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MULTIPLE SOLUTIONS FOR A FRACTIONAL p-KIRCHHOFF PROBLEM WITH SUBCRITICAL AND CRITICAL HARDY-SOBOLEV EXPONENT

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ABSTRACT. In this paper, by using the variational method and the theory of genus, we obtain the existence of multiple solutions to a fractional p-Kirchhoff problem with subcritical and critical Hardy-Sobolev exponent.

1. Introduction and statement of the main result. In this article, we consider the following Kirchhoff equation involving the fractional p-Laplacian:

(1.1)
$$\begin{cases} M([u]^p_{s,p})(-\Delta)^s_p u = \alpha f(x)|x|^{-c}|u|^{r-2}u + |x|^{-d}|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $N > sp$ with $0 < s < 1$, and

$$[u]^p_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

$(-\Delta)^s_p$ is the fractional p-Laplacian with $0 < s < 1$, and $1 \le r < p^*_s(b)$, $p < q \le p^*_s(b)$, $c < sr + N(1-r/p)$, $d < sq + N(1-q/p)$, and $\alpha \in \mathbb{R}$, with

$$p^*_s(b) = \frac{p(N-b)}{N-ps}$$

the critical Hardy-Sobolev exponent, where $0 < b < ps$, and with $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ a continuous function. Here, Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary such that $0 \in \Omega$.

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The operator $(-\Delta)_p^s$ is the fractional p-Laplacian, which may be defined, up to normalization factors, by the Riesz potential for $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s \varphi(x) := 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, \mathrm{d}y,$$

$x \in \mathbb{R}^N$, along any function $\varphi \in C_0^\infty(\Omega)$, where $B_\epsilon(x) := \{y \in \mathbb{R}^N : |y-x| < \epsilon\}$. The fractional p-Laplacian $(-\Delta)_p^s$ reduces to the fractional Laplacian $(-\Delta)^s$ if $p = 2$.

When $M \equiv 1$, equation (1.1) becomes the following fractional p-Laplacian equation

$$\begin{cases} (-\Delta)_p^s u = \alpha f(x)|x|^{-c}|u|^{r-2}u + |x|^{-d}|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

which is the fractional form of the following p-Laplacian equation

$$\begin{cases} -\Delta_p u = \alpha f(x)|x|^{-c}|u|^{r-2}u + |x|^{-d}|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

In recent years, great interest has been devoted to Kirchhoff equations of the type:

$$\begin{cases} (a + b \int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

This problem is related to the stationary analogue of the equation

$$u_{tt} + \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 \mathrm{d}x \right) u_{xx} = f(x, u) \tag{1.5}$$

proposed by Kirchhoff **[21]**. This equation extends the classical d'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Equation (1.4) received much attention only after Lions **[24]** proposed an abstract framework to the problem, see for example, **[1, 2, 3, 8, 10]**, and the references therein.

On the other hand, great attention has recently been focused on the study of elliptic equations involving the fractional Laplacian operator. This type of operator arises in many different applications, such as continuum mechanics, population dynamics, phase transition phenomena and game theory, see [**5, 6, 12, 20, 22, 26, 27, 28, 29, 30, 31, 34**]. In addition, much interest has grown on problems involving critical exponents or critical Hardy-Sobolev exponents in recent years. The interested reader is referred, for example, to [**4, 15, 18, 23, 32**], and the references therein.

Motivated by the above references, in the present paper, we investigate the existence of multiple solutions to the fractional p-Kirchhoff problem (1.1). To prove our main results, we mainly follow the ideas in [**13, 14, 17**]. In particular, our proofs are based on variational methods and the theory of genus.

Throughout this paper, we make the following assumptions on the the Kirchhoff function $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ and the function $f : \overline{\Omega} \to \mathbb{R}$:

- (M_1) the function M is continuous, and there exists an $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;
- (M_2) the function M is increasing;
- (f_1) $f \in L^\infty(\Omega)$, and there are constants ω_1 and ω_2 such that $0 < \omega_1 \leq f(x) \leq \omega_2$ for each $x \in \Omega$.

Theorem 1.1. *Suppose that* $1 \leq r < p < q < p_s^*(b)$, $c < sr + N(1 - r/p)$, $d < sq + N(1 - q/p)$, *and* (M_1) *and* (f_1) *hold. Then, there exists an* $\alpha_0 > 0$ *such that problem* (1.1) *has infinitely many solutions for each* $\alpha \in (0, \alpha_0)$.

Theorem 1.2. *Suppose that* $d = b$, $q = p_s^*(b)$, $1 \leq r < p$, $c < sr + N(1 - r/p)$, *and* (M_1), (M_2) *and* (f_1) *hold. Then, there exists an* $\alpha_1 > 0$ *such that problem* (1.1) *has infinitely many solutions for each* $\alpha \in (0, \alpha_1)$.

Theorem 1.3. *Suppose that* $d = b$, $q = p_s^*(b)$, $p < r < p_s^*(b)$, $c < sr + N(1 - r/p)$, *and* (M_1), (M_2) *and* (f_1) *hold. Then, there exists an* $\alpha_2 > 0$ *such that problem* (1.1) *has a non-trivial solution for each* $\alpha \in (\alpha_2, \infty)$.

The remainder of this paper is organized as follows. The variational framework and some preliminaries are given in Section 2. In Section 3, we consider problem (1.1) with subcritical exponent and verify Theorem 1.1. The proof of Theorem 1.2 is given in Section 4. Finally, in Section 5, using the same ideas as in [**13**], we prove Theorem 1.3.

2. Preliminaries. In this section, we give some preliminary results which will be used to prove our main results. We first provide some basic notions on the Krasnoselskii's genus that we will use in the proof of our main results. Let E be a real Banach space, and let us denote by Σ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in \Sigma$ implies $-u \in \Sigma$. For $A \in \Sigma$, we define

$$\gamma(A) = \inf\{m \in \mathbb{N} : \text{there exists a } \varphi \in C(A, \mathbb{R}^m \setminus \{0\}), \varphi(-x) = -\varphi(x)\}.$$

If there is no mapping as above for any $m \in \mathbb{N}$, then $\gamma(A) = \infty$. We list the following, main properties of the genus (see [**33**]).

Proposition 2.1. *Let* $A, B \in \Sigma$. *Then*:

(a) *if there exists an odd map* $g \in C(A, B)$, *then* $\gamma(A) \leq \gamma(B)$;
(b) *if* $A \subset B$, *then* $\gamma(A) \leq \gamma(B)$;
(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;
(d) k*-dimensional sphere* S_k *has a genus of* $k+1$ *by the Borsuk-Ulam theorem*;
(e) *suppose that* $E = \mathbb{R}^N$ *and that* $\partial\Omega$ *is the boundary of an open symmetric and bounded subset* $\Omega \subset \mathbb{R}^N$ *with* $0 \in \Omega$. *Then*, $\gamma(\partial\Omega) = N$;
(f) *if* $\gamma(B) < \infty$, *then* $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;
(g) *if* A *is compact, then* $\gamma(A) < +\infty$, *and there exists a* $\delta > 0$ *such that* $N_\delta(A) \subset \Sigma$ *and* $\gamma(N_\delta(A)) = \gamma(A)$, *here* $N_\delta(A) = \{x \in E : \|x - A\| \leq \delta\}$.

The next result, due to Clark [**9**], will be needed later.

Proposition 2.2. *Let* $\Phi \in C^1(X, \mathbb{R})$ *be a functional satisfying the Palais-Smale condition. Assume that*:

(i) Φ *is bounded from below and even*;
(ii) *there is a compact set* $K \in \Sigma$ *such that* $\gamma(K) = k$ *and* $\sup_{x \in K} \Phi(x) < \Phi(0)$.

Then, Φ *possesses at least* k *distinct pairs of critical points, and their corresponding critical values are less than* $\Phi(0)$.

Proposition 2.3. *If $K \in \Sigma$, $0 \notin K$ and $\gamma(K) \geq 2$, then K has infinitely many points.*

In this part, we introduce the basic variational framework. We first recall some definitions and basic properties of fractional Sobolev space that will be used later. Let Ω be any open set of $\mathbb{R}^N$. Following **[15]**, we define the fractional Sobolev space $Z(\Omega)$ as the closure of $C_0^\infty(\Omega)$, with respect to the norm

$$[\varphi]_{s,p} = \iint_{\mathbb{R}^{2N}} \left(\frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

Since $Z(\Omega)$ is a density space, the election of this solution space is an improvement with respect to the space

$$X_0(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{almost everywhere in } \mathbb{R}^N \setminus \Omega \right\},$$

which has been used in recent research related to nonlocal problems. In particular, the density result proven in **[16,** Theorem 6] does not hold for $X_0(\Omega)$ without assuming more restrictive conditions on the open bounded set Ω. Further, if Ω is an open, bounded subset of $\mathbb{R}^N$, then $Z(\Omega) \subset X_0(\Omega)$, with possibly $Z(\Omega) \neq X_0(\Omega)$. Note that, if Ω is any open subset of $\mathbb{R}^N$, and $\widetilde{u}$ denotes the natural extension of any $u \in Z(\Omega)$, then $\widetilde{u} \in D^{s,p}(\mathbb{R}^N)$. Thus,

$$Z(\Omega) \subset \{u \in L^{p_s^*}(\Omega) : \widetilde{u} \in D^{s,p}(\mathbb{R}^N)\}, \tag{2.1}$$

see **[15]** for more details. Therefore, the function space $Z(\mathbb{R}^N)$ reduces to $D^{s,p}(\mathbb{R}^N)$, and

$$\begin{aligned} Z(\mathbb{R}^N) &= D^{s,p}(\mathbb{R}^N) \\ &= \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}. \end{aligned}$$

Next, we state the fractional Hardy-Sobolev inequality from **[15]** that will be used later.

