i^*(1 + [S]) = (1 + [S])[S] = [S] + [S]^2. \tag{2.5}$$

- Using (2.4), we have that $c(\mathcal{N}_{S|X}) = i^*c(X) \cdot c(S)^{-1}$, and thus,

$$\begin{aligned} i_*c(\mathcal{N}_{S|X}) &= c(X) \cdot i_*(c(S)^{-1}) \\ &= [S] - i_*c_1(S) + c_2(X)[S] - i_*(c_2(S) - c_1(S)^2). \end{aligned} \tag{2.6}$$

Comparing (2.5) and (2.6), we get that

$$[S]^2 = -i_*c_1(S), \qquad c_2(X)[S] = i_*(c_2(S) - c_1(S)^2) \tag{2.7}$$

Taking the degree of the second relation in (2.7) gives us the following result.

Proposition 2.6 ([**8,** Lemma 4.4])**.** *Let X be a threefold with $c_1(X) = 0$, and S a smooth hypersurface, with associated class $[S]$. Then*

$$c_2(X)[S] = \chi_{\text{top}}(S) - K_S^2.$$

The first relation in (2.7) gives an interpretation to $[S]^2$. In order to also understand what $[S]^3$ is, we use the adjunction formula for S in X:

$$c_1(S) = i^*(c_1(X) - [S]) = -i^*[S].$$

From this relation, we have that

$$\deg c_1(S)^2 = \deg i^*[S]^2 = \deg i_*i^*[S]^2 = \deg[S]^3,$$

i.e., $K_S^2 = [S]^3$.

2.4.1. The fundamental line bundle of a Calabi-Yau elliptic fibration. Assume that $\pi : X \to B$ is an elliptic fibration with section S, where X is a Calabi-Yau threefold. We can use the first relation in (2.7) to compute the fundamental line bundle of π. In fact, since $\pi_*([S]) = B$, we have that

$$\pi_* i_* c(\mathcal{N}_{S|X}) = \pi_*([S] + [S]^2) = 1 - p_* i_* c_1(S) = 1 - c_1(B).$$

Hence, if $\mathcal{F}$ is the fundamental line bundle of π, then $c_1(\mathcal{F}) = c_1(B)$, and thus, we can embed the Weierstrass model of π in

$$\mathbb{P}(\omega_B^{-2} \oplus \omega_B^{-3} \oplus \mathcal{O}_B),$$

where ω_B is the anticanonical line bundle of B.

2.5. Proof of the Main theorem. We will split the proof of Theorem 2.1 into several steps to make it clearer. In the first step (subsection 2.5.1) we will show that possibly with the exception of a finite number of pairs (a, b), the genus one fibrations X in

$$Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$$

admit a section. In the second step (subsection 2.5.2) we will concentrate on such pairs and use the presence of the section to reduce the problem to a new problem concerning only the intersection form on the base. In the third step (subsections 2.5.3, 2.5.4) we will show that this last problem has solution only for a finite number of pairs (a, b), and this will be accomplished in two different ways, essentially, according to whether or not $\mathcal{L}$ is a rational multiple of ω_B^{-1}.

We recall here the statement of Theorem 2.1.

Main theorem. *Let B be a smooth projective surface, and $\mathcal{L}$ an ample line bundle on B. Consider the projective bundle $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, with $a \geq b \geq 0$. Then, only for a finite number of pairs (a, b), the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ is a Calabi-Yau elliptic fibration over B.*

2.5.1. Step 1. In this first step, we use the information provided by Table 1 to determine when some of the cohomology spaces where the coefficients α_{ijk} of (2.2) lie are a priori zero.

Since L is an ample divisor, there exists a suitable integer n_0 such that $nL + K_B$ is ample for any $n \geq n_0$. Fix one such n_0 (e.g., the least one), then $H^0(B, c_1(B) - nL) = 0$ for any $n \geq n_0$, for otherwise, $c_1(B) - nL = -(nL + K_B)$ would be effective. In particular, there is an infinite number of pairs (a, b) satisfying $2a - b \geq n_0$ in the octant $a \geq b \geq 0$: the divisor $(2a - b)L + K_B$ is ample; hence, by the previous argument

$$H^0(B, (b - 2a)L - K_B) = H^0(B, -((2a - b)L + K_B)) = 0,$$

and so the coefficient of x^3 in (2.2) is identically 0 (cf., Table 1). Equation (2.2) then looks like

$$F = \cancel{\alpha_{300}x^3} + \alpha_{210}x^2y + \alpha_{201}x^2z + \dots,$$

and thus, $\pi : X \to B$ has a distinguished section, given by

$$P \longmapsto (1 : 0 : 0) \in X_P. \tag{2.8}$$

Observe that there is only a finite number of pairs (a, b) in the octant $a \geq b \geq 0$ such that $2a - b < n_0$. For such pairs, the generic anticanonical hypersurface in Z is a genus 1 fibration; however, since the equation F that defines the variety is general, it is difficult to see whether or not there are sections. However, we can ignore them from now on since they are only a finite number.

