ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 6, 2018

CORRIGENDUM TO THE MINIMUM MATCHING ENERGY OF BICYCLIC GRAPHS WITH GIVEN GIRTH

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ABSTRACT. The matching energy of a graph was introduced by Gutman and Wagner in 2012 and defined as the sum of the absolute values of zeros of its matching polynomial. In **[16]**, the main result, Theorem 3.4, is in error. In this paper, the correct result is given.

1. Introduction. In **[16]**, the matching energy of bicyclic graphs with n vertices and girth g was studied. The following theorem **[16,** Theorem 3.4], where $S_{n-g}(u)\theta(1, g-3, 1)$ denotes the graph obtained by identifying the center of the star S_{n-g} with u, a vertex of degree three in $\theta(1, g-3, 1)$ was given.

Theorem 1.1. *For any graph $G \in \mathcal{B}_{n,g}$, we have*

$$G \succeq S_{n-g}(u)\theta(1, g-3, 1),$$

and thus,

$$\mathrm{ME}(G) \geq \mathrm{ME}(S_{n-g}(u)\theta(1, g-3, 1)),$$

where equality holds if and only if $G \cong S_{n-g}(u)\theta(1, g-3, 1)$.

It is easy to see that $S_{n-g}(u)\theta(1, g-3, 1)$ has girth 4, and so, $S_{n-g}(u)\theta(1, g-3, 1) \notin \mathcal{B}_{n,g}$. In this paper, the correct result of Theorem 1.1 is given, that is, the extremal graphs with minimum matching energy among all bicyclic graphs with given order and girth is given.

All graphs in this paper are finite, connected, simple and undirected. The notation and terminology that will be used can be found in **[1]**.

2010 AMS *Mathematics subject classification.* Primary 05C35, 05C50.

Keywords and phrases. Matching energy, bicyclic graph, girth.

This work was supported by NSFC, grant Nos. 11461054 and 11401348, China.

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Received by the editors on October 29, 2016, and in revised form on October 26, 2017.

DOI:10.1216/RMJ-2018-48-6-1983

Let $G = (V, E)$ be a graph with order $|V| = n$ and size $|E| = m$. A *matching* in a graph G is a set of pairwise nonadjacent edges. A matching is called a k-*matching* if it is of size k. Let $m_k(G)$ denote the number of k-matchings of G, where $m_1(G) = m$ and $m_k(G) = 0$ for $k > \lfloor n/2 \rfloor$ or $k < 0$. In addition, define $m_0(G) = 1$. The matching polynomial of graph G is defined as

$$\alpha(G) = \alpha(G, x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G. The energy of graph G [**6**] is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

An important tool of graph energy is the Coulson integral formula [**6**] (when G is a tree T):

$$E(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k \geq 0} m_k(T) x^{2k} \right] dx. \tag{1.1}$$

The graph energy has been widely studied by theoretical chemists and mathematicians. For more details, see the book on graph energy [**18**] and reviews [**8, 9**].

In 2012, Gutman and Wagner [**10**] defined the matching energy of a graph G. Let G be a simple graph, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the zeros of its matching polynomial. Then,

$$\mathrm{ME}(G) = \sum_{i=1}^{n} |\mu_i|.$$

Similarly to equation (1.1), the matching energy also has a beautiful formula, as follows [**10**]:

$$\mathrm{ME}(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k \geq 0} m_k(G) x^{2k} \right] dx. \tag{1.2}$$

By equation (1.2) and the monotonicity of the logarithm function, the matching energy of a graph G is a monotonically increasing function of any $m_k(G)$. This means that, if two graphs G and G' satisfy $m_k(G) \leq m_k(G')$ for all $k \geq 1$, then $\mathrm{ME}(G) \leq \mathrm{ME}(G')$. If, in addition,

$m_k(G) < m_k(G')$ for at least one k, then $\mathrm{ME}(G) < \mathrm{ME}(G')$. This motivates the introduction of a *quasi-order* $\succeq$ as follows: if two graphs G_1 and G_2 have the same order and size, then

$$G_1 \succeq G_2 \Longleftrightarrow m_k(G_1) \geq m_k(G_2) \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