Lemma 2.4. *Assume that $0 \leq b < ps < N$. Then, there exists a positive constant C, possibly dependent only upon N, p, s and b, such that*

$$\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(b)}}{|x|^b} \, dx < C[u]_{s,p}^{p_s^*(b)}, \tag{2.2}$$

for all $u \in Z(\mathbb{R}^N)$.

Consequently, we can define the fractional Hardy-Sobolev constant $H_b = H(p, N, s, b)$ by

$$H_b = \inf_{u \in Z(\Omega)\setminus\{0\}} \frac{[u]_{s,p}^p}{\|u\|^p_{L^{p_s^*(b)}(\Omega, |x|^{-b})}}, \tag{2.3}$$

where $L^{p_s^*(b)}(\Omega, |x|^{-b})$ is the weighted $L^{p_s^*(b)}$ space with norm:

$$\|u\|_{L^{p_s^*(b)}(\Omega,|x|^{-b})} = \left(\int_\Omega \frac{|u(x)|^{p_s^*(b)}}{|x|^b} \, \mathrm{d}x \right)^{1/p_s^*(b)}.$$

Note that, when $b = 0$, the fractional Hardy-Sobolev inequality (2.2) reduces to the fractional Sobolev inequality:

$$\|u\|^p_{p_s^*} \le C_{N,p} \frac{s(1-s)}{(N-ps)^{p-1}} [u]^p_{s,p}$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where $\|.\|_p$ denotes the usual L_p norm, and $C_{N,p}$ is a positive constant dependent only upon N and p (see **[25]**).

In deriving the following theorem we were inspired by **[7, 37]**. In particular, Theorem 2.5 implies the compact imbedding from the space $Z(\Omega)$ into some L_p spaces with weights and gives us a new version of the classical Rellich-Kondrachov compactness theorem:

Theorem 2.5. *Assume that $0 < b < ps$ and that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain with smooth boundary and $0 \in \Omega$. The embedding $Z(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if*

$$1 \le r < \frac{p(N-b)}{N-ps}, \quad \alpha < sr + N\left(1 - \frac{r}{p}\right).$$

Claim 2.6. *From the assumptions of Theorem* 2.5*, it follows that there are constants $c_*, c_\vartheta > 0$ such that, for each $u \in Z(\Omega)$ we have*:

$$\int_\Omega |x|^{-\alpha}|u|^r dx \le c_* \left(\int_\Omega |x|^{-b}|u|^{p_s^*(b)} dx \right)^{r/p_s^*(b)} \le c_* c_\vartheta ([u]_{s,p})^r. \tag{2.4}$$

Proof of Theorem 2.5. Hence, it suffices to prove the compactness part of Theorem 2.5. Let $\{u_m\}$ be a bounded sequence in $Z(\Omega)$. For any $\eta > 0$, let $B_\eta(0) \subset \Omega$ be a closed ball centered at the origin with radius η. In view of Claim 2.6, $\{u_m\} \subset L^p(\Omega \setminus B_\eta(0))$ is bounded. It can easily be seen that $\{u_m\} \subset W^{s,p}(\Omega \setminus B_\eta(0))$. Since

$$1 < r < \frac{p(N-b)}{N-ps} < \frac{pN}{N-ps},$$

the Rellich-Kondrachov compactness theorem (see **[11]**) implies the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \setminus B_\eta(0))$. By taking a diagonal sequence it may be assumed, without loss of generality, that $\{u_m\}$ converges in $L^r(\Omega \setminus B_\eta(0))$ for any $\eta > 0$.

Since

$$r < q = p_s^*(b) = \frac{p(N-b)}{N-ps},$$

from the Hölder inequality and the fractional Hardy-Sobolev inequality (2.2), for any $\eta > 0$, we have

(2.5)

$$\begin{aligned}
&\int\limits_{|x|<\eta} |x|^{-\alpha}|u_m - u_j|^r dx \\
&\quad\leq \Bigg(\int\limits_{|x|<\eta} |x|^{-(\alpha - br/q)q/(q-r)} dx\Bigg)^{(q-r)/q} \Bigg(\int\limits_{|x|<\eta} |x|^{-b}|u_m - u_j|^q dx\Bigg)^{r/q} \\
&\quad\leq C\Bigg(\int_0^\eta t^{N-1-(\alpha - br/q)q/(q-r)} dt\Bigg)^{(q-r)/q} \\
&\quad= C\eta^{[N-(\alpha - br/q)q/(q-r)]q-r/q},
\end{aligned}$$

for some constant C independent of m and j. Assumption $\alpha < sr + N(1 - r/p)$ implies that:

(2.6)

$$\begin{aligned}
N - \left(\alpha - \frac{br}{q}\right)\frac{q}{q-r} &> N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - \frac{br}{q}\right)\frac{q}{q-r} \\
&= N - \Bigg(\left(sr + N\left(1 - \frac{r}{p}\right)\right) \\
&\qquad - b + \left(b - \frac{br}{q}\right)\Bigg)\frac{q}{q-r} \\
&= N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - b\right)\frac{q}{q-r} - b
\end{aligned}$$

$$
\begin{aligned}
&= N - \left(\left(sr + N\left(1 - \frac{r}{p}\right)\right) - b\right) \\
&\quad \times \frac{p(N-b)}{p(N-b) - rN + rps} - b \\
&= 0.
\end{aligned}
$$

Thus, for a given $\varepsilon > 0$, we can choose $\eta > 0$ such that

$$
\int_{|x|<\eta} |x|^{-\alpha} |u_m - u_j|^r dx \le \varepsilon \quad \text{for all } m, j \in \mathbb{N}.
$$

Now, let $N \in \mathbb{N}$ be such that

$$
\int_{\Omega \setminus B_\eta(0)} |x|^{-\alpha} |u_m - u_j|^r dx \le c_\alpha \int_{\Omega \setminus B_\eta(0)} |u_m - u_j|^r dx \le \varepsilon
$$

for all $m, j \ge N$, where $C_\alpha = \eta^{-\alpha}$ for $\alpha \ge 0$ and $C_\alpha = (\mathrm{diam}(\Omega))^{-\alpha}$ for $\alpha < 0$. Thus,

$$
\int_\Omega |x|^{-\alpha} |u_m - u_j|^r dx \le 2\varepsilon \quad \text{for all } m, j \ge N.
$$

Therefore, $\{u_m\}$ is a Cauchy sequence in $L^r(\Omega, |x|^{-\alpha})$. Now, by considering the proof of compactness portion of Theorem 2.5, Claim 2.6 can easily be verified. Hence, the proof of Claim 2.6 is omitted. □

The energy functional associated with (1.1) is defined on $Z(\Omega)$ by

$$
\begin{aligned}
I(u) = \frac{1}{p}\widehat{M}(\|u\|^p_{Z(\Omega)}) - \frac{\alpha}{r}\int_\Omega f(x)|x|^{-c}|u|^r \mathrm{d}x \\
- \frac{1}{q}\int_\Omega |x|^{-d}|u|^q \mathrm{d}x,
\end{aligned} \tag{2.7}
$$

where $\widehat{M}(t) = \int_0^t M(\tau)\, d\tau$. Obviously, I is of class C^1, and the solutions to problem (1.1) are the critical points of the functional I in $Z(\Omega)$. In fact,

$$
\begin{aligned}
\langle I'(u), \varphi \rangle = M(\|u\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \Bigg(\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\
\cdot (\varphi(x) - \varphi(y)) \Bigg) \mathrm{d}x\, \mathrm{d}y \\
- \alpha \int_\Omega f(x)|x|^{-c}|u(x)|^{r-2}u(x)\varphi(x)\, \mathrm{d}x
\end{aligned} \tag{2.8}
$$

$$-\int_\Omega |x|^{-d}|u(x)|^{q-2}u(x)\varphi(x)\,\mathrm{d}x.$$

The functional I is even and $I(0)=0$. Considering Proposition 2.2, in order to prove Theorems 1.1 and 1.2 we need that the Euler functional I is bounded from below. Thus, following the same idea as in **[17]**, we will use a modified functional to obtain the critical points of I. Hence, we shall construct the auxiliary functional: From (M_1), (f_1) and the inequality (2.4), we obtain

