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WEIGHTED COMPOSITION OPERATORS ON THE CLASS OF SUBORDINATE FUNCTIONS

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ABSTRACT. In this article, we study the weighted composition operators preserving the class $\mathcal{P}_\alpha$ of analytic functions subordinate to $(1+\alpha z)/(1-z)$ for $|\alpha| \leq 1$, $\alpha \neq -1$. Some of its consequences and examples for some special cases are presented.

1. Introduction. Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the (metrizable) topology of uniform convergence on compact subsets of $\mathbb{D}$, and we denote the boundary of $\mathbb{D}$ by $\mathbb{T}$. A weighted composition operator is a combination of multiplication and composition operators. These operators are mainly studied in various Banach spaces or Hilbert spaces of $\mathcal{H}(\mathbb{D})$.

Recently, Arévalo, et al., **[1]** initiated the study of weighted composition operators restricted to the Carathéodory class $\mathcal{P}_1$, which consists of all $f \in \mathcal{H}(\mathbb{D})$ with positive real part and with a normalization $f(0) = 1$. Clearly, the class $\mathcal{P}_1$ is not a linear space, but it is helpful for solving some extremal problems in geometric function theory. See **[8]**.

In this article, we generalize the recent work of Arévalo, et al., **[1]** by considering weighted composition operators preserving the class $\mathcal{P}_\alpha$. This class is connected with various geometric subclasses of $\mathcal{H}(\mathbb{D})$ in univalent function theory (see **[6, 8, 12]**). Since the class $\mathcal{P}_\alpha$ is not a linear space, for a given map on $\mathcal{P}_\alpha$, questions about operator theoretic properties are not meaningful. However, one can talk about special

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classes of self-maps of $\mathcal{P}_\alpha$ and fixed points of those maps. This is the main purpose of this article.

The article is organized as follows. In Section 2, we introduce the class $\mathcal{P}_\alpha$ and list some basic properties about this class. In Section 3, we give a characterization for weighted composition operators to be self-maps of the class $\mathcal{P}_\alpha$ (see Theorem 3.5). The above situation is closely analyzed for various special cases of symbols in Section 4. In Section 5, we present some simple examples.

2. Some preliminaries about the class $\mathcal{P}_\alpha$. For f and $g \in \mathcal{H}(\mathbb{D})$, we say that f *is subordinate to* g (denoted by $f(z) \prec g(z)$ or $f \prec g$) if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that $\omega(0) = 0$ and $f = g \circ \omega$. If $f(z) \prec z$, then f is called a *Schwarz* function. For $|\alpha| \leq 1$, $\alpha \neq -1$, define h_α on $\mathbb{D}$ by $h_\alpha(z) = (1 + \alpha z)/(1 - z)$, and the half plane $\mathbb{H}_\alpha$ is described by

$$\mathbb{H}_\alpha := h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : 2\mathrm{Re}\,\{(1 + \overline{\alpha})w\} > 1 - |\alpha|^2\}.$$

In particular, if $\alpha \in \mathbb{R}$ and $-1 < \alpha \leq 1$, then

$$h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : \mathrm{Re}\, w > (1 - \alpha)/2\}$$

so that $\mathrm{Re}\, h_\alpha(z) > (1 - \alpha)/2$ in $\mathbb{D}$.

For $|\alpha| \leq 1$, $\alpha \neq -1$, it is natural to consider the class $\mathcal{P}_\alpha$ defined by

$$\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(z) \prec h_\alpha(z)\}.$$

It is worthwhile to note that, for every $f \in \mathcal{P}_\alpha$, there is a unique Schwarz function ω such that

$$f(z) = \frac{1 + \alpha\omega(z)}{1 - \omega(z)}.$$

It is well known [**12,** Lemma 2.1] that, if g is a univalent analytic function on $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. In view of this result, the class $\mathcal{P}_\alpha$ can be stated in an equivalent form as

$$\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 1,\ f(\mathbb{D}) \subset \mathbb{H}_\alpha\}.$$

We continue the discussion by stating a few basic and useful properties of the class $\mathcal{P}_\alpha$.

