ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 6, 2018

ON CHOW GROUPS OF SOME HYPERKÄHLER FOURFOLDS WITH A NON-SYMPLECTIC INVOLUTION, II

ROBERT LATERVEER

ABSTRACT. This note is about hyperkähler fourfolds X admitting a non-symplectic involution ι. The Bloch-Beilinson conjectures predict the way ι should act on certain pieces of the Chow groups of X. The main result of this note is a verification of this prediction for Fano varieties of lines on certain cubic fourfolds. This has some interesting consequences for the Chow ring of the quotient X/ι.

1. Introduction. For a smooth projective variety X over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups of X (i.e., the groups of codimension i algebraic cycles on X with $\mathbb{Q}$-coefficients, modulo rational equivalence). As explained, for instance, in **[13, 19, 30]**, the Bloch-Beilinson conjectures form a beautiful crystal ball, allowing for strikingly concrete predictions about Chow groups. In this note, we focus on one particular instance of such a prediction, concerning non-symplectic involutions on hyperkähler varieties.

Let X be a hyperkähler variety (i.e., a projective irreducible holomorphic symplectic manifold, cf., **[1, 2]**), and suppose that X has an anti-symplectic involution ι. The action of ι on the subring $H^{*,0}(X)$ is well understood: we have

$$\begin{aligned} \iota^* = -\,\mathrm{id}\colon\ H^{2i,0}(X) &\longrightarrow H^{2i,0}(X) \quad \text{for } i \text{ odd},\\ \iota^* = \mathrm{id}\colon\ H^{2i,0}(X) &\longrightarrow H^{2i,0}(X) \quad \text{for } i \text{ even}. \end{aligned}$$

The action of ι on the Chow ring $A^*(X)$ is more mysterious. To state the conjectural behavior, we will now assume the Chow ring of X

2010 AMS *Mathematics subject classification.* Primary 14C15, 14C25, 14C30.

Keywords and phrases. Algebraic cycles, Chow groups, motives, Bloch's conjecture, Bloch-Beilinson filtration, hyperkähler varieties, non-symplectic involution, multiplicative Chow-Künneth decomposition, splitting property, Calabi-Yau varieties.

Received by the editors on January 15, 2017, and in revised form on January 13, 2018.

DOI:10.1216/RMJ-2018-48-6-1925

has a bigraded ring structure $A^*_{(*)}(X)$, where each $A^i(X)$ splits into pieces
$$A^i(X)=\bigoplus_j A^i_{(j)}(X),$$
and the piece $A^i_{(j)}(X)$ is isomorphic to the graded $\mathrm{Gr}^j_F A^i(X)$ for the Bloch-Beilinson filtration that conjecturally exists for all smooth projective varieties. (It is expected that such a bigrading $A^*_{(*)}(-)$ exists for all hyperkähler varieties **[3]**.)

Since the pieces $A^i_{(i)}(X)$ and $A^{\dim X}_{(i)}(X)$ should only depend upon the subring $H^{*,0}(X)$, we arrive at the following conjecture:

Conjecture 1.1. *Let X be a hyperkähler variety of dimension $2m$, and let $\iota\in\mathrm{Aut}(X)$ be a non-symplectic involution. Then*:
$$\begin{aligned}\iota^*=(-1)^i\,\mathrm{id}\colon\ A^{2i}_{(2i)}(X)\longrightarrow A^{2i}(X),\\ \iota^*=(-1)^i\,\mathrm{id}\colon\ A^{2m}_{(2i)}(X)\longrightarrow A^{2m}(X).\end{aligned}$$

This conjecture is studied, and proven in some particular cases, in **[14, 15, 16, 17]**. The aim of this note is to provide some more examples where Conjecture 1.1 is verified, by considering Fano varieties of lines on cubic fourfolds. The main result is as follows:

Theorem 1.2. *Let $Y\subset\mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation*
$$\begin{aligned}(X_0)^2\ell_0(X_3,X_4,X_5)+(X_1)^2\ell_1(X_3,X_4,X_5)\\ +(X_2)^2\ell_2(X_3,X_4,X_5)+X_0X_1\ell_3(X_3,X_4,X_5)\\ +X_0X_2\ell_4(X_3,X_4,X_5)+X_1X_2\ell_5(X_3,X_4,X_5)\\ +g(X_3,\dots,X_5)=0,\end{aligned}$$
where the ℓ_i are linear forms and g is a homogeneous degree 3 polynomial. Let $X=F(Y)$ be the Fano variety of lines in Y. Let $\iota\in\mathrm{Aut}(X)$ be the anti-symplectic involution, induced by
$$\begin{aligned}\mathbb{P}^5(\mathbb{C})\ &\longrightarrow\mathbb{P}^5(\mathbb{C}),\\ [X_0,X_1,\dots,X_5]\ &\longmapsto[-X_0,-X_1,-X_2,X_3,X_4,X_5].\end{aligned}$$

Then,

$$\iota^* = -\operatorname{id}\colon\ A^i_{(2)}(X) \longrightarrow A^i_{(2)}(X) \quad \textit{for } i = 2, 4;$$
$$\iota^* = \operatorname{id}\colon\ A^4_{(j)}(X) \longrightarrow A^4_{(j)}(X) \quad \textit{for } j = 0, 4.$$

The notation $A^*_{(*)}(X)$ in Theorem 3.5 refers to the Fourier decomposition of the Chow ring of X constructed by Shen and Vial [**22**]. (We mention in passing that, for X as in Theorem 3.5, it is unfortunately not yet known whether $A^*_{(*)}(X)$ is a bigraded ring, cf., Remark 2.6 below.)