If $G_1 \succeq G_2$ and there exists some k such that $m_k(G_1) > m_k(G_2)$, then we write $G_1 \succ G_2$. If $G_1 \succeq G_2$, we say that G_1 is *m-greater than* G_2, or G_2 is *m-smaller than* G_1. If $G_1 \succeq G_2$ and $G_2 \succeq G_1$, the graphs G_1 and G_2 are said to be *m-equivalent*; this is denoted by $G_1 \sim G_2$. If $G_1 \succ G_2$, we say that G_1 is *strictly m-greater than* G_2. It is easy to see that

$$G_1 \succeq G_2 \Longrightarrow \mathrm{ME}(G_1) \geq \mathrm{ME}(G_2)$$

and

$$G_1 \succ G_2 \Longrightarrow \mathrm{ME}(G_1) > \mathrm{ME}(G_2).$$

Since the research of extremal graph energy is an interesting problem, the study on extremal matching energy is also interesting.

A connected simple graph with n vertices and n, $n+1$, $n+2$ edges are called *unicyclic*, *bicyclic* and *tricyclic* graphs, respectively. In **[10]**, the authors gave some elementary results on the matching energy and obtained that $\mathrm{ME}(S_n^+) \leq \mathrm{ME}(G) \leq \mathrm{ME}(C_n)$ for any unicyclic graph G, where S_n^+ is the graph obtained by adding a new edge to the star S_n. In **[12]**, Ji, et al., proved that if G is a bicyclic graph with $n \geq 10$ or $n = 8$, $\mathrm{ME}(S_n^*) \leq \mathrm{ME}(G) \leq \mathrm{ME}(P_n^{4,n-4})$. In **[13]**, the authors characterized the connected graphs (and bipartite graphs) of order n having minimum matching energy with m $(n+2 \leq m \leq 2n-4)$ edges. In particular, among all tricyclic graphs of order $n \geq 5$, $\mathrm{ME}(G) \geq \mathrm{ME}(S_n^{**})$ with equality if and only if $G \cong S_n^{**}$ or $G \cong K_4^{n-4}$. In **[3]**, extremal tricyclic graphs with maximum matching energy were given. For more results about matching energy, see **[2, 4, 11, 14, 15, 17, 19, 20, 21, 22, 23]**.

Denote by $\mathcal{B}_{n,g}$ the set of all connected bicyclic graphs with order n and girth g. Now, define two special classes of bicyclic graphs. Let $\infty_n(g,r)$ denote the graph obtained by the coalescence of two end vertices of a path $P_{n-g-r+2}$ with one vertex of two cycles C_g and C_r, respectively, and let $\theta(r,s,t)$ denote the graph obtained by fusing two triples of pendent vertices of three paths $P_{r+2}, P_{s+2}, P_{t+2}$ to two vertices, see Figure 1. Clearly, any bicyclic graph must contain either the left graph or the right graph in Figure 1 as an induced graph, called its *brace*. The set $\mathcal{B}_{n,g}$ can be partitioned into two subsets $\mathcal{B}^1_{n,g}$ and $\mathcal{B}^2_{n,g}$, where $\mathcal{B}^1_{n,g}$ is the set of all bicyclic graphs which contain a brace of the form $\infty_{n'}(g,r)$, and $\mathcal{B}^2_{n,g}$ is the set of all bicyclic graphs which contain a brace of the form $\theta(r,s,t)$.

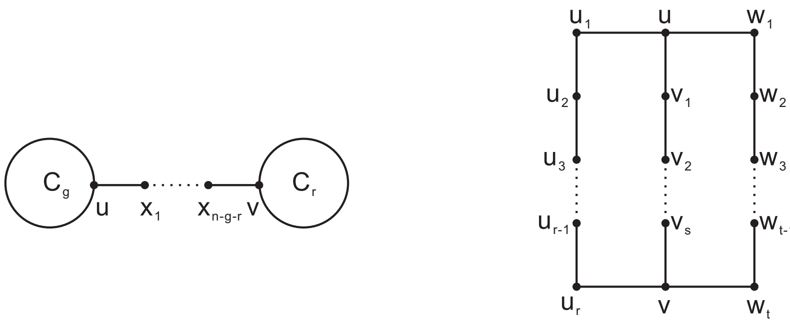


FIGURE 1. Graph $\infty_n(g,r)$ is on the left. Graph $\theta(r,s,t)$ is on the right.