Proposition 2.1. *Suppose that* $f \in \mathcal{P}_\alpha$ *and* $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$. *Then,* $|a_n| \le |\alpha + 1|$ *for all* $n \in \mathbb{N}$. *The bound is sharp as the function* $h_\alpha(z) = 1 + \sum_{n=1}^{\infty}(1+\alpha)z^n$ *shows.*

Proof. This result is an immediate consequence of Rogosinski's result [**13,** Theorem 10] (see also [**6,** page 195, Theorem 6.4(i)]) since $h_\alpha(z)$ (and hence, $(h_\alpha(z) - 1)/(1+\alpha)$) is a convex function. □

Proposition 2.2 (Growth estimate)**.** *Let* $f \in \mathcal{P}_\alpha$. *Then, for all* $z \in \mathbb{D}$, *we have*

$$\frac{1 - |\alpha z|}{1 + |z|} \le |f(z)| \le \frac{1 + |\alpha z|}{1 - |z|}.$$

Proof. This result trivially follows from clever use of the classical Schwarz lemma and the triangle inequality. □

From the 'growth estimate' and the familiar Montel's theorem on normal family, we can easily obtain the next result.

Proposition 2.3. *The class* $\mathcal{P}_\alpha$ *is a compact family in the compact-open topology, that is, topology of uniform convergence on compact subsets of* $\mathbb{D}$.

Since the half plane $\mathbb{H}_\alpha$ is convex, the following result is obvious.

Proposition 2.4. *The class* $\mathcal{P}_\alpha$ *is a convex family.*

For $p \in (0, \infty)$, the Hardy space H^p consists of analytic functions f on $\mathbb{D}$, with

$$\|f\|_p := \sup_{r \in [0,1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}$$

is finite, and H^∞ denotes the set of all bounded analytic functions on $\mathbb{D}$. The reader is referred to [**5**] for the theory of Hardy spaces. By Littlewood's subordination theorem [**11,** Theorem 2], it follows that, if $f \prec g$ and $g \in H^p$ for some $0 < p \le \infty$, then $f \in H^p$ for the same p. As a consequence, we easily have the following.

Proposition 2.5. *The class $\mathcal{P}_\alpha$ is a subset of the Hardy space H^p for each $0 < p < 1$.*

Proof. Since $(1-z)^{-1} \in H^p$ for each $0 < p < 1$, it follows easily that $h_\alpha \in H^p$ for each $0 < p < 1$ and, for $|\alpha| \leq 1$, $\alpha \neq -1$. The desired conclusion follows. □

Remark 2.6. Although $\mathcal{P}_\alpha$ does not posses the linear structure, due to being part of H^p, the results on H^p space, such as results regarding boundary behavior, are valid for functions in the class $\mathcal{P}_\alpha$.

3. Weighted composition on $\mathcal{P}_\alpha$. For an analytic self-map ϕ of $\mathbb{D}$, the composition operator C_ϕ is defined by

$$C_\phi(f) = f \circ \phi \quad \text{for } f \in \mathcal{H}(\mathbb{D}).$$

The interested reader may refer to **[4]** for the study of composition operators on various function spaces on the unit disk. Throughout this article, α denotes a complex number such that $|\alpha| \leq 1$, $\alpha \neq -1$, unless otherwise explicitly stated, and ϕ denotes an analytic self-map of $\mathbb{D}$. The next result deals with the composition operator when it is restricted to the class $\mathcal{P}_\alpha$.

Proposition 3.1. *The composition operator C_ϕ, induced by the symbol ϕ, preserves the class $\mathcal{P}_\alpha$ if and only if ϕ is a Schwarz function.*

Proof. Suppose that C_ϕ preserves the class $\mathcal{P}_\alpha$. Then, $C_\phi(h_\alpha) \in \mathcal{P}_\alpha$, and thus,

$$\frac{1+\alpha\phi(0)}{1-\phi(0)} = 1.$$

This yields that $\phi(0) = 0$, which implies that ϕ is a Schwarz function. The converse part holds trivially. □

For a given analytic self-map ϕ of $\mathbb{D}$ and analytic map ψ of $\mathbb{D}$, the corresponding weighted composition operator $C_{\psi,\phi}$ is defined by

$$C_{\psi,\phi}(f) = \psi(f \circ \phi) \quad \text{for } f \in \mathcal{H}(\mathbb{D}).$$