In order to prove Theorem 3.5, we exploit the fact that the family of cubics under consideration is sufficiently large for the method of "spread" developed by Voisin [**28, 31**] to apply. There is only one other family of cubic fourfolds with a polarized involution that is anti-symplectic on the Fano variety; this other family was treated in [**16**], using arguments very similar to those here. It is worth mentioning that the action of polarized *symplectic* automorphisms on Chow groups of Fano varieties of cubic fourfolds has already been treated by Fu [**8**], similarly using the "spread" method.

Theorem 3.5 has some rather striking consequences for the Chow ring of the quotient (this quotient is a slightly singular Calabi-Yau fourfold):

Corollaries 4.1 and 4.3. *Let* (X, ι) *be as in Theorem* 3.5. *Let* $Z := X/\iota$ *be the quotient. Then, the images of the intersection product maps*:

$$A^2(Z) \otimes A^2(Z) \longrightarrow A^4(Z),$$
$$A^3(Z) \otimes A^1(Z) \longrightarrow A^4(Z)$$

are of dimension 1.

In particular, this means that, for any two cycles $b, c \in A^2(Z)$ (or $b \in A^3(Z)$ and $c \in A^1(Z)$), the 0-cycle $b \cdot c$ is rationally trivial if and only if it has degree 0. This is similar to results for Calabi-Yau complete intersections obtained in [**7, 27**].

Corollary 4.4. *Let (X,ι) be as in Theorem* 3.5. *Let $Z:=X/\iota$ be the quotient. Then, the image of the intersection product map*

$$A^2(Z)\otimes A^1(Z)\longrightarrow A^3(Z)$$

is a finite-dimensional $\mathbb{Q}$-vector space.

A stronger statement would be expected to be true: conjecturally, for any b in the image of $A^2(Z)\otimes A^1(Z)\to A^3(Z)$, we have that b is rationally trivial if and only if b is homologically trivial. Our argument does not allow the proof of this due to the nuisance (mentioned above) that it is still unknown whether $A^*_{(*)}(X)$ is a bigraded ring (cf., Remark 4.5).

Corollaries 4.1 and 4.4 provide some (admittedly meagre) support in favor of the conjecture that

$$A^2_{\mathrm{hom}}(Z)\overset{??}{=}0.$$

(The Bloch-Beilinson conjectures imply that $A^2_{AJ}(M)=0$ for any Calabi-Yau variety M of dimension >2. As far as the author is aware, there is not a single Calabi-Yau variety M for which this is known to be true.)

Convention 1.2. In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the *Chow group of j-dimensional cycles on X with $\mathbb{Q}$-coefficients*; for X smooth of dimension n, the notation $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notation $A^j_{\mathrm{hom}}(X)$, $A^j_{AJ}(X)$ will be used to indicate the *subgroups of homologically trivial*, respectively, *Abel-Jacobi trivial cycles.* For a morphism $f\colon X\to Y$, we will write $\Gamma_f\in A_*(X\times Y)$ for the *graph* of f. The *contravariant category of Chow motives* (i.e., *pure motives with respect to rational equivalence* as in **[19, 21]**) will be denoted $\mathcal{M}_{\mathrm{rat}}$. We will write $H^j(X)$ to indicate singular cohomology $H^j(X,\mathbb{Q})$.

2. Preliminaries.

2.1. MCK decomposition.

Definition 2.1 ([**18**])**.** Let X be a smooth projective variety of dimension n. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_0 + \pi_1 + \cdots + \pi_{2n} \quad \text{in } A^n(X \times X)\ ,$$

such that the π_i are mutually orthogonal idempotents and $(\pi_i)_* H^*(X) = H^i(X)$.

"CK decomposition" is shorthand for "Chow-Künneth decomposition."

Remark 2.2. The existence of a CK decomposition for any smooth projective variety is part of Murre's conjectures [**13, 18**].

Definition 2.3 ([**22**])**.** Let X be a smooth projective variety of dimension n. Let $\Delta_X^{\mathrm{sm}} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{\mathrm{sm}} := \big\{(x,x,x) \mid x \in X\big\} \subset X \times X \times X.$$

An MCK *decomposition* is a CK decomposition $\{\pi_i^X\}$ of X that is *multiplicative*, i.e., it satisfies

$$\pi_k^X \circ \Delta_X^{\mathrm{sm}} \circ (\pi_i^X \times \pi_j^X) = 0 \quad \text{in } A^{2n}(X \times X \times X) \text{ for all } i+j \neq k.$$

"MCK decomposition" is shorthand for "multiplicative Chow-Künneth decomposition."

A *weak* MCK *decomposition* is a CK decomposition $\{\pi_i^X\}$ of X that satisfies

$$\big(\pi_k^X \circ \Delta_X^{\mathrm{sm}} \circ (\pi_i^X \times \pi_j^X)\big)_*(a \times b) = 0 \quad \text{for all } a, b \in\ A^*(X).$$

Remark 2.4. The small diagonal (seen as a correspondence from $X \times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{\mathrm{sm}} \colon h(X) \otimes h(X) \longrightarrow h(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

Suppose that X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

By definition, this decomposition is multiplicative if, for any i, j, the composition

$$h^i(X) \otimes h^j(X) \longrightarrow h(X) \otimes h(X) \xrightarrow{\Delta_X^{\mathrm{sm}}} h(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}$$

factors through $h^{i+j}(X)$.