In Figure 1, this fact is shown in the particular case where $B = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$.

Remark 2.7. Exploiting this argument and comparing with the first four rows in Table 1, it is then easy to see that, if (a, b) satisfy

$$\begin{cases} 2a - b \geq n_0 \\ a \geq n_0 \\ b \geq n_0 \\ 2b - a \geq n_0 \\ a \geq b \geq 0 \end{cases} \longrightarrow \frac{1}{2}a + \frac{1}{2}n_0 \leq b \leq a, \tag{2.9}$$

then the coefficients α_{ij0} are all necessarily identically zero. In particular, equation (2.2) factors as $F(x, y, z) = z \cdot f(x, y, z)$, and thus,

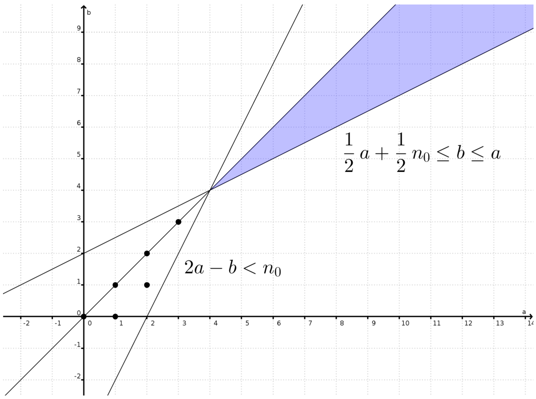

FIGURE 1. The finitely many cases with $2a - b < n_0$. The picture refers to the particular case where $B = \mathbb{P}^2$ and L is the class of a line so that $n_0 = 4$. The shaded area corresponds to the bounds given in (2.9).

$F = 0$ cannot define a smooth variety. However, this is not enough to conclude the proof of our Main theorem, since an infinite number of pairs (a, b)s remains.

2.5.2. Step 2. It follows from the first step that, in the infinitely many cases where $2a - b \geq n_0$, the generic anticanonical hypersurface of Z admits the presence of a section, as defined in (2.8). In this step, we want to use the relation in Proposition 2.6 to drop the problem down to B.

Let S be the image of the section (2.8). By Proposition 2.6, we have that

$$c_2(X)[S] = c_2(S) - c_1(S)^2 = c_2(B) - c_1(B)^2,$$

and thus, we need to compute the term on the left.

Let $i : X \hookrightarrow Z$ be the inclusion: by (2.3), we have that $c_2(X) = i^*\psi$, where

$$\psi = 3\xi^2 + p^*(2(a+b)L + 3c_1(B))\xi + p^*((a+b)Lc_1(B) + abL^2 + c_2(B)),$$

and hence,

$$\deg c_2(X)[S] = \deg i^*\psi \cdot [S] = \deg i_*(i^*\psi \cdot [S]) = \deg \psi \cdot i_*[S].$$

In order to compute $i_*[S]$, which is the class of S in the Chow ring of Z, we recall that S is defined in Z by $y = z = 0$ and that this intersection is transverse. Therefore,

$$i_*[S] = (\xi + bp^*L)\xi = \xi^2 + bp^*L\xi,$$

and the relation $\deg \psi \cdot i_*[S] = c_2(B) - c_1(B)^2$ reduces to

$$a(a-b)L^2 + (b-2a)c_1(B)L + c_1(B)^2 = 0. \tag{2.10}$$

Observe now that we have a problem concerning only the base and its intersection theoretic properties. Letting $(a, b) \in \mathbb{R}^2$, equation (2.10) defines a plane conic, which is reducible if and only if

$$L^2 = 0 \quad \text{or} \quad (c_1(B)L)^2 = L^2 c_1(B)^2.$$

The first case is impossible since we are assuming that L is ample.

By the Hodge index theorem, $(c_1(B)L)^2 \geq L^2 c_1(B)^2$ and

$$(c_1(B)L)^2 = L^2 c_1(B)^2 \Longleftrightarrow rL \equiv sc_1(B) \tag{2.11}$$

for suitable integers r and s (where $\equiv$ denotes numerical equivalence), and $s \neq 0$ since L is ample.