(2.9)
$$\begin{aligned} I(u) &= \frac{1}{p}\widehat{M}(\|u\|^p_{Z(\Omega)}) - \frac{\alpha}{r}\int_\Omega f(x)|x|^{-c}|u|^r\mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-d}|u|^q\mathrm{d}x \\ &\geq \frac{m_0}{p}\|u\|^p_{Z(\Omega)} - \alpha\frac{C_1}{r}\|u\|^r_{Z(\Omega)} - \frac{C_2}{q}\|u\|^q_{Z(\Omega)}. \end{aligned}$$

Now, we define

$$Q_\alpha(t) = \frac{m_0}{p}\,t - \alpha\frac{C_1}{r}\,t^{r/p} - \frac{C_2}{q}t^{q/p}.$$

Thus, $I(u) \geq Q_\alpha(\|u\|^p_{Z(\Omega)})$. If $r<p<q\leq p_s^*(b)$, then we have $\lim_{t\to+\infty} Q_\alpha(t) = -\infty$. Thus, I is not bounded from below. However, there exists an $\alpha_0>0$ such that, for any $\alpha\in(0,\alpha_0)$, there exist $T_1, T_2\in(0,+\infty)$ such that $Q_\alpha(t)<0$ for $0<t<T_1$, $Q_\alpha(t)>0$ for $T_1<t<T_2$, $Q_\alpha(t)<0$ for $t>T_2$, and hence, the function $Q_\alpha(t)$ achieves a positive maximum, and $Q_\alpha(T_1)=Q_\alpha(T_2)=0$. Now, we consider $\phi\in C_0^1([0,+\infty))$ with $0\leq\phi\leq 1$, $\phi(t)=1$ if $t\leq T_1$, and $\phi(t)=0$ if $t\geq T_2$, $\phi'(t)\leq 0$ for all $t\in[0,+\infty)$. Furthermore, we define the function $\overline{Q}_\alpha : [0,+\infty)\to\mathbb{R}$ as

$$\overline{Q}_\alpha(t) = \frac{m_0}{p}t - \alpha\frac{C_1}{r}\,t^{r/p} - \frac{C_2}{q}\phi(t)t^{q/p}.$$

We have $\overline{Q}_\alpha(0)=0$, and it can easily be seen that $\overline{Q}_\alpha(t)\geq 0$ for all $t\geq T_1$. In addition, it is clear that $\lim_{t\to+\infty}\overline{Q}_\alpha(t)=+\infty$. In order to prove Theorem 1.1 we define the following auxiliary functional on $Z(\Omega)$ by

$$\begin{aligned} J(u) = \frac{1}{p}\widehat{M}(\|u\|^p_{Z(\Omega)}) - \frac{\alpha}{r}\int_\Omega f(x)x^{-c}|u|^r\mathrm{d}x \\ - \frac{\phi(\|u\|^p_{Z(\Omega)})}{q}\int_\Omega x^{-d}|u|^q\mathrm{d}x, \end{aligned} \tag{2.10}$$

where $0 < \alpha < \alpha_0$. From $J(u) \geq \overline{Q}_\alpha(\|u\|^p_{Z(\Omega)})$, we obtain that J is coercive in $Z(\Omega)$. Thus, J is bounded from below in $Z(\Omega)$, and we can apply this functional to prove Theorem 1.1.

3. Subcritical case. We recall that, given E a real Banach space and $I \in C^1(E, R)$, we say that I satisfies the Palais-Smale condition on the level $h \in \mathbb{R}$ denoted by $(PS)_h$, if every sequence $\{u_n\} \subset E$ such that $I(u_n) \to h$ and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

In the sequel, we will need the following lemmas.

Lemma 3.1. *Suppose that $0 < \alpha < \alpha_0$. Assume that v_0 is a critical point of J with $J(v_0) < 0$. Then, v_0 is a critical point of I.*

Proof. Note that $J(u) \geq \overline{Q}_\alpha(\|u\|^p_{Z(\Omega)})$ for each $u \in Z(\Omega)$; hence, $\overline{Q}_\alpha(\|v_0\|^p_{Z(\Omega)}) < 0$. On the other hand, since $\overline{Q}_\alpha(t) \geq 0$, for all $t \geq T_1$, we conclude that $\|v_0\|^p_{Z(\Omega)} < T_1$. Since J is continuous, there exists an $R > 0$ such that $J(u) < 0$ for each $u \in B(v_0, R) \subset Z(\Omega)$. Therefore, $\phi(\|u\|^p_{Z(\Omega)}) = 1$ for all $u \in B(v_0, R)$, and this implies that $J(u) = I(u)$ for all $u \in B(v_0, R)$. $\square$

Lemma 3.2. *Suppose that $1 \leq r < p < q < p_s^*(b)$, $c < sr + N(1 - r/p)$, $d < sq + N(1 - q/p)$, and (M_1) and (f_1) hold. Then, J satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset Z(\Omega)$ be a Palais-Smale sequence at level $h \in \mathbb{R}$ for $J(u)$. Then, since $J(u)$ is coercive, we deduce that $\{u_n\} \subset Z(\Omega)$ is bounded. Therefore, we can assume, going if necessary to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u, && \text{in } Z(\Omega), \\ &u_n \to u, && \text{in } L^\tau(\Omega, |x|^{-\nu}), \\ &u_n(x) \to u(x) && \text{almost everywhere in } \Omega, \\ &\|u_n\| \longrightarrow \eta_0 \geq 0, && \end{aligned} \tag{3.1}$$

where $1 \leq \tau < p_s^*(b)$ and $\nu < s\tau + N(1 - \tau/p)$. Therefore,

(3.2)

$$\begin{aligned}
&\langle J'(u_n), u_n - u\rangle \\
&\quad = M(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+ps}} \\
&\qquad\qquad \cdot ((u_n - u)(x) - (u_n - u)(y))\bigg) \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad - \alpha \int_\Omega |x|^{-c} f(x) |u_n(x)|^{r-2} u_n(x)(u_n - u)(x)\,\mathrm{d}x \\
&\qquad - \phi(\|u_n\|^p_{Z(\Omega)}) \int_\Omega |x|^{-d} |u_n(x)|^{q-2} u_n(x)(u_n - u)(x)\,\mathrm{d}x \\
&\qquad - \frac{p}{q}\phi'(\|u_n\|^p_{Z(\Omega)}) \int_\Omega |x|^{-d}|u_n(x)|^q \mathrm{d}x \\
&\qquad \cdot \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+ps}} \\
&\qquad\qquad \cdot ((u_n - u)(x) - (u_n - u)(y))\bigg) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = o_n(1).
\end{aligned}$$

It follows from the Hölder inequality, (3.1) and since ϕ is continuous, that

$$\lim_{n\to\infty} \int_\Omega |x|^{-d} |u_n(x)|^{q-2} u_n(x)(u_n - u)(x)\,\mathrm{d}x = 0, \tag{3.3}$$

and

$$\lim_{n\to\infty} \phi(\|u_n\|^p_{Z(\Omega)}) \int_\Omega |x|^{-d} |u_n(x)|^{q-2} u_n(x)(u_n - u)(x)\,\mathrm{d}x = 0. \tag{3.4}$$

In addition, from (f_1), the Hölder inequality and (3.1), we get

$$\begin{aligned}
\lim_{n\to\infty} \bigg| \int_\Omega |x|^{-c} f(x) |u_n|^{r-2} u_n (u_n - u)\,\mathrm{d}x \bigg| \\
\leq \lim_{n\to\infty} \omega_2 \int_\Omega |x|^{-c} |u_n|^{r-1} |u_n - u|\,\mathrm{d}x = 0.
\end{aligned} \tag{3.5}$$

From (3.2)–(3.5), we conclude that

(3.6)
$$\lim_{n\to\infty} \Psi(u_n) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot ((u_n-u)(x)-(u_n-u)(y)) \bigg)\, \mathrm{d}x\,\mathrm{d}y = 0,$$

where

$$\Psi(u_n) = \bigg(M(\|u_n\|^p_{Z(\Omega)}) - \frac{p}{q}\phi'(\|u_n\|^p_{Z(\Omega)}) \int_\Omega |x|^{-d}|u_n(x)|^q \mathrm{d}x \bigg).$$

Note that M and ϕ' are continuous and $M(t) \geq m_0$ and $\phi'(t) \leq 0$ for all $t \geq 0$. Thus,

$$m_0 \leq \Psi(u_n) \leq C_3;$$

therefore,

(3.7)
$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot ((u_n-u)(x)-(u_n-u)(y)) \bigg)\, \mathrm{d}x\,\mathrm{d}y = 0.$$