If X has a weak MCK decomposition, then, setting

$$A^i_{(j)}(X) := (\pi^X_{2i-j})_* A^i(X),$$

a bigraded ring structure is obtained on the Chow ring, that is, the intersection product sends $A^i_{(j)}(X) \otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

It is expected (but not proven) that, for any X with a weak MCK decomposition, we have

$$A^i_{(j)}(X) \stackrel{??}{=} 0 \quad \text{for } j < 0, \qquad A^i_{(0)}(X) \cap A^i_{\mathrm{hom}}(X) \stackrel{??}{=} 0;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [**18**].

The property of having an MCK decomposition is severely restrictive and is closely related to Beauville's "(weak) splitting property" [**3**]. For more ample discussion, and examples of varieties with an MCK decomposition, the reader is referred to [**22,** Section 8], as well as [**11, 23, 25**].

In what follows, we will make use of the following:

Theorem 2.5 ([**22**]). *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let $X := F(Y)$ be the Fano variety of lines in Y. There exists a* CK *decomposition $\{\pi^X_i\}$ for X, and*

$$(\pi^X_{2i-j})_* A^i(X) = A^i_{(j)}(X),$$

where the right-hand side denotes the splitting of the Chow groups defined in terms of the Fourier transform, as in [**22,** Theorem 2].

Moreover, we have

$$A^i_{(j)}(X) = 0 \quad \textit{for } j < 0 \textit{ and for } j > i.$$

In the case where Y is very general, the Fourier decomposition $A^*_{(*)}(X)$ forms a bigraded ring, and hence, $\{\pi_i^X\}$ is a weak MCK *decomposition.*

Proof. A remark on notation: what we have denoted $A^i_{(j)}(X)$ is denoted $CH^i(X)_j$ in [**22**]. The existence of a CK decomposition $\{\pi_i^X\}$ is [**22,** Theorem 3.3], combined with the results in [**22,** Section 3] to ensure that the hypotheses of [**22,** Theorem 3.3] are satisfied. (Alternatively, the existence of a CK decomposition is also established in [**10,** Proposition A.6].) According to [**22,** Theorem 3.3], the given CK decomposition agrees with the Fourier decomposition of the Chow groups. The "moreover" part is due to the fact that the $\{\pi_i^X\}$ are shown to satisfy Murre's conjecture B [**22,** Theorem 3.3].

The statement for very general cubics is [**22,** Theorem 3]. $\square$

Remark 2.6. Unfortunately, it is not yet known whether the Fourier decomposition of [**22**] induces a bigraded ring structure on the Chow ring for *all* Fano varieties of smooth cubic fourfolds. For instance, it has not yet been proven that

$$A^2_{(0)}(X) \cdot A^2_{(0)}(X) \overset{??}{\subset} A^4_{(0)}(X)$$

for the Fano variety of a given (not necessarily very general) cubic fourfold (cf., [**22,** subsection 22.3] for a discussion).

To prove that $A^*_{(*)}()$ is a bigraded ring for all Fano varieties of smooth cubic fourfolds, it would suffice to construct an MCK decomposition for the Fano variety of the very general cubic fourfold.

2.2. A multiplicative result. Let X be the Fano variety of lines on a smooth cubic fourfold. As we have seen (Theorem 2.5), the Chow ring of X splits into pieces $A^i_{(j)}(X)$. The work [**22**] contains a thorough analysis of the multiplicative behavior of these pieces. Here are the relevant results which will be necessary:

Theorem 2.7 (**[22]**). *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let $X := F(Y)$ be the Fano variety of lines in Y.*

(i) *There exists an $\ell \in A^2_{(0)}(X)$ such that intersecting with ℓ induces an isomorphism*

$$\cdot\ell\colon A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X).$$

(ii) *Intersection product induces a surjection*

$$A^2_{(2)}(X) \otimes A^2_{(2)}(X) \twoheadrightarrow A^4_{(4)}(X).$$

Proof. Statement (i) is [**22,** Theorem 4]. Statement (ii) is [**22,** Proposition 20.3]. □

2.3. The involution.

Lemma 2.8. *Let $\iota_{\mathbb{P}} \in \mathrm{Aut}(\mathbb{P}^5(\mathbb{C}))$ be the involution, defined as*

$$[X_0, X_1, \ldots, X_5] \longmapsto [-X_0, -X_1, -X_2, X_3, X_4, X_5].$$

The cubic fourfolds, invariant under $\iota_{\mathbb{P}}$, are exactly those defined by an equation

$$\begin{aligned}(X_0)^2\ell_0(X_3,X_4,X_5) + (X_1)^2\ell_1(X_3,X_4,X_5) + (X_2)^2\ell_2(X_3,X_4,X_5)\\ + X_0X_1\ell_3(X_3,X_4,X_5) + X_0X_2\ell_4(X_3,X_4,X_5)\\ + X_1X_2\ell_5(X_3,X_4,X_5) + g(X_3,\ldots,X_5) = 0,\end{aligned}$$

where the ℓ_i are linear forms, and g is a homogeneous degree 3 polynomial.

Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic invariant under $\iota_{\mathbb{P}}$, and let $\iota_Y \in \mathrm{Aut}(Y)$ be the involution induced by $\iota_{\mathbb{P}}$. Let $X = F(Y)$ be the Fano variety of lines in Y, and let $\iota \in \mathrm{Aut}(X)$ be the involution induced by ι_Y. The involution ι is anti-symplectic.