Our next step is to study the conic defined in (2.10) when it is irreducible (subsection 2.5.3) and when it is reducible (subsection 2.5.4), and to show that, in each of these two cases, we have only a finite number of integral points (a, b) in the octant $a \geq b \geq 0$ on the conic (2.10).

2.5.3. Step 3**, Case** 1**.** We concentrate first on the case when the conic (2.10) is irreducible; it is a hyperbola, with asymptotes

$$a = \frac{c_1(B)L}{L^2} \quad \text{and} \quad b = a - \frac{c_1(B)L}{L^2}.$$

Observe that, if we multiply (2.10) by L^2, then it can be written as

$$(L^2 a - c_1(B)L)(L^2(a-b) - c_1(B)L) = (c_1(B)L)^2 - c_1(B)^2 L^2,$$

and thus, the integral points of (2.10) are the integral pairs (a_i, b_i) having

$$a_i = \frac{d_i + c_1(B)L}{L^2}, \qquad b_i = \frac{d_i - d_i'}{L^2} = \frac{d_i^2 + c_1(B)^2L^2 - (c_1(B)L)^2}{L^2 d_i},$$

where d_i runs through all the divisors of $(c_1(B)L)^2 - c_1(B)^2L^2$ and $d_i' = [(c_1(B)L)^2 - c_1(B)^2L^2]/d_i$. Hence, it is clear that they are finite.

2.5.4. Step 3, Case 2. Now, we concentrate on the case where the conic (2.10) is reducible, i.e., the case where $(c_1(B)L)^2 = L^2c_1(B)^2$.

The equation for the conic (2.10) is

$$(L^2a - c_1(B)L)(L^2a - L^2b - c_1(B)L) = 0.$$

By (2.11), $rL \equiv sc_1(B)$ implies $(c_1(B)L)/L^2 = r/s$. We have two further subcases, according to whether r/s is or is not a positive integer.

If $r/s \notin \mathbb{N}$, the two lines

$$a = \frac{c_1(B)L}{L^2} \quad \text{and} \quad b = a - \frac{c_1(B)L}{L^2}$$

have no integral points in the octant $a \geq b \geq 0$. This means that we have no new smooth Calabi-Yau fibrations.

If, instead, $r/s \in \mathbb{N}$, then, in the range $a \geq b \geq 0$, we have a finite number of pairs (a, b) on the line $a = (c_1(B)L)/L^2$, namely, $(c_1(B)L)/L^2 + 1 = r/s + 1$, and an infinite number of (a, b) on the line $b = a - (c_1(B)L)/L^2$. To give a limitation on the number of the latter ones, we look at the coefficients of the first monomials in equation (2.2), listed in Table 2, and use the integer n_0 introduced in subsection 2.5.1. It was defined by the property that $nL - c_1(B)$ is ample for any $n \geq n_0$.

Table 2: Weight of α_{ij0} on the line $b = a - (c_1(B)L)/L^2$.

Monomial	Weight of the coefficient
x^3	$c_1(B) - \left(b + 2\frac{r}{s}\right) L$
x^2y	$c_1(B) - \left(b + \frac{r}{s}\right) L$
xy^2	$c_1(B) - bL$
y^3	$c_1(B) - \left(b - \frac{r}{s}\right) L$

Arguing as in Remark 2.7, we now find a bound: if $b \geq n_0 + r/s$, we have that all of the bundles listed in Table 2 are anti-ample. Hence, the coefficients of x^3, x^2y, xy^2 and y^3 in (2.2) are necessarily identically zero, and thus, the equation F for the variety factors as $F(x,y,z) = z \cdot f(x,y,z)$. Then, $F = 0$ cannot define a smooth variety.

Observe that, in this case, $z = 0$ defines a divisor whose class is ξ, while $f(x,y,z) = 0$ defines a divisor of class $p^*c_1(B) + (a+b)p^*L + 2\xi$, which is neither a Calabi-Yau variety nor an elliptic fibration.