If $\psi \equiv 1$, then $C_{\psi,\phi}$ reduces to a composition operator C_ϕ, and, if $\phi(z) = z$ for all $z \in \mathbb{D}$, then $C_{\psi,\phi}$ reduce to a multiplication operator M_ψ. For a given analytic map ψ of $\mathbb{D}$, the corresponding multiplication operator M_ψ is then defined by

$$M_\psi(f) = \psi f \quad \text{for } f \in \mathcal{H}(\mathbb{D}).$$

The characterization of M_ψ that preserves the class $\mathcal{P}_\alpha$ is given in Section 4.

Banach begun the study of weighted composition operators. In [**2**], Banach proved the classical Banach-Stone theorem, that is, the surjective isometries between the spaces of continuous real-valued functions on a closed and bounded interval are certain weighted composition operators. In [**7**], Forelli proved that the isometric isomorphism of the Hardy space H^p, $p \neq 2$, are also weighted composition operators. The same result for the case of the Bergman space was proven by Kolaski in [**9**].

The study of weighted composition operators can be viewed as a natural generalization of the well-known field in analytic function theory, namely, composition operators. Moreover, weighted composition operators appear in applied areas, such as dynamical systems and evolution equations. For example, classification of dichotomies in certain dynamical systems is connected to weighted composition operators, see [**3**].

In this section, we discuss the weighted composition operator that preserves $\mathcal{P}_\alpha$. Before doing so, we recall some useful results from the theory of extreme points.

Lemma 3.2 ([**8,** Theorem 5.7]). *Extreme points of the class $\mathcal{P}_\alpha$ consist of functions, given by*

$$f_\lambda(z) = \frac{1+\alpha\lambda z}{1-\lambda z}, \quad |\lambda| = 1.$$

A point p of a convex set A is called an *extreme point* if p is not a interior point of any line segment which entirely lies in A. We denote the set of all extreme points of the class $\mathcal{P}_\alpha$ by $\mathcal{E}_\alpha$, that is, $\mathcal{E}_\alpha = \{f_\lambda : |\lambda| = 1\}$. Now, we recall a well-known result by Krein and Milman [**10**].

Lemma 3.3 ([**8,** Theorem 4.4])**.** *Let X be a locally convex, topological vector space, and let A be a convex, compact subset of X. Then, the closed convex hull of extreme points of A is equal to A.*

The original version of this was proven in [**10**]. On $\mathcal{H}(\mathbb{D})$, f_n converges to f, denoted by

$$f_n \xrightarrow{\text{u.c}} f.$$

It is easy to see that

$$C_{\psi,\phi}(f_n) \xrightarrow{\text{u.c}} C_{\psi,\phi}(f)$$

whenever $f_n \xrightarrow{\text{u.c}} f$. Thus, $C_{\psi,\phi}$ is continuous on $\mathcal{H}(\mathbb{D})$ (in particular, on $\mathcal{P}_\alpha$).

Proposition 3.4. *Suppose that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then, ϕ is a Schwarz function and, there exists a Schwarz function ω such that*

$$\psi = h_\alpha \circ \omega = \frac{1+\alpha\omega}{1-\omega}.$$

Proof. Suppose that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Take $f \equiv 1$ to be a constant function, which belongs to $\mathcal{P}_\alpha$. Thus, $C_{\psi,\phi}(f) = \psi \in \mathcal{P}_\alpha$, and hence, there exists a Schwarz function ω such that

$$\psi = h_\alpha \circ \omega = \frac{1+\alpha\omega}{1-\omega}.$$

In particular, $\psi(0) = 1$.

Since $h_\alpha \in \mathcal{P}_\alpha$, we have $\psi(0)(h_\alpha(\phi(0))) = 1$, which gives $\phi(0) = 0$. Hence, ϕ will be a Schwarz function. □

In view of the above result, from now on, we will assume that $\psi = h_\alpha \circ \omega = (1+\alpha\omega)/(1-\omega)$ and ϕ, ω are Schwarz functions.