Now we define the functional $A : Z(\Omega) \to (Z(\Omega))^*$ by setting, for all $u, v \in Z(\Omega)$,

(3.8)
$$\langle A(u), v\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, \mathrm{d}x\,\mathrm{d}y.$$

By a standard argument (see, for example, [**19**]), we can easily see that $Z(\Omega)$ is uniformly convex, and the functional A enjoys the (S)-property, that is, whenever $\{u_n\}$ is a sequence in $Z(\Omega)$ such that $u_n \rightharpoonup u$ in $Z(\Omega)$ and $\langle A(u_n), u_n - u\rangle \to 0$, then $u_n \to u$ in $Z(\Omega)$. Hence, from (3.1), (3.7) and, since the functional A satisfies the (S)-property, we conclude that $u_n \to u$ in $Z(\Omega)$. The proof is complete. □

Proof of Theorem 1.1. Note that $Z(\Omega)$ is a uniformly convex Banach space. Thus, $Z(\Omega)$ is reflexive. Note also that $Z(\Omega)$ is separable. Then, for any $k \in N$, there is a k-dimensional linear subspace X_k of $Z(\Omega)$ such that $X_k \subset C_0^\infty(\Omega)$. Since all norms on X_k are equivalent, there exists a constant $\delta(k) > 0$ that depends upon k such that $r\delta(k)\|u\|^r \leq \omega_1 \|u\|^r_{L^r(\Omega,|x|^{-c})}$ for each $u \in X_k$. Thus, if $u \in X_k$, then, by (f_1), we obtain

$$\frac{1}{r}\int_{\Omega} f(x)|x|^{-c}|u|^r \mathrm{d}x \geq \frac{\omega_1}{r}\int_{\Omega} |x|^{-c}|u|^r \mathrm{d}x \geq \delta(k)\|u\|^r.$$

Therefore, in view of the continuity of M, we can find $C > 0$ such that, for each $u \in X_k$ with $\|u\| < 1$, the following holds:

$$J(u) \leq C\|u\|^p - \alpha\delta(k)\|u\|^r.$$

Since $1 \leq r < p$, there is a $\zeta > 0$ such that, for all $u \in X_k$ with $\|u\| = \zeta$, we have

$$J(u) < 0 = J(0). \tag{3.9}$$

We set $\Lambda = \{u \in X_k : \|u\| = \zeta\}$. Note that X_k and $\mathbb{R}^k$ are isomorphic, and Λ and S^{k-1} are homeomorphic. We conclude that $\gamma(\Lambda) = k$. On the other hand, J is coercive, and, by Lemma 3.2, satisfies the Palais-Smale condition. Therefore, Proposition 2.2 implies that J contains at least k pairs of different critical points. Since k is arbitrary, we conclude that J has infinitely many critical points. Now, from (3.9) and Lemma 3.1, we deduce that I has infinitely many critical points. The proof is complete. □

4. Proof of Theorem 1.2. In this section, we shall prove Theorem 1.2. Thus, we set $d = b$ and $q = p_s^*(b)$ in equation (1.1). Hence, throughout this section, we have $1 \leq r < p < p_s^*(b)$. In addition, it follows from (M_2) that $M(t)$ is increasing. Since we deal with critical growth, and behavior of the nonlocal operator M at infinity is unknown, we are required to make a truncation on the function M. Since $p < p_s^*(b)$, there exists a $\theta \in (p, p_s^*(b))$ and, due to the fact that M is increasing, there exists a $t_0 > 0$ such that $m_0 \leq M(0) < M(t_0) < (\theta/p)m_0$. We define

$$M_0(t) = \begin{cases} M(t) & \text{if } 0 \leq t \leq t_0; \\ M(t_0) & \text{if } t \geq t_0. \end{cases}$$

By (M_2), we have

$$m_0 \leq M_0(t) \leq \frac{\theta}{p} m_0. \tag{4.1}$$

In order to prove Theorem 1.2, we first need to investigate the solutions of the following related equation:

$$\begin{cases} M_0([u]_{s,p}^p)(-\Delta)_p^s u \\ \quad = \alpha f(x)|x|^{-c}|u|^{r-2}u + |x|^{-b}|u|^{p_s^*(b)-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.2}$$

The energy functional $I_\alpha : Z(\Omega) \to \mathbb{R}$ associated with problem (4.2) is as in the following

$$\begin{aligned} I_\alpha(u) = {} & \frac{1}{p}\widehat{M_0}(\|u\|_{Z(\Omega)}^p) - \frac{\alpha}{r}\int_\Omega f(x)|x|^{-c}|u|^r \mathrm{d}x \\ & - \frac{1}{p_s^*(b)}\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\mathrm{d}x, \end{aligned} \tag{4.3}$$

where $\widehat{M_0}(t) = \int_0^t M_0(\tau)\, d\tau$. Obviously, I_α is of class C^1 and, for each $\varphi \in Z(\Omega)$, we have

$$\begin{aligned} \langle I_\alpha'(u), \varphi\rangle = {} & M_0(\|u\|_{Z(\Omega)}^p) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}} \\ & \cdot (u(x)-u(y))(\varphi(x)-\varphi(y)) \bigg) \mathrm{d}x\, \mathrm{d}y \\ & - \alpha \int_\Omega f(x)|x|^{-c}|u(x)|^{r-2}u(x)\varphi(x)\, \mathrm{d}x \\ & - \int_\Omega |x|^{-b}|u(x)|^{p_s^*(b)-2}u(x)\varphi(x)\, \mathrm{d}x. \end{aligned} \tag{4.4}$$

Let $T_1(\alpha)$ be the first root of the function $Q_\alpha(t)$, defined as in previous sections. Since $Q_\alpha(T_1(\alpha)) = 0$ and $Q_\alpha'((T_1(\alpha)) > 0$, the following lemma can easily be deduced, and we omit the proof.

Lemma 4.1. *Let $1 \le r < p$. Assume that $T_1(\alpha)$ is the first root of the function $Q_\alpha(t)$, defined as in previous sections. Then,* $\lim_{\alpha\to 0} T_1(\alpha) = 0$.

Similar to the proof of Theorem 1.1, we define an auxiliary functional $J_\alpha : Z(\Omega) \to \mathbb{R}$ as

$$J_\alpha(u) = \frac{1}{p}\widehat{M_0}(\|u\|_{Z(\Omega)}^p) - \frac{\alpha}{r}\int_\Omega f(x)|x|^{-c}|u|^r \mathrm{d}x \tag{4.5}$$

$$- \frac{\phi(\|u\|^p_{Z(\Omega)})}{q} \int_\Omega |x|^{-b}|u|^q \mathrm{d}x,$$

where $q = p_s^*(b)$ and $\widehat{M_0}(t) = \int_0^t M_0(\tau)\, d\tau$.

In order to study the behavior of the Palais-Smale sequences of the energy functional J_α, we shall need the following concentration-compactness principle:

Lemma 4.2 ([**15**]). *Let Ω be an open, bounded subset of $\mathbb{R}^N$, and let $0 < b \le ps$. Let $\{u_j\}$ be a weakly convergent sequence in $Z(\Omega)$ with weak limit u. Then, there exist two finite positive measures μ and ν in $\mathbb{R}^N$ such that*

$$\left(\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \,\mathrm{d}y\right) \mathrm{d}x \rightharpoonup \mu \textit{ weakly } * \textit{ in } M(\mathbb{R}^N),$$

$$\frac{|u_n|^{p_s^*(b)}}{|x|^b} \mathrm{d}x \rightharpoonup \nu \textit{ weakly } * \textit{ in } M(\mathbb{R}^N).$$

Furthermore, there exist two nonnegative numbers μ_0 and ν_0 such that

$$\nu = \frac{|u|^{p_s^*(b)}}{|x|^b} \,\mathrm{d}x + \nu_0 \delta_0 \tag{4.6}$$

$$\mu \ge \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \,\mathrm{d}y\right) \mathrm{d}x + \mu_0 \delta_0, \tag{4.7}$$

and

$$0 \le H_b \nu_0^{p/p_s^*(b)} \le \mu_0,$$

where δ_0 is the Dirac mass at $0 \in \Omega$, and H_b is the Hardy-Sobolev constant defined in (2.3).

The proof of the next lemma may be found in [**36**].