Proof. The only necessary explanation is the last phrase; this is proven in [**5,** Section 7]. The idea is that there is an isomorphism of Hodge structures, compatible with the involution

$$H^2(X) \cong H^4(Y).$$

The action of ι_Y on $H^{3,1}(Y)$ is minus the identity since $H^{3,1}(Y)$ is generated by the meromorphic form

$$\sum_{i=0}^{5}(-1)^i X_i \frac{dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_5}{f^2},$$

where f is an equation for Y. □

2.4. Spread.

Lemma 2.9 ([**28, 31**])**.** *Let M be a smooth projective variety of dimension $n+1$ and L a very ample line bundle on M. Let*

$$\pi\colon \mathcal{X} \longrightarrow B$$

denote a family of hypersurfaces, where $B \subset |L|$ is a Zariski open. Let

$$p\colon \widetilde{\mathcal{X} \times_B \mathcal{X}} \longrightarrow \mathcal{X} \times_B \mathcal{X}$$

denote the blow-up of the relative diagonal. Then, $\widetilde{\mathcal{X} \times_B \mathcal{X}}$ is Zariski open in V, where V is a projective bundle over $\widetilde{M \times M}$, the blow-up of $M \times M$ along the diagonal.

Proof. This is [**28,** Proof of Proposition 3.13], [**31,** Lemma 1.3]. The idea is to define V as

$$V := \{((x,y,z),\sigma) | \sigma|_z = 0\} \subset \widetilde{M \times M} \times |L|.$$

The very ampleness assumption ensures that $V \to \widetilde{M \times M}$ is a projective bundle. □

This is used in the following key proposition:

Proposition 2.10 ([**31**])**.** *Assumptions hold as in Lemma 2.9. Assume, moreover, that M has trivial Chow groups. Let $R \in A^n(V)$. Suppose that, for all $b \in B$, we have*

$$H^n(X_b)_{\mathrm{prim}} \neq 0 \quad \textit{and} \quad R|_{\widetilde{X_b \times X_b}} = 0 \in H^{2n}(\widetilde{X_b \times X_b}).$$

Then, there exists a $\gamma \in A^n(M \times M)$ such that

$$(p_b)_*(R|_{\widetilde{X_b \times X_b}}) = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b)$$

for all $b \in B$. (Here p_b denotes the restriction of p to $\widetilde{X_b \times X_b}$, which is the blow-up of $X_b \times X_b$ along the diagonal.)

Proof. This is [**31,** Proposition 1.6]. □

Next is an equivariant version of Proposition 2.10:

Proposition 2.11 ([**31**])**.** *Let M and L be as in Proposition 2.10. Let $G \subset \mathrm{Aut}(M)$ be a finite group. Assume the following:*

(i) *The linear system $|L|^G := \mathbb{P}(H^0(M,L)^G)$ has no base-points, and the locus of points in $\widetilde{M \times M}$ parametrizing triples (x,y,z) such that the length 2 subscheme z imposes only one condition on $|L|^G$ is contained in the union of (proper transforms of) graphs of non-trivial elements of G, plus some loci of codimension $> n+1$.*

(ii) *Let $B \subset |L|^G$ be the open parametrizing smooth hypersurface, and let $X_b \subset M$ be a hypersurface for $b \in B$ general. There is no non-trivial relation*

$$\sum_{g \in G} c_g \Gamma_g + \gamma = 0 \quad \textit{in } H^{2n}(X_b \times X_b),$$

where $c_g \in \mathbb{Q}$ and γ is a cycle in $\mathrm{Im}(A^n(M \times M) \to A^n(X_b \times X_b))$.

Let $R \in A^n(\mathcal{X} \times_B \mathcal{X})$ be such that

$$R|_{X_b \times X_b} = 0 \in H^{2n}(X_b \times X_b) \quad \textit{for all } b \in B.$$

Then, there exists a $\gamma \in A^n(M \times M)$ such that

$$R|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b) \quad \textit{for all } b \in B.$$

Proof. This is not stated verbatim in [**31**]; however, it is contained in the proof of [**31,** Proposition 3.1, Theorem 3.3]. We briefly review the argument. We consider

$$V := \{((x,y,z),\sigma) | \sigma|_z = 0\} \subset \widetilde{M \times M} \times |L|^G.$$

The problem is that this is no longer a projective bundle over $\widetilde{M \times M}$. However, as explained in the proof of [**31,** Theorem 3.3], hypothesis (i) ensures that we can obtain a projective bundle after blowing up the graphs Γ_g, $g \in G$, plus some loci of codimension $> n+1$. Let

$M' \to \widetilde{M \times M}$ denote the result of these blow-ups, and let $V' \to M'$ denote the projective bundle obtained by base-changing.

Analyzing the situation as in [**31,** Proof of Theorem 3.3], we obtain

$$R|_{X_b \times X_b} = R_0|_{X_b \times X_b} + \sum_{g \in G} \lambda_g \Gamma_g \quad \text{in } A^n(X_b \times X_b),$$

where $R_0 \in A^n(M \times M)$ and $\lambda_g \in \mathbb{Q}$ (this is [**31,** equation (15)]). By assumption, $R|_{X_b \times X_b}$ is homologically trivial. Using hypothesis (ii), this implies that all λ_g must be 0. $\square$

3. Main result. This section contains the proof of the main result of this note, Theorem 3.5. The proof is split into two parts. In the first part, we prove a statement (Theorem 3.1) about the action of the involution on 1-cycles on the cubic Y. The proof is an application of the technique of "spread" of cycles in a family, as developed by Voisin [**28, 29, 30, 31**] (more precisely, the results recalled in subsection 2.4).