Theorem 3.5. *Let ϕ, ω and ψ be as above. Then, $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if*

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \quad \text{on } \mathbb{D}, \tag{3.1}$$

where $P(\omega) = |\alpha\omega|^2 - |1+(\alpha-1)\omega|^2$ and $Q(\omega) = |(\alpha-1)|\omega|^2 + \overline{\omega} - \alpha\omega|$.

Proof. First, we prove that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, which is equivalent to the inclusion $C_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. In order to do so, we suppose that $C_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha$ is a convex family, we obtain

$$C_{\psi,\phi} \text{ (convex hull } (\mathcal{E}_\alpha)) \subset \mathcal{P}_\alpha.$$

Now, by the Krein-Milman theorem and the fact that $C_{\psi,\phi}$ is continuous on a compact family $\mathcal{P}_\alpha$, we see that $C_{\psi,\phi}(\mathcal{P}_\alpha) \subset \mathcal{P}_\alpha$. The converse part is trivial.

Next, we prove that $C_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$ if and only if

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \quad \text{on } \mathbb{D}.$$

Assume that $C_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. This gives $\psi(f_\lambda \circ \phi) \in \mathcal{P}_\alpha$ for all $|\lambda| = 1$. Thus, for all $|\lambda| = 1$, there exists a Schwarz function ω_λ such that $\psi(f_\lambda \circ \phi) = h_\alpha \circ \omega_\lambda$, that is,

$$\frac{1+\alpha\omega}{1-\omega}\frac{1+\alpha\lambda\phi}{1-\lambda\phi} = \frac{1+\alpha\omega_\lambda}{1-\omega_\lambda}.$$

Solving this equation for ω_λ, we obtain that

$$\omega_\lambda = \frac{\omega+\lambda\phi+(\alpha-1)\lambda\omega\phi}{1+\alpha\lambda\phi\omega}.$$

For each $|\lambda| = 1$, ω_λ is a Schwarz function if and only if

$$|\omega+\lambda\phi+(\alpha-1)\lambda\omega\phi|^2 < |1+\alpha\lambda\phi\omega|^2 \quad \text{for all } |\lambda| = 1,$$

which is equivalent to

$$2\mathrm{Re}\,(\lambda\phi\{(\alpha-1)|\omega|^2+\overline{\omega}-\alpha\omega\}) < (1-|\omega|^2)+|\phi|^2(|\alpha\omega|^2-|1+(\alpha-1)\omega|^2),$$

for all $|\lambda| = 1$. By taking the supremum over λ on both sides, the last inequality gives (3.1). The converse part follows by repeating the above arguments in the reverse direction. $\square$

Remark 3.6. Suppose that $\alpha = a+ib$ and $\omega(z) = u(z)+iv(z)$. Then

$$\begin{aligned} P(\omega) &= |\alpha\omega|^2 - |1+(\alpha-1)\omega|^2 \\ &= a(|\omega|^2-1) + (a-1)|\omega-1|^2 + 2bv. \end{aligned}$$

Set $q(\omega) = (\alpha-1)|\omega|^2+\overline{\omega}-\alpha\omega$ so that $Q(\omega) = |q(\omega)|$. Upon simplifying, we obtain that

$$q(\omega) = (\alpha - 1)\,\overline{\omega}\,(\omega - 1) - 2i\alpha\, v = (\alpha - 1)(|\omega|^2 - \omega) - 2iv,$$

and thus,

$$q(\omega) = [(a - 1)(|\omega|^2 - u) + bv] + i[b(|\omega|^2 - u) - v(a + 1)]. \tag{3.2}$$

In addition, it is easy to see that

$$-q(\omega) = |1 - \omega|^2\psi + (|\omega|^2 - 1) \quad \text{with } \psi = \frac{1 + \alpha\omega}{1 - \omega}. \tag{3.3}$$

4. Special cases. In this section, first we recall some familiar results on the Hardy space H^p, which will aid in the smooth traversing of this article.