The main result of this paper is the following theorem which gives the graph in $\mathcal{B}_{n,g}$ with minimum matching energy. Let $\theta(a,b,b)(u)S_{n-g-b+1}$ be the graph obtained by identifying the vertex u of $\theta(a,b,b)$ with the center of star $S_{n-g-b+1}$, see Figure 2 (b).

Theorem 1.2. *Let $g \geq 3$ be an integer, $a = \lfloor (g-2)/2 \rfloor$ and $b = g-2-a$. For any $G \in \mathcal{B}_{n,g}$, we have $G \succeq \theta(a,b,b)(u)S_{n-g-b+1}$, and thus,*

$$\mathrm{ME}(G) \geq \mathrm{ME}(\theta(a,b,b)(u)S_{n-g-b+1}),$$

where equality holds if and only if $G \cong \theta(a,b,b)(u)S_{n-g-b+1}$.

2. Preliminaries. We now exhibit some basic results which will be used later.

Lemma 2.1 (**[10]**). *Let G be a graph and e one of its edges. Let $G-e$ be the subgraph obtained by deleting from G the edge e, but keeping all of the vertices of G. Then*:

$$\mathrm{ME}(G-e) < \mathrm{ME}(G).$$

In **[5, 7]**, two fundamental identities are established as follows.

Lemma 2.2. *Let G be a graph. Then, for any edge $e = uv$ and $N(u) = \{v_1(= v), v_2, \dots, v_t\}$, we have the following two identities*:

$$m_k(G) = m_k(G - uv) + m_{k-1}(G - u - v), \tag{2.1}$$

$$m_k(G) = m_k(G - u) + \sum_{i=1}^{t} m_{k-1}(G - u - v_i). \tag{2.2}$$

Let G_i $(i = 1, 2, \dots, n-1)$ be the graph obtained from G $(v \in V(G))$ by adding $n-1$ new vertices to G in the following manner: at v, attach $i-1$ pendent edges and a path of length $n-i$. It is easy to see that the next lemma holds.

Lemma 2.3 ([**16**]). $G_1 \succ G_2 \succ \cdots \succ G_{n-1}$.

3. Proof of the main result. The following theorem and lemma are from [**16,** Theorems 3.1, 3.2]. Let $G(u)H$ be the graph obtained by identifying a common vertex of G and H. Let $S_n(g, g)$ be the graph in $\mathcal{B}^1_{n,g}$ with $n + 1 - 2g$ pendent edges attached at the common vertex of two C_g (see Figure 2 (a)). Let $\theta(r, s, t)(u)S_{n-g-t+1}$ be the graph obtained by identifying the vertex u of $\theta(r, s, t)$ with the center of star $S_{n-g-t+1}$ (see Figure 2 (b)).

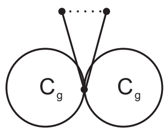
Theorem 3.1 ([**16**]). *For any graph $G \in \mathcal{B}^1_{n,g}$, we have $G \succeq S_n(g, g)$ with equality if and only if $G \cong S_n(g, g)$.*

Lemma 3.2 ([**16**]). *Let r, s, t be three integers with $r + s + 2 = g$. For any $G \in \mathcal{B}^2_{n,g}$ with $\theta(r, s, t)$ as its brace, $G \succeq \theta(r, s, t)(u)S_{n-g-t+1}$ with equality if and only if $G \cong \theta(r, s, t)(u)S_{n-g-t+1}$.*

Theorem 3.3. *Let $g \geq 3$ be an integer, $a = \lfloor (g-2)/2 \rfloor$ and $b = g - 2 - a$. For any graph $G \in \mathcal{B}^2_{n,g}$, we have $G \succeq \theta(a, b, b)(u)S_{n-g-b+1}$ with equality if and only if $G \cong \theta(a, b, b)(u)S_{n-g-b+1}$.*

Proof. Suppose that G has $\theta(r, s, t)$ as its brace where $r + s + 2 = g$ and $r \leq s \leq t$. From Lemma 3.2, $G \succeq \theta(r, s, t)(u)S_{n-g-t+1}$.