In the second part, we deduce from this our main result, Theorem 3.5. This second part builds on the structural results of Shen and Vial [**22**] (notably the results recalled in subsections 2.1 and 2.2).

3.1. First part.

Theorem 3.1. *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation*

$$\begin{aligned}(X_0)^2\ell_0(X_3, X_4, X_5) + (X_1)^2\ell_1(X_3, X_4, X_5) + (X_2)^2\ell_2(X_3, X_4, X_5) \\ + X_0X_1\ell_3(X_3, X_4, X_5) + X_0X_2\ell_4(X_3, X_4, X_5) \\ + X_1X_2\ell_5(X_3, X_4, X_5) + g(X_3, \ldots, X_5) = 0,\end{aligned}$$

where the ℓ_i are linear forms, and g is a homogeneous degree 3 polynomial.

Let $\iota_Y \in \mathrm{Aut}(Y)$ be the involution of Lemma 2.8. Then,

$$(\iota_Y)^* = -\,\mathrm{id}\colon A^3_{hom}(Y) \longrightarrow A^3(Y).$$

Proof. We have seen (Proof of Lemma 2.8) that

$$(\iota_Y)^* = -\,\mathrm{id}\colon H^{3,1}(Y) \longrightarrow H^{3,1}(Y).$$

Let $H^4_{tr}(Y)$ denote the orthogonal complement (under the cup-product pairing) of $N^2H^4(Y)$ (which coincides with $H^{2,2}(Y,\mathbb{Q})$ since the Hodge conjecture is true for Y). Since $H^4_{tr}(Y)\subset H^4(Y)$ is the smallest Hodge substructure containing $H^{3,1}(Y)$, we must also have

$$(\iota_Y)^* = -\operatorname{id}\colon H^4_{tr}(Y)\longrightarrow H^4_{tr}(Y). \tag{3.1}$$

This implies that there is a decomposition

$${}^t\Gamma_{\iota_Y} = -\Delta_Y + \gamma \quad \text{in } H^8(Y\times Y), \tag{3.2}$$

where $\gamma\in A^4(Y\times Y)$ is a "completely decomposed" cycle, i.e.,

$$\gamma=\gamma_0+\gamma_2+\gamma_4+\gamma_6+\gamma_8,$$

and γ_{2i} has support on $V_i\times W_i\subset Y\times Y$ with $\dim V_i = i$ and $\dim W_i = 4-i$. (Indeed, the cycle γ is obtained by considering

$$\gamma_{2i} := ({}^t\Gamma_{\iota_Y}+\Delta_Y)\circ\pi_{2i}\in H^8(Y\times Y),$$

where π_i denotes the Künneth component. For $i\neq 4$, the claimed support condition is obviously satisfied since it is satisfied by π_i. For $i=4$, we use (3.1) to see that γ_4 is supported on $N^2H^4(Y)\otimes N^2H^4(Y)\subset H^8(Y\times Y)$.) $\square$

Now, we consider things family-wise. Let

$$\mathcal{Y}\longrightarrow B$$

denote the universal family of all smooth cubic fourfolds, defined by an equation as in Theorem 3.1. Let $Y_b\subset\mathbb{P}^5(\mathbb{C})$ denote the fibre over $b\in B$.

The involution $\iota_{\mathbb{P}}$ defines, by restriction, an involution $\iota_{\mathcal{Y}}\in\operatorname{Aut}(\mathcal{Y})$. Let $\Delta_{\mathcal{Y}}\in A^4(\mathcal{Y}\times_B\mathcal{Y})$ denote the relative diagonal. Obviously, the argument leading to decomposition (3.2) applies to each fibre Y_b. This means that, for each $b\in B$, there exists a completely decomposed cycle $\gamma_b\in A^4(Y_b\times Y_b)$ such that

$$\big({}^t\Gamma_{\iota_{\mathcal{Y}}}+\Delta_{\mathcal{Y}}\big)|_{Y_b\times Y_b}=\gamma_b \quad \text{in } H^8(Y_b\times Y_b).$$

Applying the "spread" result [**28,** Proposition 3.7], we can find a "completely decomposed" relative correspondence $\gamma\in A^4(\mathcal{Y}\times_B\mathcal{Y})$ such that

$$\big({}^t\Gamma_{\iota_{\mathcal{Y}}}+\Delta_{\mathcal{Y}}-\gamma\big)|_{Y_b\times Y_b}=0 \quad \text{in } H^8(Y_b\times Y_b) \text{ for all } b\in B.$$

(By this, we mean the following: there exist subvarieties $\mathcal{V}_i, \mathcal{W}_i \subset \mathcal{Y}$ for $i = 0, 2, 4, 6, 8$, with

$$\operatorname{codim} \mathcal{V}_i + \operatorname{codim} \mathcal{W}_i = 4,$$

and such that the cycle γ is supported on

$$\cup_i \mathcal{V}_i \times_B \mathcal{W}_i \subset \mathcal{Y} \times_B \mathcal{Y}.$$

Actually, for $i \neq 4$, this is obvious since the π_i, $i \neq 4$, obviously exist relatively. The recourse to [**28,** Proposition 3.7] can thus be limited to $i = 4$.)