In particular, we have only a finite number of pairs (a,b) on the line $b = a - (c_1(B)L)/L^2$ such that the generic anticanonical hypersurface in

$$\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$$

could define a Calabi-Yau elliptic fibration over B, and a limitation is

$$\frac{r}{s} \leq a \leq n_0 + 2\frac{r}{s} - 1, \quad 0 \leq b \leq n_0 + \frac{r}{s} - 1.$$

We can be even more precise (see, also, Remark 3.1) since, up to numerical equivalence, we have $rL/s \equiv c_1(B)$, and thus, $nL - c_1(B) \equiv (n - r/s)L$ is ample if $n \geq r/s - 1$. This means that we can choose $n_0 = r/s - 1$, which gives us the limitations

$$\frac{r}{s} \leq a \leq 3\frac{r}{s}, \quad 0 \leq b \leq 2\frac{r}{s}. \tag{2.12}$$

Remark 2.8. It is interesting to observe that the "extreme" case of limitation (2.12) occur. In fact, choosing $(a,b) = (3r/s, 2r/s)$, from the relation $rL \equiv sc_1(B)$, we obtain

$$3\frac{r}{s}L \equiv 3c_1(B), \qquad 2\frac{r}{s}L \equiv 2c_1(B),$$

and thus, we are dealing with the projective bundle

$$\mathbb{P}(\omega_B^{-3} \oplus \omega_B^{-2} \oplus \mathcal{O}_B),$$

where we can find all of the Weierstrass models of elliptic fibrations over B, whose total space is a Calabi-Yau manifold (cf., subsection 2.4.1).

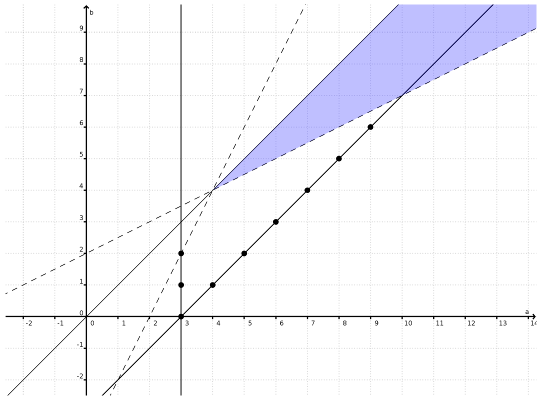

FIGURE 2. If $B = \mathbb{P}^2$ and L is the class of a line, then we are in the case described in subsection 2.5.4, to which the picture corresponds.

If $r/s \in \mathbb{N}$, then we have at most

$$3\,\frac{r}{s} + 1 = \underbrace{\left(\frac{r}{s} + 1\right)}_{\substack{\text{pairs on the line} \\ a = (c_1(B)L)/L^2}} + \underbrace{\left(2\frac{r}{s} + 1\right)}_{\substack{\text{pairs on the line} \\ b = a - (c_1(B)L)/L^2}} - \underbrace{1}_{\substack{\text{the common case} \\ (a,b) = \left((c_1(B)L)/L^2, 0\right)}} \tag{2.13}$$

such pairs (a, b).

2.5.5. Conclusion. Only for a finite number of pairs (a, b) is the generic anticanonical hypersurface in $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ a smooth Calabi-Yau elliptic fibration, which completes the proof of Theorem 2.1.

We summarize the results obtained in Table 3.

Remark 2.9. We want to stress that we have proved that the number of genus 1 fibrations whose total space is smooth lies in a finite number of $\mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$; however, we do not know a priori whether all of them are elliptic fibrations. In the finite number of cases detected in subsection 2.5.1, it is unclear, in fact, if there is at least a section.

Remark 2.10. We can also read our result in another way: only for a finite number of $Z = \mathbb{P}(\mathcal{L}^{\otimes a} \oplus \mathcal{L}^{\otimes b} \oplus \mathcal{O}_B)$ the generic element of the

TABLE 3. Summary of the results.

$(2a-b)L+K_B$ is not ample	$(2a-b)L+K_B$ is ample		
	$(K_BL)^2 \neq K_B^2L^2$	$(K_BL)^2 = K_B^2L^2$	
		$r/s \notin \mathbb{N}$	$r/s \in \mathbb{N}$
Finite number of cases, which are a priori only genus one fibrations. It is not clear if they have at least one section or not.	The conic (2.10) is irreducible, and we have a finite number of Calabi-Yau elliptic fibrations.	No pairs	Finite number of Calabi-Yau elliptic fibrations, at most $3r/s+1$.

anticanonical system $|-K_Z|$ is a *smooth* hypersurface. We now focus on the infinite number of cases where this does not hold: in view of Bertini's theorem, we can then claim that, for such ambient spaces Z, the linear system $|-K_Z|$ is not base point free.

3. Examples. We want to run this program in two cases of interest: the case where the base B is a del Pezzo surface and L is a rational multiple of an anticanonical divisor, and the case where B is a Hirzebruch surface and L is any ample line bundle.