Proposition 4.1 ([**5,** Theorem 1.3]). *For every bounded analytic function f on $\mathbb{D}$, the radial limit* $\lim_{r\to 1} f(\mathrm{r}e^{i\theta})$ *exists almost everywhere.*

In view of Proposition 4.1, every Schwarz function has radial limit almost everywhere, and, using the fact that the function h_α has radial limit almost everywhere, it is easy to see that, every function $f \in \mathcal{P}_\alpha$ has radial limit almost everywhere on $\mathbb{T}$. In addition, it is well known that (see [**5,** Section 2.3])

$$\sup_{|z|<1} |f(z)| = \operatorname*{ess\,sup}_{0\le\theta\le 2\pi} |f(e^{i\theta})|,$$

for every $f \in H^\infty$. Now, we will state a classical theorem of Nevanlinna.

Proposition 4.2 ([**5,** Theorem 2.2]). *If $f \in H^p$ for some $p > 0$ and its radial limit $f(e^{i\theta}) = 0$ on a set of positive measure, then $f \equiv 0$.*

Since every Schwarz function f belongs to H^∞, and every $f \in \mathcal{P}_\alpha$ belongs to H^p for $0 < p < 1$, the above result is valid for functions in the class $\mathcal{P}_\alpha$ and for Schwarz functions.

An analytic function f on $\mathbb{D}$ is said to be an *inner function* if $|f(z)| \le 1$ for all $z \in \mathbb{D}$ and its radial limit $|f(\zeta)| = 1$ almost everywhere on $|\zeta| = 1$.

Theorem 4.3. *Suppose that ϕ and ω are Schwarz functions, ϕ is inner and $\psi = (1+\alpha\omega)/(1-\omega)$. Then, $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if $\psi \equiv 1$, i.e., $\omega \equiv 0$.*

Proof. If $\psi \equiv 1$, then $C_{\psi,\phi}$ becomes a composition operator C_ϕ, and thus, $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ since ϕ is a Schwarz function.

Conversely, suppose that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then, by Theorem 3.5, we have the inequality

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \quad \text{on } \mathbb{D}.$$

With abuse of notation, we denote the radial limits of ϕ, ω and ψ, again by ϕ, ω and ψ, respectively. Also, let $\alpha = a + ib$ and $\omega(z) = u(z) + iv(z)$. By allowing $|z| \to 1$ in (3.1), we obtain that

$$2Q(\omega) \leq (1-|\omega|^2) + P(\omega) \quad \text{almost everywhere on } \mathbb{T},$$

which, after computation, is equivalent to

$$Q(\omega) \leq (a-1)(|\omega|^2 - u) + bv \quad \text{almost everywhere on } \mathbb{T}.$$

In view of (3.2) in Remark 3.6, the above inequality can rewritten as

$$|q(\omega)| \leq \mathrm{Re}\,[q(\omega)] \quad \text{almost everywhere on } \mathbb{T},$$

which gives that $\mathrm{Im}\,[q(\omega)] = 0$ almost everywhere on $\mathbb{T}$. Again, by using (3.3), we have

$$|1-\omega|^2 \mathrm{Im}\,(\psi) = 0 \quad \text{almost everywhere on } \mathbb{T}.$$

Analyzing the function ω through the classical theorem of Nevanlinna (see Proposition 4.2), we can get that $\mathrm{Im}\,\psi = 0$ almost everywhere on $\mathbb{T}$. Now, the proof of $\psi \equiv 1$ is as follows.

Consider the analytic map $f = e^{-i(\psi-1)}$. Then, $|f| = e^{\mathrm{Im}\,\psi} = 1$ almost everywhere on $\mathbb{T}$ and

$$1 = f(0) \leq \sup_{|z|<1} |f(z)| = \operatorname*{ess\,sup}_{0\leq\theta\leq 2\pi} |f(e^{i\theta})| = 1.$$

Hence, by the maximum modulus principle, we obtain that $f \equiv 1$, which yields $\psi \equiv 1$. $\square$

Corollary 4.4. *M_ψ preserves the class $\mathcal{P}_\alpha$ if and only if $\psi \equiv 1$.*

Proof. The desired result follows if we set $\phi(z) \equiv z$ in Theorem 4.3. □

Theorem 4.5. *Suppose that α is a real number, ϕ and ω are Schwarz functions, ω is an inner function, and $\psi = (1+\alpha\omega)/(1-\omega)$. Then, $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if ϕ is identically zero.*