Claim 3.4. *If $s < t$, then $\theta(r, s, t)(u)S_{n-g-t+1} \succ \theta(r, s, s)(u)S_{n-g-s+1}$.*



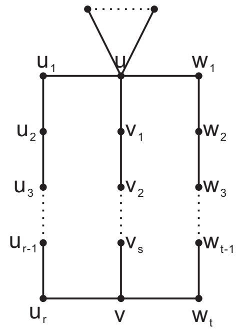


FIGURE 2. (a) The graph $S_n(g,g)$. (b) The graph $\theta(r,s,t)(u)S_{n-g-t+1}$.

Proof. Let $G_1 = \theta(r,s,t)(u)S_{n-g-t+1}$ and $G_2 = \theta(r,s,s)(u)S_{n-g-s+1}$. From equation (2.1), we obtain

$$m_k(G_1) = m_k(G_1 - vw_t) + m_{k-1}(G_1 - v - w_t)$$

and

$$m_k(G_2) = m_k(G_2 - vw_s) + m_{k-1}(G_2 - v - w_s).$$

Note that $G_1 - vw_t \cong C_g(u)S_{n-g-t+1}(u)P_{t+1}$ and $G_2 - vw_s \cong C_g(u)S_{n-g-s+1}(u)P_{s+1}$. Since $s < t$, from Lemma 2.3, $m_k(G_1 - vw_t) \geq m_k(G_2 - vw_s)$, and the inequality is strict for some k.

Also, note that $G_1 - v - w_t \cong P_{r+1}(u)P_{s+1}(u)P_t(u)S_{n-g-t+1}$ and $G_2 - v - w_s \cong P_{r+1}(u)P_{s+1}(u)P_s(u)S_{n-g-s+1}$. Since $s < t$, from Lemma 2.3, $m_{k-1}(G_1 - v - w_t) \geq m_{k-1}(G_2 - v - w_s)$, and the inequality is strict for some k. $\square$

Claim 3.5. *If $s - r \geq 2$, then $\theta(r,s,s)(u)S_{n-g-s+1} \succ \theta(r+1, s-1, s-1)(u)S_{n-g-s+2}$.*

Proof. Let $G_2 = \theta(r,s,s)(u)S_{n-g-s+1}$ and $G_3 = \theta(r+1, s-1, s-1)(u)S_{n-g-s+2}$. From equation (2.1), we obtain

$$m_k(G_2) = m_k(G_2 - vw_s) + m_{k-1}(G_2 - v - w_s)$$

and

$$m_k(G_3) = m_k(G_3 - vw_{s-1}) + m_{k-1}(G_3 - v - w_{s-1}).$$

Note that $G_2 - vw_s \cong C_g(u)S_{n-g-s+1}(u)P_{s+1}$ and $G_3 - vw_{s-1} \cong C_g(u)S_{n-g-s+2}(u)P_s$. From Lemma 2.3,

$$m_k(G_2 - vw_s) \geq m_k(G_3 - vw_{s-1}),$$

and the inequality is strict for some k. Furthermore, note that $G_2 - v - w_s \cong P_{r+1}(u)P_{s+1}(u)P_s(u)S_{n-g-s+1}$ and $G_3 - v - w_{s-1} \cong P_{r+2}(u)P_s(u)P_{s-1}(u)S_{n-g-s+2}$. From equation (2.1), we obtain

$$\begin{aligned} m_{k-1}(G_2 - v - w_s) = m_{k-1}&(G_2 - v - w_s - v_s v_{s-1}) \\ &+ m_{k-2}(G_2 - v - w_s - v_s - v_{s-1}) \end{aligned}$$

and

$$\begin{aligned} m_{k-1}(G_3 - v - w_{s-1}) = m_{k-1}&(G_3 - v - w_{s-1} - u_{r+1}u_r) \\ &+ m_{k-2}(G_3 - v - w_{s-1} - u_{r+1} - u_r). \end{aligned}$$

In addition, note that $G_2 - v - w_s - v_s v_{s-1} \cong P_{r+1}(u)P_s(u)P_s(u)S_{n-g-s+1}$ and $G_3 - v - w_{s-1} - u_{r+1}u_r \cong P_{r+1}(u)P_s(u)P_{s-1}(u)S_{n-g-s+2}$. From Lemma 2.3, $m_{k-1}(G_2 - v - w_s - v_s v_{s-1}) \geq m_{k-1}(G_3 - v - w_{s-1} - u_{r+1}u_r)$ and the inequality is strict for some k.