Lemma 4.3. *Assume that $\{u_n\} \subset Z(\Omega)$ is the sequence given by Lemma* 4.4*. For $\varepsilon > 0$, let $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ be a smooth, cut-off function centered at x_j such that $0 \le \phi \le 1$, $\phi_\varepsilon \equiv 1$ in $B(x_j, \varepsilon)$, $\phi_\varepsilon \equiv 0$ in $\mathbb{R}^N \setminus B(x_j, 2\varepsilon)$ and $|\nabla \phi_\varepsilon(x)| \le 2/\varepsilon$ for all $x \in \mathbb{R}^N$. Then:*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^p |u_n(y)|^p}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Lemma 4.4. *Assume that* $\{u_n\}$ *is a bounded sequence in* $Z(\Omega)$ *such that* $I_\alpha(u_n) \to h$ *and* $I'_\alpha(u_n) \to 0$ *in* $Z(\Omega)$ *as* $n \to \infty$. *Suppose that* $1 \leq r < p$, $q = p_s^*(b)$, $c < sr + N(1 - r/p)$, (M_1), (M_2) *and* (f_1) *hold, and*

$$\begin{aligned} h < &\left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)(m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)} \\ &- \xi \left[\frac{\alpha\omega_2\omega_3(1/r + 1/\theta)}{1/\theta - 1/p_s^*(b)}\right]^{p_s^*(b)/(p_s^*(b)-r)} \\ &\cdot \left[\left(\frac{r}{p_s^*(b)}\right)^{r/(p_s^*(b)-r)} - \left(\frac{r}{p_s^*(b)}\right)^{p_s^*(b)/(p_s^*(b)-r)}\right], \end{aligned} \tag{4.8}$$

where

$$\omega_3 = \left(\int_\Omega |x|^{-(c - br/p_s^*(b))(p_s^*(b)/(p_s^*(b)-r))} \mathrm{d}x\right)^{(p_s^*(b)-r)/p_s^*(b)},$$

and

$$\xi = \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right).$$

Then, $\{u_n\}$ *has a convergent subsequence.*

Proof. Since $\{u_n\} \subset Z(\Omega)$ is bounded, we can assume, going if necessary to a subsequence, that

$$\begin{aligned} &u_n \rightharpoonup u, && \text{in } Z(\Omega), \\ &u_n \to u, && \text{in } L^r(\Omega, |x|^{-c}), \\ &u_n(x) \to u(x) && \text{almost everywhere in } \Omega, \\ &\|u_n\| \longrightarrow \eta_0 \geq 0. \end{aligned} \tag{4.9}$$

From Lemma 4.2, there exist two finite positive measures μ and ν in $\mathbb{R}^N$ such that

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}y\right) \mathrm{d}x \rightharpoonup \mu \text{ weakly } * \quad &\text{in } M(\mathbb{R}^N), \\ \frac{|u_n|^{p_s^*(b)}}{|x|^b} \mathrm{d}x \rightharpoonup \nu \text{ weakly } * \quad &\text{in } M(\mathbb{R}^N). \end{aligned}$$

Furthermore, there exist two nonnegative numbers μ_0 and ν_0 such that

$$\nu = \frac{|u|^{p_s^*(b)}}{|x|^b}\,\mathrm{d}x + \nu_0\delta_0, \tag{4.10}$$

$$\mu \geq \left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y\right)\mathrm{d}x + \mu_0\delta_0, \tag{4.11}$$

and

$$0 \leq H_b \nu_0^{p/(p_s^*(b))} \leq \mu_0. \tag{4.12}$$

Let $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi \leq 1$, $\phi_\varepsilon \equiv 1$ in $B(0,\varepsilon)$, $\phi_\varepsilon \equiv 0$ in $\mathbb{R}^N \setminus B(0,2\varepsilon)$ and $|\nabla\phi_\varepsilon(x)| \leq 2/\varepsilon$ in Ω. Then, it is seen that $\{u_n\phi_\varepsilon\}$ is bounded in $Z(\Omega)$ (see **[15]**, for example). Therefore, $\lim_{n\to\infty}\langle I'_\alpha(u_n),\ u_n\phi_\varepsilon\rangle = 0$, i.e.,

$$\begin{aligned} M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(&\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\ &\cdot (u_n(x)\phi_\varepsilon(x) - u_n(y)\phi_\varepsilon(y))\bigg)\,\mathrm{d}x\,\mathrm{d}y + o_n(1) \\ = \bigg(\alpha\int_\Omega f(x)|x|^{-c}|u_n(x)|^r\phi_\varepsilon(x)\,\mathrm{d}x &+ \int_\Omega |x|^{-b}|u_n(x)|^{p_s^*(b)}\phi_\varepsilon(x)\,\mathrm{d}x\bigg) \end{aligned} \tag{4.13}$$

Now, we estimate the first term of the left-hand side of (4.13).

$$\begin{aligned} M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(&\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\ &\cdot (u_n(x)\phi_\varepsilon(x) - u_n(y)\phi_\varepsilon(y))\bigg)\,\mathrm{d}x\,\mathrm{d}y \\ = M_0(\|u_n\|^p_{Z(\Omega)}) &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\phi_\varepsilon(x)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ + M_0(\|u_n\|^p_{Z(\Omega)}) &\iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\ &\cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y)\bigg)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.14}$$

From (4.13) and (4.14), we obtain

(4.15)

$$
\begin{aligned}
&M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\
&\qquad\qquad \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y) \bigg) \,\mathrm{d}x\,\mathrm{d}y + o_n(1) \\
&\quad = -M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p \phi_\varepsilon(x)}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \bigg(\alpha \int_\Omega f(x)|x|^{-c}|u_n(x)|^r \phi_\varepsilon(x)\,\mathrm{d}x \\
&\qquad\qquad + \int_\Omega |x|^{-b}|u_n(x)|^{p_s^*(b)} \phi_\varepsilon(x)\,\mathrm{d}x \bigg).
\end{aligned}
$$

By (4.10), (4.11), (M_1), and, since $\phi_\varepsilon(0)=1$, we conclude that

$$
\begin{aligned}
&M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\
&\qquad\qquad \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y) \bigg) \,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le -m_0 \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p \phi_\varepsilon(x)}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y - m_0\mu_0 \\
&\qquad + \alpha \int_\Omega f(x)|x|^{-c}|u_n(x)|^r \phi_\varepsilon(x)\,\mathrm{d}x \\
&\qquad + \int_\Omega |x|^{-b}|u(x)|^{p_s^*(b)} \phi_\varepsilon(x)\,\mathrm{d}x + \nu_0 + o_n(1).
\end{aligned} \tag{4.16}
$$

Since $u_n \to u$ in $L^r(\Omega, |x|^{-c})$, we use (f_1) and the dominated convergence theorem to obtain

$$
\lim_{n\to\infty} \int_\Omega f(x)|x|^{-c}|u_n(x)|^r \phi_\varepsilon(x)\,\mathrm{d}x = \int_\Omega f(x)|x|^{-c}|u(x)|^r \phi_\varepsilon(x)\,\mathrm{d}x.
$$

Thus, we have

(4.17)

$$
\begin{aligned}
&\limsup_{n\to\infty} M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \\
&\qquad\qquad \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y) \bigg) \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

$$\leq -m_0 \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p \phi_\varepsilon(x)}{|x-y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y - m_0\mu_0$$
$$+ \alpha \int_\Omega f(x)|x|^{-c}|u(x)|^r \phi_\varepsilon(x) \, \mathrm{d}x$$
$$+ \int_\Omega |x|^{-b}|u(x)|^{p_s^*(b)} \phi_\varepsilon(x) \, \mathrm{d}x + \nu_0.$$

Using the dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p \phi_\varepsilon(x)}{|x-y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y = 0,$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega f(x)|x|^{-c}|u|^r \phi_\varepsilon \, \mathrm{d}x = 0,$$

$$\lim_{\varepsilon\to 0} \int_\Omega |x|^{-b}|u|^{p_s^*(b)} \phi_\varepsilon \, \mathrm{d}x = 0.$$

Consequently,
(4.18)
$$\lim_{\varepsilon\to 0} (\limsup_{n\to\infty} M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y) \bigg) \, \mathrm{d}x \, \mathrm{d}y$$
$$\leq \nu_0 - m_0\mu_0.$$

Now, we show that
(4.19)
$$\lim_{\varepsilon\to 0} (\limsup_{n\to\infty} (M_0(\|u_n\|^p_{Z(\Omega)}) \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y) \bigg) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

Note that

$$\left|\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right| \\ \leq \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)^{(p-1)/p} \\ \cdot \left(\iint_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \\ \leq C\left(\iint_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)^{(p-1)/p}. \tag{4.20}$$

Since $\{u_n\}$ is bounded in $Z(\Omega)$ and M_0 is continuous, we get

$$M_0(\|u_n\|^p_{Z(\Omega)})\iint_{\mathbb{R}^{2N}} \left(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y)\right)\mathrm{d}x\,\mathrm{d}y \\ \leq CL\left(\iint_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)^{(p-1)/p}. \tag{4.21}$$

On the other hand, by Lemma 4.3, we conclude that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x)-\phi_\varepsilon(y)|^p|u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.22}$$

Therefore, by (4.21) and (4.22), we conclude that

$$\lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty}\left(M_0(\|u_n\|^p_{Z(\Omega)}) \cdot \iint_{\mathbb{R}^{2N}}\left(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}} \cdot (\phi_\varepsilon(x)-\phi_\varepsilon(y))u_n(y)\right)\mathrm{d}x\,\mathrm{d}y\right)\right) = 0. \tag{4.23}$$