The reason why del Pezzo surfaces are interesting is provided by the following observation.

Remark 3.1. Let B be a surface and L an ample divisor on B. Assume that, at the end of step 2 (subsection 2.5.2), the conic (2.10) is reducible. It easily follows from (2.11) that B is a del Pezzo surface and L is (numerically) a rational multiple of $c_1(B)$.

Before dealing with the general case in subsection 3.2, it is worthwhile separately studying the subcase $B = \mathbb{P}^2$ (subsection 3.1).

The motivation for our interest in Hirzebruch surfaces is the following.

Remark 3.2. Assume that $\pi : X \longrightarrow B$ is a smooth, elliptic Calabi-Yau threefold, with B a smooth minimal surface. It follows from [**10,** Corollary 3.3] and [**10,** Theorem 3.1] that, either B is birationally ruled, or B is a K3 or Enriques surface and the j-invariant function is constant. In the first case, it follows from the discussion following [**10,** Corollary 3.3] that B can either be $\mathbb{P}^2$ or a geometrically ruled surface with Sakai invariant e bounded by $0 \leq e \leq 12$. Finally, from [**20,** Main theorem], we deduce that B is rational; hence, it is $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_e$ (with $e \neq 1$).

Hirzebruch surfaces will be dealt with in subsection 3.3.

3.1. The case of $B = \mathbb{P}^2$. Observe that, if B is a smooth surface with $\operatorname{Pic} B \simeq \mathbb{Z}$, then we are necessarily in the case described in subsection 2.5.4.

Take $B = \mathbb{P}^2$, and $L = dl$ for $d \in \mathbb{N}$, l a line in $\mathbb{P}^2$ (Figures 1, 2 correspond to the choice $d = 1$). Now, we compute the least integer n_0 such that $n_0 L + K_{\mathbb{P}^2}$ is ample:

$$n_0 = \begin{cases} 4 & \text{if } d = 1 \\ 2 & \text{if } d = 2, 3 \\ 1 & \text{if } d \geq 4; \end{cases}$$

thus, the cases satisfying $2a - b < n_0$ (subsection 2.5.1) are

$$\begin{array}{ll} (0,0), (1,0), (1,1), (2,1), (2,2), (3,3) & \text{if } d = 1 \\ (0,0), (1,1) & \text{if } d = 2, 3 \\ (0,0) & \text{if } d \geq 4. \end{array}$$

Since $c_1(\mathbb{P}^2) = 3l$, we have

$$rdl = 3sl \iff rd = 3s \iff \frac{r}{s} = \frac{3}{d}.$$

We only have two cases where the ratio r/s is an integer, which correspond to

$$d = 1 \quad \text{and} \quad d = 3,$$

i.e., $L = l$ or $L = -K_{\mathbb{P}^2}$. For all of the other cases, the only possible pair is then $(a, b) = (0, 0)$, with the exception of $L = 2l$, which also has $(a, b) = (1, 1)$.

For $d = 3$, there are five possibilities: besides the two we already know, on the reducible conic (2.10), we also have the pairs $(a, b) = (1, 1), (2, 1), (2, 3)$.

Table 4: Summary of cases with $B = \mathbb{P}^2$, $L = dl$ and $d \geq 2$.

d	Possible (a, b)
2	$(0, 0)$, $(1, 1)$
3	$(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 1)$, $(2, 3)$
≥ 4	$(0, 0)$

The only remaining case is $d = 1$, in the situation of subsection 2.5.4. We must count the integral points on the conic

$$(a - 3)(a - b - 3) = 0,$$

which are in the first octant and have $b \leq 6$ (estimate (2.12)). On the line $a = 3$, we have the points $(3, 2)$, $(3, 1)$ and $(3, 0)$, while on the line $b = a - 3$, we have the points $(4, 1)$, $(5, 2)$, $(6, 3)$, $(7, 4)$, $(8, 5)$ and $(9, 6)$.

Then, the pairs (a, b) such that the generic anticanonical hypersurface in the bundle

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2})$$

could be a smooth Calabi-Yau elliptic fibration are the following 15:

$$\begin{array}{cccccc} (0,0), & (1,0), & (1,1), & (2,1), & (2,2), & (3,3), \\ & (3,2), & (3,1), & (3,0), & & \\ (4,1), & (5,2), & (6,3), & (7,4), & (8,5), & (9,6). \end{array}$$

Remark 3.3. Some of these families are already known. For example, the families corresponding to $(a, b) = (3, 3)$ and $(6, 3)$ were analyzed in [**1**], while the one corresponding to $(a, b) = (6, 3)$ and $(3, 0)$ was analyzed in [**3**].