The relative correspondence

$$\Gamma := {}^t\Gamma_{\iota_{\mathcal{Y}}} + \Delta_{\mathcal{Y}} - \gamma \in A^4(\mathcal{Y} \times_B \mathcal{Y})$$

is fibrewise homologically trivial:

$$\Gamma|_{Y_b \times Y_b} = 0 \quad \text{in } H^8(Y_b \times Y_b) \text{ for all } b \in B.$$

At this point, we note that the family $\mathcal{Y} \to B$ is large enough to verify the hypotheses of Proposition 2.11; this will be proven in Lemma 3.2 below. Applying Proposition 2.11 to the relative correspondence Γ, we find that there exists a $\delta \in A^4(\mathbb{P}^5 \times \mathbb{P}^5)$ such that

$$\Gamma|_{Y_b \times Y_b} + \delta|_{Y_b \times Y_b} = 0 \quad \text{in } A^4(Y_b \times Y_b) \text{ for all } b \in B.$$

However,

$$(\delta|_{Y_b \times Y_b})_* = 0 \colon A^3_{\text{hom}}(Y_b) \longrightarrow A^3(Y_b) \quad \text{for all } b \in B$$

(indeed, the action factors over $A^4_{\text{hom}}(\mathbb{P}^5)$ which is 0). In addition, we have

$$(\gamma|_{Y_b \times Y_b})_* = 0 \colon A^3_{\text{hom}}(Y_b) \longrightarrow A^3(Y_b) \quad \text{for general } b \in B$$

(indeed, for general $b \in B$, the restriction $\gamma|_{Y_b \times Y_b}$ is a completely decomposed cycle; such cycles do not act on A^3_{hom} for dimension reasons).

By definition of Γ, this means that

$$\left({}^t\Gamma_{\iota_{Y_b}} + \Delta_{Y_b}\right)_* = 0 \colon A^3_{\text{hom}}(Y_b) \longrightarrow A^3(Y_b) \quad \text{for general } b \in B.$$

This proves Theorem 3.1 for general $b \in B$. In order to extend to all $b \in B$, it can be argued as in [**8,** Lemma 3.1].

It only remains to check that the hypotheses of Voisin's result are satisfied:

Lemma 3.2. *Let $\mathcal{Y} \to B$ be the family of smooth cubic fourfolds as in Theorem* 3.1*, i.e.,*

$$B \subset \big(\mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))\big)^G$$

is the open subset parametrizing smooth G-invariant cubics, where $G = \{id, \iota_{\mathbb{P}}\} \subset \mathrm{Aut}(\mathbb{P}^5)$ is as above. This set-up verifies the hypotheses of Proposition 2.11*.*

Proof. We first prove hypothesis (i) of Proposition 2.11 is satisfied. Toward this end, we consider the quotient morphisms

$$p\colon \mathbb{P}^5 \longrightarrow P := \mathbb{P}^5/G \longrightarrow P' := \mathbb{P}(2^3, 1^3) = \mathbb{P}^5/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}),$$

where $P' := \mathbb{P}(2^3, 1^3)$ denotes a weighted projective space.

We write $\iota_0, \iota_1, \iota_2$ for the involutions of $\mathbb{P}^5$

$$\begin{aligned}
\iota_0[X_0 : \ldots : X_5] &:= [-X_0 : X_1 : \ldots : X_5],\\
\iota_1[X_0 : \ldots : X_5] &:= [X_0 : -X_1 : X_2 : \ldots : X_5],\\
\iota_2[X_0 : \ldots : X_5] &:= [X_0 : X_1 : -X_2 : X_3 : X_4 : X_5].
\end{aligned}$$

(We note that $\iota_{\mathbb{P}} = \iota_0 \circ \iota_1 \circ \iota_2$, and the weighted projective space P' is $\mathbb{P}^5/\langle \iota_0, \iota_1, \iota_2\rangle$.)

The sections in $(\mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))^G$ are in bijection with sections coming from P and contain the sections coming from P':

$$\big(\mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))\big)^G \supset \mathbb{P}H^0\big(P', \mathcal{O}_{P'}(3)\big).$$

Let us now assume that $x, y \in \mathbb{P}^5$ are two points such that

$$(x, y) \notin \Delta_{\mathbb{P}^5} \cup \bigcup_{0 \le r_0, r_1, r_2 \le 1} \Gamma_{(\iota_0)^{r_0} \circ (\iota_1)^{r_1} \circ (\iota_2)^{r_2}}.$$

Then,

$$p(x) \neq p(y) \quad \text{in } P',$$

and thus (using Lemma 3.3 below), there exists a $\sigma \in \mathbb{P}H^0(P', \mathcal{O}_{P'}(3))$ containing $p(x)$ but not $p(y)$. The pullback $p^*(\sigma)$ contains x but not y, and so, these points (x, y) impose two independent conditions on $(\mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))^G$.

It remains to check that a generic element

$$(x,y) \in \bigcup_{0\le r_0,r_1,r_2\le 1} \Gamma_{(\iota_0)^{r_0}\circ(\iota_1)^{r_1}\circ(\iota_2)^{r_2}} \setminus \Gamma_{\iota_{\mathbb{P}}}$$

also imposes two independent conditions. We first assume that (x,y) is generic on Γ_{ι_0}. Let us write $x = [a_0 : a_1 : \ldots : a_5]$. By genericity, we may assume that all a_i are $\neq 0$ (intersections of Γ_{ι_0} with a coordinate hyperplane have codimension $> n+1$ and thus need not be considered for hypothesis (i) of Proposition 2.11). We can, thus, write

$$x = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5],$$
$$y = [-a_0 : a_1 : a_2 : a_3 : a_4 : a_5], \quad a_i \neq 0.$$

The cubic

$$(a_1)^3(X_0)^3 - (a_0)^3(X_1)^3 = 0$$

is G-invariant and contains x while avoiding y, and thus, the element (x,y) again imposes two independent conditions.