By (4.18) and (4.23), we get

$$m_0\mu_0 \leq \nu_0. \tag{4.24}$$

Now, inequality (4.12) implies that

$$\nu_0 \geq (m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)}, \tag{4.25}$$

Now, we claim that (4.25) cannot occur. Indeed, suppose otherwise. Note that $m_0 \leq M_0(t) \leq (\theta/p)m_0$; thus, from (f_1), we obtain that

$$\begin{aligned} h &= I_\alpha(u_n) - \frac{1}{\theta}\langle I_\alpha'(u_n), u_n\rangle + o_n(1) \\ &\geq \left(\frac{m_0}{p} - \frac{\theta m_0}{p\theta}\right)\|u_n\|_{Z(\Omega)}^p \\ &\quad - \alpha\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega f(x)|x|^{-c}|u_n|^r \mathrm{d}x \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)\int_\Omega |x|^{-b}|u_n|^{p_s^*(b)}\mathrm{d}x + o_n(1) \\ &\geq \left(\frac{m_0}{p} - \frac{\theta m_0}{p\theta}\right)\|u_n\|_{Z(\Omega)}^p - \omega_2\alpha\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega |x|^{-c}|u_n|^r \mathrm{d}x \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)\int_\Omega |x|^{-b}|u_n|^{p_s^*(b)}\phi_\varepsilon \,\mathrm{d}x + o_n(1). \end{aligned} \tag{4.26}$$

Letting $n \to \infty$, we have

$$\begin{aligned} h &\geq -\alpha\omega_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega |x|^{-c}|u|^r \mathrm{d}x \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\phi_\varepsilon \,\mathrm{d}x + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)\nu_0 \\ &\geq -\alpha\omega_2\left(\frac{1}{r} + \frac{1}{\theta}\right)\int_\Omega |x|^{-c}|u|^r \mathrm{d}x \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\phi_\varepsilon \,\mathrm{d}x \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)(m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)}. \end{aligned} \tag{4.27}$$

Using the Hölder inequality, we obtain

$$\int_\Omega |x|^{-c}|u|^r \mathrm{d}x \leq \omega_3\left(\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\mathrm{d}x\right)^{r/(p_s^*(b))},$$

where

$$\omega_3 = \left(\int_\Omega |x|^{-(c-br/p_s^*(b))(p_s^*(b)/(p_s^*(b)-r))} \mathrm{d}x \right)^{(p_s^*(b)-r)/p_s^*(b)} < \infty.$$

Thus, letting $\varepsilon \to \infty$, we obtain

$$\begin{aligned} h \geq & -\alpha\omega_2\omega_3 \left(\frac{1}{r} + \frac{1}{\theta} \right) \\ & \cdot \left(\int_\Omega |x|^{-b} |u|^{p_s^*(b)} \mathrm{d}x \right)^{r/p_s^*(b)} \\ & + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)} \right) \int_\Omega |x|^{-b} |u|^{p_s^*(b)} \mathrm{d}x \\ & + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)} \right) (m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)}. \end{aligned} \tag{4.28}$$

Here, we consider the function $\chi : \mathbb{R}^+ \to \mathbb{R}$, given by

$$\begin{aligned} \chi(t) = & -\alpha\omega_2\omega_3 \left(\frac{1}{r} + \frac{1}{\theta} \right) t^r + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)} \right) t^{p_s^*(b)} \\ & + \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)} \right) (m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)}. \end{aligned}$$

The function χ attains its absolute minimum at the point

$$s_0 = \left[\frac{\alpha r \omega_2 \omega_3 ((1/r) + (1/\theta))}{p_s^*(b)(1/\theta - 1/p_s^*(b))} \right]^{1/(p_s^*(b)-r)}.$$

Hence, we have

$$\begin{aligned} h \geq & \, \chi(s_0) \\ = & \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)} \right) (m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)} \\ & - \left[\frac{\alpha\omega_2\omega_3((1/r) + (1/\theta))}{1/\theta - 1/p_s^*(b)} \right]^{p_s^*(b)/(p_s^*(b)-r)} \\ & \cdot \left[\xi \left(\frac{r}{p_s^*(b)} \right)^{r/(p_s^*(b)-r)} - \xi \left(\frac{r}{p_s^*(b)} \right)^{p_s^*(b)/(p_s^*(b)-r)} \right], \end{aligned} \tag{4.29}$$

which is a contradiction. Thus, $\nu_0 = \mu_0 = 0$, and we obtain

$$\lim_{n\to\infty} \int_\Omega |x|^{-b}|u_n|^{p_s^*(b)}\mathrm{d}x = \int_\Omega |x|^{-b}|u|^{p_s^*(b)}\mathrm{d}x, \tag{4.30}$$

and then the Brézis-Lieb lemma yields that

$$\lim_{n\to\infty} \int_\Omega |x|^{-b}|u_n - u|^{p_s^*(b)}\mathrm{d}x = 0. \tag{4.31}$$

We are ready to show that $u_n \to u$ in $Z(\Omega)$. Using the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega |x|^{-b}|u_n|^{p_s^*(b)-2}u_n(u_n - u)\,\mathrm{d}x = 0,$$
$$\lim_{n\to\infty} \int_\Omega f(x)|x|^{-c}|u_n|^{r-2}u_n(u_n - u)\,\mathrm{d}x = 0.$$

Since $\{u_n\}$ is bounded in $Z(\Omega)$, $\langle I'_\alpha(u_n), u_n - u\rangle \to 0$, $\|u_n\| \to \eta_0 \geq 0$, and M is continuous and positive, we deduce that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \bigg(\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+ps}} \cdot ((u_n - u)(x) - (u_n - u)(y)) \bigg)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.32}$$

Now, since the functional A, defined in the previous section, satisfies the (S)-property, we conclude that $u_n \to u$ in $Z(\Omega)$. The proof is complete. $\square$

Remark 4.5. From Lemma 4.1, there is an α_0 such $T_1 = T_1(\alpha) < t_0$ for each $\alpha \in (0, \alpha_0)$. Note also that there is a $\widetilde{\alpha} < \alpha_0$ such that, for each $0 < \alpha < \widetilde{\alpha}$, we have that

(4.33)
$$\left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)(m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)} - \xi\left[\frac{\alpha\omega_2\omega_3(1/r+1/\theta)}{(1/\theta)-(1/p_s^*(b))}\right]^{p_s^*(b)/(p_s^*(b)-r)} \cdot \left[\left(\frac{r}{p_s^*(b)}\right)^{r/(p_s^*(b)-r)} - \left(\frac{r}{p_s^*(b)}\right)^{p_s^*(b)/(p_s^*(b)-r)}\right] > 0.$$

Lemma 4.6. *Assume that $J_\alpha(u) < 0$. Then, $\|u\|^p_{Z(\Omega)} < T_1$ and, for each v in a sufficiently small neighborhood of u, we have $J_\alpha(u) = I_\alpha(u)$.*

In addition, J_α satisfies a local Palais-Smale condition for $h < 0$ and for all $\alpha \in (0, \widetilde{\alpha})$.

Proof. Note that $0 > J_\alpha(u) \geq \overline{Q}_\alpha(\|u\|^p_{Z(\Omega)})$. Using the same arguments of Lemma 3.1, we conclude that $\|u\|^p_{Z(\Omega)} < T_1$, and $J_\alpha(u) = I_\alpha(u)$ for all $v \in B(u, R)$. Hence, if $\{u_n\}$ is a sequence such that $J_\alpha(u_n) \to h < 0$ and $J'_\alpha(u_n) \to 0$, then we have $I_\alpha(u_n) = J_\alpha(u_n) \to h < 0$ and $I'_\alpha(u_n) = J'_\alpha(u_n) \to 0$. Since J_α is coercive, we conclude that $\{u_n\}$ is bounded in $Z(\Omega)$. From Remark 4.5, for $\alpha \in (0, \widetilde{\alpha})$, we conclude that

$$\begin{aligned} h < 0 < & \left(\frac{1}{\theta} - \frac{1}{p_s^*(b)}\right)(m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)} \\ & - \xi \left[\frac{\alpha\omega_2\omega_3(1/r + 1/\theta)}{1/\theta - 1/p_s^*(b)}\right]^{p_s^*(b)/(p_s^*(b)-r)} \\ & \cdot \left[\left(\frac{r}{p_s^*(b)}\right)^{r/p_s^*(b)-r} - \left(\frac{r}{p_s^*(b)}\right)^{p_s^*(b)/(p_s^*(b)-r)}\right]. \end{aligned} \tag{4.34}$$

Hence, we deduce from Lemma 4.4 that $\{u_n\}$ has a convergent subsequence. The proof is complete. □

Now, we will use the min-max procedure to prove the existence of a sequence of critical values of J_α. First, we prove the following lemma.