3.2. The case of del Pezzo surfaces. Let B denote a del Pezzo surface and $\mathcal{L}$ a rational multiple of the anticanonical bundle, say, $\mathcal{L}^r = \omega_B^{-s}$ (this is the natural setting by Remark 3.1). Let $n_0 = [r/s]+1$. Then, $nL + K_B$ is ample for all $n \geq n_0$. With the notation of subsection 2.5.1, the number of pairs (a, b) for which we cannot ensure the presence of a section, i.e., those satisfying the system

$$\begin{cases} a \geq b \geq 0 \\ 2a - b < n_0, \end{cases}$$

is

$$\begin{cases} \dfrac{n_0(n_0+2)}{4} & \text{for } n_0 \text{ even,} \\ \dfrac{n_0^2+4n_0-1}{4} & \text{for } n_0 \text{ odd.} \end{cases} \tag{3.1}$$

If the ratio r/s is not an integer, then these are the only cases among which we can find elliptic fibrations.

Remark 3.4. In particular, for $r < s$ we only have the pair $(a, b) = (0, 0)$.

If the ratio r/s is an integer m, then $r = ms$, and thus, $mL = -K_B$, i.e., L is a submultiple of $-K_B$. In this case, $n_0 = m + 1$, and we must also count the points on the reducible conic (2.10): in view of estimate (2.13), these are $3m$ since the point $(a, b) = (m, m)$ was already taken into account. However, then, the number of families of elliptic Calabi-Yau threefolds over B is bounded by

$$\begin{cases} \dfrac{m^2+18m+4}{4} & \text{for } m \text{ even,} \\ \dfrac{m^2+16m+3}{4} & \text{for } m \text{ odd.} \end{cases} \tag{3.2}$$

Remark 3.5. Observe that these results agree with those found in subsection 3.1 for the plane $\mathbb{P}^2$. Let l be the class of a line. Then:

(1) For $L = l$, we have $r = 3$, $s = 1$ and so we can use (3.2) with $m = 3$; we have 15 cases.
(2) For $L = 2l$, we have $r = 3$, $s = 2$ and so we can use (3.1) with $n_0 = 2$; we have 2 cases.
(3) For $L = 3l$, we have $r = s = 1$ and so we can use (3.2) with $m = 1$; we have 5 cases.
(4) For $L = kl$, with $k \geq 4$, we have $r/s < 1$ and so we can use (3.1) with $n_0 = 1$; we have only 1 case.

3.3. The case of Hirzebruch surfaces. Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1})$ be a Hirzebruch surface. Then, the Picard group of $\mathbb{F}_e$ is generated by two classes, C and f, with $C^2 = -e$, $C \cdot f = 1$ and $f^2 = 0$.

The canonical divisor of $\mathbb{F}_e$ is $K_{\mathbb{F}_e} = -2C - (e+2)f$, and a divisor $L = \alpha C + \beta f$ is ample if and only if (cf., [**13,** Corollary V.2.18])

$$\begin{cases} \alpha > 0 \\ \beta > \alpha e. \end{cases} \tag{3.3}$$

It is then easy to see that $-K_{\mathbb{F}_e}$ is ample if and only if $e < 2$, and thus, the only minimal Hirzebruch surface which is also a del Pezzo surface is

$$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1.$$

In what follows, we will then assume that $e \geq 2$.

Following along the lines of the proof of Theorem 2.1, we first compute the less integer n_0 such that $K_{\mathbb{F}_e} + nL$ is ample for every $n \geq n_0$. Due to (3.3), we have that

$$n_0 = \begin{cases} 3 & \text{if } \alpha = 1 \\ 2 & \text{if } \alpha = 2 \\ 1 & \text{if } \alpha \geq 3. \end{cases}$$

Hence, as a first result, the pairs (a, b) satisfying $2a - b < n_0$ are

$$\begin{cases} (0,0), (1,0), (1,1), (2,2) & \text{if } \alpha = 1 \\ (0,0), (1,1) & \text{if } \alpha = 2 \\ (0,0) & \text{if } \alpha \geq 3. \end{cases}$$