The argument for the other r_i is similar: consider, for instance, a generic element (x,y) in $\Gamma_{\iota_0\circ\iota_1}$. By genericity, we can write

$$x = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5],$$
$$y = [-a_0 : -a_1 : a_2 : a_3 : a_4 : a_5], \quad a_i \neq 0.$$

The cubic

$$(a_2)^3(X_0)^3 - (a_0)^3(X_2)^3 = 0$$

is G-invariant and contains x while avoiding y, and thus, the element (x,y) again imposes two independent conditions. This proves hypothesis (i) of Proposition 2.11 is satisfied.

In order to establish hypothesis (ii) of Proposition 2.11, let $Y = Y_b$ be a cubic as in Theorem 3.1, and let us suppose that there is a relation

$$c\,\Delta_Y + d\,\Gamma_\iota + \delta = 0 \quad \text{in } H^8(Y \times Y),$$

where $c, d \in \mathbb{Q}$ and $\delta \in \mathrm{Im}(A^4(\mathbb{P}^5 \times \mathbb{P}^5) \to A^4(Y_b \times Y_b))$. Looking at the action on $H^{3,1}(Y)$, we find that, necessarily, $c = d$ (indeed, δ does not act on $H^{3,1}(Y)$, and ι acts as minus the identity on $H^{3,1}(Y)$).

On the other hand, looking at the action on $(H^4(Y)_{\mathrm{prim}})^\iota$ (which is non-zero due to Lemma 3.4), we find that $c = -d$. We conclude that $c = d = 0$, and thus, hypothesis (ii) is satisfied.

Lemma 3.3. *Let $P' = \mathbb{P}(2^3, 1^3)$. Let $r, s \in P'$ and $r \neq s$. Then, there exists a $\sigma \in \mathbb{P}H^0(P', \mathcal{O}_{P'}(3))$ containing r but avoiding s.*

Proof. It follows from Delorme's work [**6,** Proposition 2.3(3)] that the locally free sheaf $\mathcal{O}_{P'}(2)$ is very ample. This means that there exists a $\sigma' \in \mathbb{P}H^0(P, \mathcal{O}_P(2))$ containing r but avoiding s. Taking the union of σ' with a hyperplane avoiding s, we obtain σ, as required. $\square$

Lemma 3.4. *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic as in Theorem* 3.1*. Then*

$$\dim H^4(Y)^{\iota_Y} > 1.$$

Proof. Griffiths' description of the cohomology of a hypersurface [**26,** Section 18] implies that there is an isomorphism, given by the residue map

$$H^5(\mathbb{P}^5 \setminus Y) \supset \ H_{\leq 3} \xrightarrow{\cong} H^{2,2}(Y)_{\text{prim}},$$

where $H_{\leq 3}$ is, by definition, the subspace of meromorphic forms with poles of order ≤ 3 along Y. In order to prove the lemma, it thus suffices to exhibit an $\iota_{\mathbb{P}}$-invariant meromorphic five-form with a pole of order 3. Let $f = f(X_0, \ldots, X_5)$ be an equation defining Y. The meromorphic form

$$\frac{(X_0)^3}{f^3} \sum_{j=0}^{5} X_j \, dX_0 \wedge \cdots \wedge \widehat{dX_j} \wedge \cdots \wedge dX_5 \in H_{\leq 3}$$

does the job.

Another proof of Lemma 3.4[1] is as follows: the cubic Y contains the plane $P := \{x_3 = x_4 = x_5 = 0\}$. The plane P is not proportional to the class h^2, where $h \in A^1(Y)$ is a hyperplane class. Indeed, if P were equal to mh^2 in $H^4(Y)$ for some integer m, the intersection $P \cdot h^2$ would be a multiple of 3, whereas $P \cdot h^2 = 1$. Since both P and h^2 are ι_Y-invariant, this proves the lemma. $\square$

This finishes the proof of Lemma 3.2, and hence, of Theorem 3.1. $\square$

3.2. Second part.

Theorem 3.5. *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, defined by an equation*

$$\begin{aligned}(X_0)^2\ell_0(X_3,X_4,X_5) + (X_1)^2\ell_1(X_3,X_4,X_5) + (X_2)^2\ell_2(X_3,X_4,X_5)\\ + X_0X_1\ell_3(X_3,X_4,X_5) + X_0X_2\ell_4(X_3,X_4,X_5)\\ + X_1X_2\ell_5(X_3,X_4,X_5) + g(X_3,\dots,X_5) = 0,\end{aligned}$$

where the ℓ_i are linear forms, and g is a homogeneous degree 3 polynomial.