Lemma 4.7. *Denote $J^{-\varepsilon} := \{u \in Z(\Omega) : J_\alpha(u) \leq -\varepsilon\}$. Given $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$, such that $\gamma(J^{-\varepsilon}) \geq k$.*

Proof. Let X_k be a k-dimensional subspace of $Z(\Omega)$. Thus, there exists a constant $\delta(k) > 0$ that depends upon k such that

$$r\delta(k)\|u\|^r \leq \omega_1 \|u\|^r_{L^r(\Omega,|x|^{-c})}$$

for each $u \in X_k$. We choose $\varrho > 0$ small enough such that, for $u \in Z(\Omega)$ with $\|u\| = \varrho$, we have $\|u\|^p < T_1$, and thus, $I_\alpha(u) = J_\alpha(u)$. Using the same arguments as in the proof of Theorem 1.1, there is an $R > 0$ such that $I_\alpha(u) < -\varepsilon$ for each $u \in \Lambda := \{u \in X_k : \|u\| = s_1\}$, where $s_1 < \min\{\varrho, R\}$. Therefore, $\Lambda \subset J^{-\varepsilon}$, and since $J^{-\varepsilon}$ is closed and symmetric, we deduce from Proposition 2.1 that $\gamma(J^{-\varepsilon}) \geq \gamma(\Lambda) = k$. □

We define

$$\Sigma_k \left\{A \subset Z(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A, \gamma(A) \geq k\right\},$$

$$K_h = \{u \in Z(\Omega) : J'_\alpha(u) = 0, J_\alpha(u) = h\}$$

and

$$h_k = \inf_{A \in \Sigma_k} \sup_{u \in A} J_\alpha(u).$$

Lemma 4.8. *Assume that $k \in \mathbb{N}$. Then, $h_k < 0$.*

Proof. By Lemma 4.7, there exists an $\varepsilon > 0$ such that $\gamma(J^{-\varepsilon}) \geq k$. Note that $0 \notin J^{-\varepsilon}$ and $J^{-\varepsilon} \in \Sigma_k$. In addition, it follows from the definition that $\sup_{u \in J^{-\varepsilon}} J_\alpha(u) \leq -\varepsilon$; hence, we obtain

$$-\infty < h_k = \inf_{A \in \Sigma_k} \sup_{u \in A} J_\alpha(u) \leq \sup_{u \in J^{-\varepsilon}} J_\alpha(u) \leq -\varepsilon. \qquad \square$$

Lemma 4.9. *Assume that $\alpha \in (0, \widetilde{\alpha})$. Then, all h_k are critical values of J_α. Moreover, if $h = h_k = h_{k+1} = \cdots = h_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\gamma(K_h) \geq r + 1.$$

Proof. By Lemma 4.6 and a standard argument, as in **[33]**, it follows that all h_k are critical values of J_α. Now, let $\{u_n\}$ be a sequence in K_h. From Lemma 4.6, we deduce that $\{u_n\}$ has a convergent subsequence. Thus, K_h is a compact set. Furthermore, $-K_h = K_h$. By contradiction, assume that $\gamma(K_h) \leq r$. Hence, by Proposition 2.1, there exists a closed and symmetric set U such that $K_h \subset U$ and $\gamma(U) = \gamma(K_h) \leq r$. From the deformation lemma (see **[35]**), there exist $\epsilon > 0$ $(h + \epsilon < 0)$ and an odd homeomorphism $\eta : Z(\Omega) \to Z(\Omega)$ such that

$$\eta(J^{h+\epsilon} \setminus U) \subset J^{h-\epsilon}.$$

Therefore, $J^{h+\epsilon} \subset J^0$, and from the definition of $h = h_{k+r}$, it follows that there exists an $A \in \Sigma_{k+r}$ such that $\sup_{u \in A} J_\alpha(u) < h + \epsilon$. Thus, $A \subset J^{h+\epsilon}$. By the properties of γ, we obtain

$$\gamma(\overline{A \setminus U}) \geq \gamma(A) - \gamma(U) \geq k, \qquad \gamma(\eta(\overline{A \setminus U})) \geq k.$$

Hence, we have $\eta\overline{(A \setminus U)} \in \Sigma_k$. Consequently,

$$\sup_{u \in \eta\overline{(A\setminus U)}} J_\alpha(u) \geq h_k > h - \epsilon,$$

a contradiction; thus, $\gamma(K_h) \geq r + 1$. □

Lemma 4.10. *Assume that $\alpha \in (0, \widetilde{\alpha})$. Then, I_α has infinitely many critical points.*

Proof. If $-\infty < h_1 < h_2 < \cdots < h_k < \cdots < 0$, then, since each h_k is a critical value of J_α, we obtain infinitely many critical points of J_α. Now, assume that, for two constants h_k and h_{k+r}, we have $h_k = h_{k+r}$. Then, $h = h_k = h_{k+1} = \cdots = h_{k+r}$ for some $r \in \mathbb{N}$. Thus, by Lemma 4.9, we have $\gamma(K_h) \geq r + 1 \geq 2$. Therefore, by Proposition 2.3, we conclude that K_h has infinitely many points. Hence, J_α has infinitely many critical points. Hence, it follows from Lemma 4.6 that I_α has infinitely many critical points. □

Proof of Theorem 1.2. Suppose that $\widetilde{\alpha}$ is as in Remark 4.5. Assume that $\alpha \in (0, \widetilde{\alpha})$, and that u_α is a non-trivial critical point of I_α, found in Lemma 4.10. In particular, $J_\alpha(u_\alpha) = I_\alpha(u_\alpha) < 0$, and it follows from Lemma 4.6 that $\|u_\alpha\|^p < T_1 < t_0$. Thus,

$$M_0(\|u_\alpha\|^p) = M(\|u_\alpha\|^p),$$

and u_α is a solution to problem (1.1). The proof is complete. □

5. Proof of Theorem 1.3. In this section, inspired by the ideas used in **[13]**, we investigate equation (1.1) for the case of $p < r < p_s^*(b)$, $d = b$, $q = p_s^*(b)$. First, by similar arguments to those in Section 4, we make a truncation on the function $M(t)$. Since $p < r < p_s^*(b)$, we can obtain $\eta \in (p, r)$. Note that M is increasing. Hence, there exists a $t_0 > 0$ such that $m_0 \leq M(0) < M(t_0) < (\eta/p)m_0$. We define

$$M_0(t) = \begin{cases} M(t) & \text{if } 0 \leq t \leq t_0; \\ M(t_0) & \text{if } t \geq t_0. \end{cases}$$

From (M_2), we have

$$m_0 \leq M_0(t) \leq \frac{\eta}{p} m_0. \tag{5.1}$$

In order to prove Theorem 1.3, we first investigate the solutions of the following, related equation:

$$\begin{cases} M_0([u]_{s,p}^p)(-\Delta)_p^s u \\ \quad = \alpha f(x)|x|^{-c}|u|^{r-2}u + |x|^{-b}|u|^{p_s^*(b)-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.2}$$

The energy functional $I_\alpha : Z(\Omega) \to \mathbb{R}$ associated with problem (5.2) is as in the following:

$$\begin{aligned} I_\alpha(u) = {} & \frac{1}{p}\widehat{M_0}(\|u\|_{Z(\Omega)}^p) - \frac{\alpha}{r}\int_\Omega f(x)|x|^{-c}|u|^r \mathrm{d}x \\ & - \frac{1}{q}\int_\Omega |x|^{-b}|u|^{p_s^*(b)}\mathrm{d}x, \end{aligned} \tag{5.3}$$

where $\widehat{M_0}(t) = \int_0^t M_0(\tau)\,d\tau$. Note that I is of class C^1 and, for each $\varphi \in Z(\Omega)$, we have

(5.4)

$$\begin{aligned} \langle I_\alpha'(u), \varphi\rangle = M_0(\|u\|_{Z(\Omega)}^p) \iint_{\mathbb{R}^{2N}} \bigg(& \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\ & \cdot (\varphi(x) - \varphi(y))\bigg)\,\mathrm{d}x\,\mathrm{d}y \\ - \alpha \int_\Omega & f(x)|x|^{-c}|u(x)|^{r-2}u(x)\varphi(x)\,\mathrm{d}x \\ - \int_\Omega & |x|^{-b}|u(x)|^{p_s^*(b)-2}u(x)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

The next lemma implies that I_α possesses the mountain-pass structure.

Lemma 5.1.

(i) *Let the constant η be defined as above. Assume that conditions (M_1), (M_2) and (f_1) hold. Then, there exist positive numbers ρ and ϑ such that*:

$$I_\alpha(u) \geq \vartheta > 0 \quad \textit{for all } u \in Z(\Omega) \textit{ with } \|u\|_{Z(\Omega)} = \rho;$$

(ii) *for all $\alpha > 0$, there exists an $e \in Z(\Omega)$ such that $I_\alpha(e) < 0$ and $\|e\|_{Z(\Omega)} > \rho$.*

Proof.