Next, we must consider the conic (2.10) and find its integral points in the octant $a \geq b \geq 0$. Observe that (2.10) can also be written as

$$(K_{\mathbb{F}_e} + aL)^2 = bL(K_{\mathbb{F}_e} + aL),$$

which is easier to deal with. Before finding the integral points on this conic, we make a small digression, giving some useful estimates for some intersection numbers. We have that

$$-\frac{K_{\mathbb{F}_e} \cdot L}{L^2} = \frac{2\beta - e\alpha - 2\alpha}{\alpha(2\beta - e\alpha)} = \frac{1}{\alpha} + \frac{2}{2\beta - e\alpha}.$$

Observe, then, that:

(1) If $\alpha = 1$, we have $\beta > e \geq 2$, and thus, $\beta \geq 3$. As a consequence,

$$2\beta - e = \beta + (\beta - e) \geq 3 + 1 = 4;$$

thus, we deduce that

$$1 < -\frac{K_{\mathbb{F}_e} \cdot L}{L^2} \leq 1 + \frac{1}{2} = \frac{3}{2}.$$

(2) If $\alpha \geq 2$, arguing as above, we deduce that $\beta \geq 5$, and thus,

$$2\beta - e\alpha = \beta + (\beta - e\alpha) \geq 5 + 1 = 6.$$

This means that

$$0 < -\frac{K_{\mathbb{F}_e} \cdot L}{L^2} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

With these estimates, we can then prove the next lemma.

Lemma 3.6. *Let L be an ample line bundle on the Hirzebruch surface $\mathbb{F}_e$ with $e \geq 2$. Then, the conic* (2.10) *has no integral points* (a, b) *with $a \geq 3$.*

Proof. Write $L = \alpha C + \beta f$, as above, and observe that we have $-K_{\mathbb{F}_e} \cdot L \geq 6$. This means, in particular, that the *intersection of any ample divisor with the canonical divisor is strictly negative.* We split the proof into two parts, according to whether $\alpha = 1$ or $\alpha \geq 2$.

If $\alpha = 1$, the oblique asymptote $b = a + (K_{\mathbb{F}_e}/L^2)$ has $-3/2 \leq (K_{\mathbb{F}_e}/L^2) < -1$, and thus, it suffices to show that, given any integer $a \geq 3$, the b-coordinate of the point (a, b) on the conic (2.10) satisfies the inequality $b > a - 2$. This means that this point lies between the asymptote and the closest integral point below it; therefore, this point cannot be integral. Since we can write our conic as

$$b = \frac{(K_{\mathbb{F}_e} + aL)^2}{L(K_{\mathbb{F}_e} + aL)}$$

and, for $a \geq 3$, we have that $(K_{\mathbb{F}_e} + aL)$ is ample, we see that $b > a - 2$ is equivalent to

$$(K_{\mathbb{F}_e} + aL)(K_{\mathbb{F}_e} + 2L) > 0.$$

However, explicitly writing this product, we find that it turns out to be $(2\beta - e - 2)(a-2)$, which is positive since L is ample and $a \geq 3$. Thus, we are finished in this case.

In order to deal with the case $\alpha \geq 2$, we argue in the same manner but, since we have

$$-\frac{5}{6} \leq \frac{K_{\mathbb{F}_e} \cdot L}{L^2} < 0,$$

we want to show that

$$b = \frac{(K_{\mathbb{F}_2} + aL)^2}{L(K_{\mathbb{F}_e} + aL)} > a - 1.$$

This is equivalent to $(K_{\mathbb{F}_e} + aL)(K_{\mathbb{F}_e} + L) > 0$, which is true if $\alpha \geq 3$ since it is the intersection of two ample divisors. It remains to show that the inequality also holds when $\alpha = 2$. Explicitly writing the intersection product, we find that

$$(K_{\mathbb{F}_e} + aL)(K_{\mathbb{F}_e} + L) = 2(a-1)(\beta - e - 2),$$

which is positive since L is ample and $a \geq 3$. $\square$

Due to the previous lemma, it remains to deal with only five integral points in the plane.

(1) The point $(2,0)$ belongs to the conic (2.10) if and only if $(K_{\mathbb{F}_e} + 2L)^2 = 0$. Thus, if $\alpha \geq 2$, it cannot be a point of the conic, as $K_{\mathbb{F}_e} + 2L$ is ample. On the contrary, if $\alpha = 1$, then

$$K_{\mathbb{F}_e} + 2L = (2\beta - e - 2)f,$$

and thus, $(K_{\mathbb{F}_e} + 2L)^2 = 0$. Therefore, we do have an integral point.