Now, we see that

$$G_2 - v - w_s - v_s - v_{s-1} \cong P_{r+1}(u)P_{s-1}(u)P_s(u)S_{n-g-s+1}$$

and

$$G_3 - v - w_{s-1} - u_{r+1} - u_r \cong P_r(u)P_s(u)P_{s-1}(u)S_{n-g-s+2}.$$

From Lemma 2.3, $m_{k-2}(G_2 - v - w_s - v_s - v_{s-1}) \geq m_{k-2}(G_3 - v - w_{s-1} - u_{r+1} - u_r)$, and the inequality is strict for some k. $\square$

First apply Claim 3.4 to obtain the form $\theta(r,s,s)(u)S_{n-g-s+1}$, and then repeatedly apply Claim 3.5 until we have $r \leq s \leq r+1$. From here, it is easy to verify that this works for $r = a = \lfloor (g-2)/2 \rfloor$ and $s = b = g - 2 - a$, establishing the result and completing the proof. $\square$

Theorem 3.6. *Let $g \geq 3$ be an integer, $a = \lfloor (g-2)/2 \rfloor$ and $b = g - 2 - a$. Then, $S_n(g,g) \succ \theta(a,b,b)(u)S_{n-g-b+1}$.*

Proof. Let $G = S_n(g,g)$ and $H = \theta(a,b,b)(u)S_{n-g-b+1}$. Let u_0' be the common vertex of the two copies of C_g in $G = S_n(g,g)$. Denote the vertices of one C_g in G by $u'_0, u'_1, u'_2, \ldots, u'_{g-1}$ subsequently. From equation (2.1), we obtain

$$m_k(G) = m_k(G - u'_1u'_2) + m_{k-1}(G - u'_1 - u'_2)$$

and

$$m_k(H) = m_k(H - vw_b) + m_{k-1}(H - v - w_b).$$

Note that

$$G - u'_1u'_2 \cong C_g(u_0)S_{n+3-2g}(u_0)P_{g-1}$$

and

$$H - vw_b \cong C_g(u)S_{n-g-b+1}(u)P_{b+1}.$$

When $g = 3$, $g - 1 = b + 1$; thus, $m_k(G - u'_1u'_2) = m_k(H - vw_b)$. When $g \geq 4$, $g - 1 > b + 1$, and, from Lemma 2.3, we get $m_k(G - u'_1u'_2) \geq m_k(H - vw_b)$ such that the inequality is strict for some k. Furthermore, note that $G - u'_1 - u'_2 \cong C_g(u_0)S_{n+2-2g}(u_0)P_{g-2}$ and $H - v - w_b \cong P_{a+1}(u)P_{b+1}(u)P_b(u)S_{n-g-b+1}$. $G - u'_1 - u'_2$ has one more edge than $H - v - w$, and $P_{a+1}(u_0)P_{b+2}(u_0)S_{n+2-2g}(u_0)P_{g-2}$ is a proper subgraph of $G - u'_1 - u'_2$. Finally, we see that $g - 2 \geq b$, and, from Lemmas 2.1 and 2.3, we obtain that

$$\begin{aligned} m_{k-1}(G - u'_1 - u'_2) &> m_{k-1}(P_{a+1}(u_0)P_{b+2}(u_0)S_{n+2-2g}(u_0)P_{g-2}) \\ &\geq m_{k-1}(H - v - w_b). \end{aligned}$$

The inequality is strict for some k. This completes the proof. $\square$

Theorem 1.2 now follows from Theorem 3.1, Lemma 3.2, Theorem 3.3 and Theorem 3.6.

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