(i) By (M_1), (f_1) and inequality (2.4), we get

$$I_\alpha(u) \geq \frac{m_0}{p}\|u\|^p_{Z(\Omega)} - \alpha C_4\|u\|^r_{Z(\Omega)} - \frac{1}{p_s^*(b)}C_5\|u\|^{p_s^*(b)}_{Z(\Omega)}.$$

Since $p < r < p_s^*(b)$, by choosing $\rho > 0$ small enough, we obtain the desired result.

(ii) Choose $v_0 \in Z(\Omega)$ with $v_0 \geq 0$ in Ω and $\|v_0\|_{Z(\Omega)} = 1$. From (5.1) and (f_1), we conclude that:

$$\begin{aligned} I_\alpha(tv_0) \leq \frac{\eta m_0}{p^2}t^p\|v_0\|^p_{Z(\Omega)} - \frac{\omega_1\alpha}{r}t^r\int_\Omega |x|^{-c}|v_0|^r\mathrm{d}x \\ - \frac{1}{p_s^*(b)}t^{p_s^*(b)}\int_\Omega |x|^{-b}|v_0|^{p_s^*(b)}\mathrm{d}x. \end{aligned}$$

Since $p < r < p_s^*(b)$, we deduce that $\lim_{t\to\infty} I_\alpha(tv_0) = -\infty$. Hence, for a $\bar{t} > 0$ large enough, the result follows if we set $e = \bar{t}v_0$. □

By a version of the Mountain pass theorem, due to Ambrosetti and Rabinowitz [**35**], without the (PS) condition, we conclude that there exists a sequence $\{u_n\} \subset Z(\Omega)$ such that

$$I_\alpha(u_n) \to h_\alpha, \quad \text{and} \quad I'_\alpha(u_n) \to 0 \text{ in } (Z(\Omega))^{-1},$$

where

$$h_\alpha = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\alpha(\gamma(t)) > 0$$

and

$$\Gamma := \{\gamma \in C([0,1], Z(\Omega)) : \gamma(0) = 0, \quad \gamma(1) = e\}.$$

Lemma 5.2. *Let the constant η be defined as above. Assume that conditions (M_1), (M_2) and (f_1) hold. Then, we have* $\lim_{\alpha\to+\infty} h_\alpha = 0$.

Proof. Since the functional I_α has the mountain pass structure, there exists a $t_\alpha > 0$ such that $I_\alpha(t_\alpha v_0) = \max_{t\geq 0} I_\alpha(tv_0)$, where v_0 is given by Lemma 5.1. Thus, by (f_1) and inequality (5.1), we have

$$\begin{aligned} 0 = \langle I'_\alpha(t_\alpha v_0),\ t_\alpha v_0\rangle \leq \frac{\eta m_0}{p}t_\alpha^p - \alpha\omega_1 t_\alpha^r\int_\Omega |x|^{-c}|v_0|^r\mathrm{d}x \\ - t_\alpha^{p_s^*(b)}\int_\Omega |x|^{-b}|v_0|^{p_s^*(b)}\mathrm{d}x; \end{aligned} \tag{5.5}$$

thus,

$$\frac{\eta m_0}{p} t_\alpha^p \geq t_\alpha^{p_s^*(b)} \int_\Omega |x|^{-b} |v_0|^{p_s^*(b)} \mathrm{d}x.$$

Hence, $\{t_\alpha\}$ is bounded. Therefore, there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \to +\infty$ and $\kappa_0 \geq 0$ such that $\lim_{n\to\infty} t_{\alpha_n} = \kappa_0$. Thus, there exists an $R_1 > 0$ such that

$$\frac{\eta m_0}{p} t_{\alpha_n}^p \leq R_1 \quad \text{for all } n \in \mathbb{N}.$$

It follows from (5.5) that

$$\alpha_n \omega_1 t_{\alpha_n}^r \int_\Omega |x|^{-c} |v_0|^r \mathrm{d}x + t_{\alpha_n}^{p_s^*(b)} \int_\Omega |x|^{-b} |v_0|^{p_s^*(b)} \mathrm{d}x \leq R_1 \quad \text{for all } n \in \mathbb{N}.$$

If $\kappa_0 > 0$, then

$$\lim_{n\to\infty} \left(\alpha_n \omega_1 t_{\alpha_n}^r \int_\Omega |x|^{-c} |v_0|^r \mathrm{d}x + \frac{1}{p_s^*(b)} t_{\alpha_n}^{p_s^*(b)} \int_\Omega |x|^{-b} |v_0|^{p_s^*(b)} \mathrm{d}x \right) = +\infty,$$

which is a contradiction. Thus, we have $\kappa_0 = 0$.

Next, we consider the path $\gamma_1(t) = te$ for $t \in [0,1]$. It is clear that $\gamma_1 \in \Gamma$, Hence, we may have

$$0 < h_\alpha \leq \sup_{t\in[0,1]} I_\alpha(\gamma_1(t)) = I_\alpha(t_\alpha v_0) \leq \frac{\eta m_0}{p^2} t_\alpha^p.$$

On the other hand, the sequence $\{h_\alpha\}$ is monotone; thus, we conclude that $\lim_{\alpha\to+\infty} h_\alpha = 0$. □

Lemma 5.3.

(i) *Under the assumptions of Lemma* 5.2, *there exists an* α_1 *such that*

$$h_\alpha < \left(\frac{1}{p} m_0 - \frac{1}{\eta} M_0(t_0) \right) t_0 \quad \textit{for all } \alpha > \alpha_1.$$

(ii) *Assume that* (M_1), (M_2) *and* (f_1) *hold, and that* $\alpha > \alpha_1$, *where* α_1 *is given in item* (i). *Let* $\{u_n\} \subset Z(\Omega)$ *be a bounded sequence such that*

$$I_\alpha(u_n) \longrightarrow h_\alpha \quad \textit{and} \quad I'_\alpha(u_n) \longrightarrow 0 \quad \textit{in } (Z(\Omega))^{-1}.$$

Then, there is an $N_0 \in \mathbb{N}$ *such that, for all* $n \geq N_0$, *we have*

$$\|u_n\|_{Z(\Omega)}^p \leq t_0.$$

Proof. Lemma 5.3 (i) is a direct consequence of Lemma 5.2.

(ii) Arguing by contradiction, we assume that there is a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}$ such that $\|u_{n_k}\|^p_{Z(\Omega)} > t_0$ for all $k \in \mathbb{N}$. Hence, for each $\alpha > \alpha_1$, we have, from the definition of $M_0(t)$ and η, that

$$\begin{aligned} h_\alpha &= I_\alpha(u_{n_k}) - \frac{1}{\eta}\langle I'_\alpha(u_{n_k}),\ u_{n_k}\rangle + o_k(1) \\ &\geq \frac{1}{p}\widehat{M_0}(\|u_{n_k}\|^p_{Z(\Omega)}) - \frac{1}{\eta}M_0(t_0)\|u_{n_k}\|^p_{Z(\Omega)} + o_k(1) \\ &\geq \left(\frac{1}{p}m_0 - \frac{1}{\eta}M_0(t_0)\right)\|u_{n_k}\|^p_{Z(\Omega)} + o_k(1). \end{aligned} \tag{5.6}$$

Since $m_0 < M_0(t) < (\eta/p)m_0$, we deduce that $(1/p)m_0-(1/\eta)M_0(t_0) > 0$. Thus, we have

$$h_\alpha \geq \left(\frac{1}{p}m_0 - \frac{1}{\eta}M_0(t_0)\right)t_0 > 0,$$

and this is a contradiction. Hence, we conclude that there is an $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$, we have

$$\|u_n\|^p_{Z(\Omega)} \leq t_0. \qquad \square$$

Proof of Theorem 1.3. Using Lemma 5.2, there exists an $\widetilde{\alpha} > \alpha_1 > 0$ such that, for all $\alpha > \widetilde{\alpha}$, we have

$$h_\alpha < \left(\frac{1}{\eta} - \frac{1}{p_s^*(b)}\right)(m_0 H_b)^{p_s^*(b)/(p_s^*(b)-p)}. \tag{5.7}$$

Now, we fix $\alpha > \widetilde{\alpha}$. From Lemma 5.1, we conclude that there exists a bounded sequence $\{u_n\} \subset Z(\Omega)$ such that $I_\alpha(u_n) \to h_\alpha$ and $I'_\alpha(u_n) \to 0$, as $n \to \infty$. By an argument similar to Lemma 4.4, and from (5.7), we conclude that, up to a subsequence, $u_n \to u_\alpha$. Hence, u_α is a weak solution of problem (5.2). Finally, it follows from Lemma 5.3 that u_α is a weak solution of problem (1.1). The proof is complete. $\square$

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