(2) The point $(2,1)$ belongs to the conic (2.10) if and only if $(K_{\mathbb{F}_e} + 2L)(K_{\mathbb{F}_2} + L) = 0$. We can assume that $\alpha \neq 1$ since we know that, in this case, the conic passes through the point $(2,0)$. We can also discard all of the cases with $\alpha \geq 3$ since the intersection on the left is the intersection of two ample divisors. Therefore, we are only left with the case where $\alpha = 2$, in which case, we have

$$(K_{\mathbb{F}_e} + 2L)(K_{\mathbb{F}_2} + L) = 2(\beta - e - 2) = 0.$$

The only possible line bundle, then, is

$$L = 2C + (e-2)f = -K_{\mathbb{F}_e};$$

however, we must discard this possibility as $-K_{\mathbb{F}_e}$ is not ample.

(3) The point $(2,2)$ belongs to the conic (2.10) if and only if $K_{\mathbb{F}_e}\cdot(K_{\mathbb{F}_e}+2L)=0$. As before, we can assume that $\alpha\geq 2$, in which case, $K_{\mathbb{F}_e}+2L$ is ample. However, then, as pointed out in the proof of Lemma 3.6, its intersection with an anticanonical divisor is negative. Hence, we do not have new integral points.

(4) The point $(1,0)$ belongs to the conic (2.10) if and only if $(K_{\mathbb{F}_e}+L)^2=0$. Recall that the points we are now considering must also satisfy $b\leq 2a-n_0$; thus, we can assume that $\alpha\geq 2$. As before, if $\alpha\geq 3$, we have the self-intersection of an ample divisor. Hence, it cannot be zero. If $\alpha=2$, then

$$K_{\mathbb{F}_e}+L=(\beta-e-2)f,$$

and thus, $(K_{\mathbb{F}_e}+L)^2=0$. This means that we have an integral point.

(5) The point $(1,1)$ belongs to the conic (2.10) if and only if

$$K_{\mathbb{F}_e}\cdot(K_{\mathbb{F}_e}+L)=0.$$

Due to the limitation $b\leq 2a-n_0$, we can restrict to $\alpha\geq 3$. In this case, $K_{\mathbb{F}_e}+L$ is ample, and thus, its intersection with an anticanonical divisor is negative. Hence, we do not have new integral points.

We can now sum up these results in the next proposition.

Proposition 3.7. *Let $\mathbb{F}_e$ be a Hirzebruch surface, with $e\geq 2$. Let $L=\alpha C+\beta f$ be an ample divisor on $\mathbb{F}_e$, corresponding to the line bundle $\mathcal{L}$. Then, the generic anticanonical divisor in*

$$\mathbb{P}(\mathcal{L}^a\oplus\mathcal{L}^b\oplus\mathcal{O}_{\mathbb{F}_e})$$

defines a smooth Calabi-Yau elliptic fibration over $\mathbb{F}_e$ only if (a,b) is one in the following list:

$$\begin{array}{ll} (0,0),(1,0),(1,1),(2,2);(2,0) & \textit{if } \alpha=1,\\ (0,0),(1,1);(1,0) & \textit{if } \alpha=2,\\ (0,0) & \textit{if } \alpha\geq 3. \end{array}$$

Remark 3.8. Concerning the surface

$$\mathbb{F}_0=\mathbb{P}^1\times\mathbb{P}^1,$$

we have that $\operatorname{Pic}\mathbb{F}_0$ is generated by two classes, f_1 and f_2, with intersections $f_1^2=f_2^2=0$ and $f_1\cdot f_2=1$. The canonical divisor is

$$K_{\mathbb{F}_0} = -2f_1 - 2f_2,$$

and $-K_{\mathbb{F}_0}$ is ample. Therefore, we must distinguish two cases.

(1) The line bundle L is $L = f_1 + f_2$. In this case, we can apply the arguments of subsection 3.2, and we see that the possible pairs (a, b) are:

$$\begin{gathered}(0,0),(1,0),(1,1),(2,2),\\(2,0),(2,1),\\(3,1),(4,2),(5,3),(6,4).\end{gathered}$$

(2) The line bundle $L = \alpha f_1 + \beta f_2$ is not a rational multiple of $-K_{\mathbb{F}_0}$. In this case, up to switch f_1 and f_2, it is not restrictive to assume that $\beta > \alpha > 0$, and, arguing as we did previously in this section, we can conclude that the possible pairs (a, b) are the following:

$$\begin{array}{ll}(0,0),(1,0),(1,1),(2,2);(2,0) & \text{if } \alpha = 1,\\(0,0),(1,1);(1,0) & \text{if } \alpha = 2,\\(0,0) & \text{if } \alpha \geq 3.\end{array}$$

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