Proof. If $\phi \equiv 0$, then $C_{\psi,\phi}$ becomes a constant map ψ, and hence, it preserves $\mathcal{P}_\alpha$. Conversely, suppose that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then, by Theorem 3.5,

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \quad \text{on } \mathbb{D}.$$

By allowing $|z| \to 1$, we get that

$$2|1-\omega|^2|\psi|\,|\phi| \le (2\mathrm{Im}\,\alpha\,\mathrm{Im}\,\omega + (\mathrm{Re}\,\alpha - 1)|1-\omega|^2)|\phi|^2 \\ \text{almost everywhere on } \mathbb{T},$$

from which we obtain that

$$|1-\omega|^2|\psi|\,|\phi| \le 0 \quad \text{almost everywhere on } \mathbb{T}.$$

By the hypothesis on ω and ψ, as well as the classical theorem of Nevanlinna, we find that $\phi \equiv 0$. □

Next is an easy consequence of Theorem 4.5.

Corollary 4.6. *Let α be a real number, ϕ and ω are Schwarz functions and $\phi \not\equiv 0$. Suppose that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, and*

$$E = \{\zeta \in \mathbb{T} : |\omega(\zeta)| = 1\}.$$

Then, the Lebesgue arc length measure of the set E is zero, i.e., $m(E) = 0$.

5. Examples for special cases. In this section, we give specific examples of ϕ and ψ so that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. For a bounded analytic function on $\mathbb{D}$, we denote $\sup_{|z|<1}|f(z)|$ by $\|f\|$.

Example 5.1. Suppose that $\|\phi\| < 1$. If

$$\|\omega\| < \frac{1-\|\phi\|}{1+\|\phi\|},$$

then $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, for $\alpha \in [0,1]$.

Proof. In view of Theorem 3.5, it suffices to verify the inequality (3.1). This inequality can be rewritten as

$$2|(1-\alpha)\,\overline{\omega}\,(\omega-1)+2i\alpha\,\mathrm{Im}\,\omega|\,|\phi|+(1-\alpha)(|\omega|^2+|1-\omega|^2|\phi|^2-1) < \alpha(1-|\omega|^2)(1-|\phi|^2).$$

We may set $\|\omega\|=A$ and $\|\phi\|=B$. Thus, it is sufficient to verify that

$$2(1-\alpha)A(A+1)B+4\alpha AB + (1-\alpha)(A^2+(1+A)^2B^2-1) < \alpha(1-A^2)(1-B^2),$$

which is equivalent to

$$[A+B+AB-1][(1-\alpha)(A+B+AB+1)+\alpha(A+B-AB+1)]<0.$$

This yields the condition $A+B+AB-1<0$, which means that $A<(1-B)/(1+B)$, and the desired conclusion follows. □

Since the condition $A+B+AB-1<0$ gives $B<(1-A)/(1+A)$, we have the next result.

Example 5.2. Suppose that $\|\omega\|<1$. If

$$\|\phi\|<\frac{1-\|\omega\|}{1+\|\omega\|},$$

then $C_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, for $\alpha\in[0,1]$.

Example 5.3. Suppose that $\phi(z)=z(az+b)$, where a and b are non-zero real numbers such that $|a|+|b|=1$. Take $\omega(z)=z(cz+d)$ with

$$c=-\frac{ab}{K} \quad \text{and} \quad d=\frac{1-(a^2+b^2)}{K} \text{ for } K>2+\sqrt{5}.$$

Then, $C_{\psi,\phi}$ preserves the class $\mathcal{P}_1$.

Proof. Clearly, $|\phi(z)|^2\le a^2+b^2+2abx$ for $z=x+iy$, and thus,

$$0<1-(a^2+b^2)-2abx\le(1-|\phi|^2).$$

In addition, note that

$$|\mathrm{Im}\,\omega|\le|2cx+d|=\frac{1-(a^2+b^2)-2abx}{K}$$

and

$$|\omega(z)| \leq |c| + |d| = \frac{1 - |ab| - (|a| - |b|)^2}{K} \leq \frac{1}{K}.$$

The inequality (3.1) for $\alpha = 1$ reduces to

$$4|\phi|\,|\mathrm{Im}\,\omega| < (1 - |\omega|^2)(1 - |\phi|^2).$$

Since $4|\phi|\,|\mathrm{Im}\,\omega| \leq 4|\mathrm{Im}\,\omega| \leq 4|2cx + d|$ and

$$\left(1 - \frac{1}{K^2}\right) K|2cx + d| \leq (1 - |\omega|^2)(1 - |\phi|^2),$$

to verify the inequality (3.1), it suffices to verify the inequality

$$\frac{4}{K} < 1 - \frac{1}{K^2}, \quad \text{i.e., } K^2 - 4K - 1 > 0.$$

This yields the condition $K > 2 + \sqrt{5}$, and the proof is complete. □

Remark 5.4. By letting $\alpha = 0$ in Theorem 3.5, we see that $C_{\psi,\phi}$ preserves the class $\mathcal{P}_0$ if and only if $|1 - \omega|\,|\phi| + |\omega| < 1$ on $\mathbb{D}$.

Example 5.5. If $|\phi| \leq |\omega| < \sqrt{2} - 1$ on $\mathbb{D}$, then $C_{\psi,\phi}$ preserves $\mathcal{P}_0$.

Proof. In view of Remark 5.4 and the assumption that $|\phi| \leq |\omega|$, it is sufficient to show that $|\omega|\,|1 - \omega| < 1 - |\omega|$ which, by squaring and then simplifying, is seen to be equivalent to the inequality

$$|\omega|^4 - 2\mathrm{Re}\,\omega|\omega|^2 + 2|\omega| - 1 < 0.$$

In order to verify this, we observe that

$$\begin{aligned} |\omega|^4 - 2\mathrm{Re}\,\omega|\omega|^2 + 2|\omega| - 1 &\leq |\omega|^4 + 2|\omega|^3 + 2|\omega| - 1 \\ &= (|\omega|^2 + 1)(|\omega|^2 + 2|\omega| - 1) \end{aligned}$$

which is negative whenever $|\omega|^2 + 2|\omega| - 1 < 0$, i.e., $|\omega| < \sqrt{2} - 1$. The desired result follows. □

Example 5.6. If $|\phi| \leq |\omega| < s_0$ or $|\omega| \leq |\phi| < s_0$ on $\mathbb{D}$, then $C_{\psi,\phi}$ preserves $\mathcal{P}_\alpha$ for every α with $-1 < \alpha < 0$, where s_0 (≈ 0.2648) is the unique positive root of the polynomial $P(x) = 2x^4 + 8x^3 + 12x^2 - 1$.

Proof. Without loss of generality, we assume that $|\phi| \le |\omega|$. In view of Remark 3.6 and the assumption that $\alpha \in (-1, 0)$, the inequality (3.1) can be rewritten as

$$2|(1-\alpha)\,\overline{\omega}\,(\omega-1) + 2i\alpha\,\mathrm{Im}\,\omega|\,|\phi| \\ + (1-\alpha)(|\omega|^2 + |1-\omega|^2|\phi|^2 - 1) - \alpha(1-|\omega|^2)(1-|\phi|^2) < 0.$$

By setting $\|\omega\| = A$ and $\|\phi\| = B$ (so that $B \le A$), it suffices to verify that

$$2(1-\alpha)A(A+1)B - 4\alpha AB + (1-\alpha)(A^2 + (1+A)^2B^2 - 1) - \alpha < 0,$$

which is equivalent to

$$-\alpha[(A+B+AB)^2 + 4AB] + (A+B+AB)^2 - 1 < 0.$$

Since $B \le A$ and $\alpha \in (-1, 0)$, the last inequality holds if

$$(2A+A^2)^2 + 4A^2] + (2A+A^2)^2 - 1 = 2A^4 + 8A^3 + 12A^2 - 1 < 0.$$

Clearly, the function $P(x) = 2x^4 + 8x^3 + 12x^2 - 1$ is strictly increasing on $(0, \infty)$, and thus, $P(x) < 0$ for $0 \le x < s_0$, where s_0 is the unique positive root of $P(x)$. The desired result follows. $\square$

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