Let $X = F(Y)$ be the Fano variety of lines in Y, and let $\iota \in \mathrm{Aut}(X)$ be the anti-symplectic involution of Lemma 2.8. Then,

$$\begin{aligned}\iota^* = -\,\mathrm{id}\colon A^i_{(2)}(X) \longrightarrow A^i_{(2)}(X) \quad \textit{for } i = 2,4;\\ \iota^* = \mathrm{id}\colon A^4_{(j)}(X) \longrightarrow A^4_{(j)}(X) \quad \textit{for } j = 0,4.\end{aligned}$$

Proof. First, note that

$$A^2_{(2)}(X) = I_* A^4_{\mathrm{hom}}(X),$$

where $I \subset X \times X$ is the incidence correspondence [**22,** Proof of Proposition 21.10]. On the other hand,

$$I = {}^tP \circ P \quad \text{in } A^2(X \times X),$$

where $X \leftarrow P \rightarrow Y$ denotes the universal family of lines on Y [**22,** Lemma 17.2]. Hence,

$$A^2_{(2)}(X) = ({}^tP)_* P_* A^4_{\mathrm{hom}}(X).$$

However, $P_*\colon A^4_{\mathrm{hom}}(X) \to A^3_{\mathrm{hom}}(Y)$ is surjective [**20**], and thus,

$$A^2_{(2)}(X) = ({}^tP)_* A^3_{\mathrm{hom}}(Y).$$

It is readily verified that the diagram

$$\begin{array}{ccc} A^3_{\mathrm{hom}}(Y) & \xrightarrow{({}^tP)_*} & A^2(X) \\ {\scriptstyle (\iota_Y)^*}\downarrow & & \downarrow{\scriptstyle \iota^*} \\ A^3_{\mathrm{hom}}(Y) & \xrightarrow{({}^tP)_*} & A^2(X) \end{array}$$

is commutative (this is since the involution extends to an involution on P). Using this diagram, Theorem 3.1 implies that ι acts as minus the identity on $({}^tP)_* A^3_{\mathrm{hom}}(Y) = A^2_{(2)}(X)$.

Since intersection product induces a surjection

$$A^2_{(2)}(X) \otimes A^2_{(2)}(X) \longrightarrow A^4_{(4)}(X)$$

(Theorem 2.7 (ii)), it follows that ι acts as the identity on $A^4_{(4)}(X)$.

Next, we want to exploit the fact that there is an isomorphism

$$\cdot\ell\colon A^2_{(2)}(X) \xrightarrow{\cong} A^4_{(2)}(X)$$

(Theorem 2.7 (i)). Since $\iota^*(\ell) = \ell$ (Proposition 3.6 below), this implies that ι acts as minus the identity on $A^4_{(2)}(X)$.

Proposition 3.6. *Let X be the variety of lines on a smooth cubic fourfold $Y \subset \mathbb{P}^5(\mathbb{C})$, and let $\iota \in \mathrm{Aut}(X)$ be an involution induced by an involution $\iota_Y \in \mathrm{Aut}(Y)$. Let $\ell \in A^2(X)$ be the class of Theorem* 2.7 (i)*. Then*:

$$\iota^*(\ell) = \ell \quad \textit{in } A^2(X).$$

Proof. We give two proofs of this fact. The first proof has the benefit of brevity; the second proof will be useful in proving another result (Lemma 3.7 below).

First proof. It is known that

$$\ell = \frac{5}{6} c_2(X) \quad \text{in } A^2(X)$$

(where the right-hand side denotes the second Chern class of the tangent bundle T_X of X) **[22,** equation (108)]. Since

$$\iota^* c_2(X) = c_2(\iota^* T_X) = c_2(X) \quad \text{in } A^2(X),$$

this proves the proposition. □

Second proof. Shen and Vial define the class $L \in A^2(X \times X)$ (lifting the Beauville-Bogomolov class $\mathfrak{B} \in H^4(X \times X)$) as

$$L := \frac{1}{3}\Big((g_1)^2 + \frac{3}{2} g_1 g_2 + (g_2)^2 - c_1 - c_2 \Big) - I \in A^2(X \times X)$$

[**22,** equation (107)]. Here, I is the incidence correspondence, and

$$\begin{aligned} g &:= -c_1(\mathcal{E}_2) \in \ A^1(X), \\ c &:= c_2(\mathcal{E}_2) \in \ A^2(X), \\ g_i &:= (p_i)^*(g) \in A^1(X \times X) \quad (i = 1,2), \\ c_i &:= (p_i)^*(c) \in \ A^2(X \times X) \quad (i = 1,2), \end{aligned}$$

where $\mathcal{E}_2$ is the rank 2 vector bundle coming from the tautological bundle on the Grassmannian, and $p_i \colon X \times X \to X$ denotes the two projections.

Clearly, we have

$$(\iota \times \iota)^*(I) = I, \qquad (\iota \times \iota)^*(c_i) = c_i, \qquad (\iota \times \iota)^*(g_i) = g_i.$$

In view of the definition of L, it follows that

$$(\iota \times \iota)^*(L) = L \quad \text{in } A^2(X \times X).$$

Using Lieberman's lemma [**24,** Lemma 3.3], plus the fact that ${}^t\Gamma_\iota = \Gamma_\iota$, this means that there is a commutativity relation

$$L \circ \Gamma_\iota = \Gamma_\iota \circ L \quad \text{in } A^2(X \times X). \tag{3.3}$$

The class ℓ is defined as $\ell := (i_\Delta)^*(L) \in A^2(X)$. We now find that

$$\begin{aligned} \iota^*(\ell) &= \iota^*(i_\Delta)^*(L) \\ &= (i_\Delta)^*(\iota \times \iota)^*(L) \\ &= (i_\Delta)^*(\Gamma_\iota \circ L \circ \Gamma_\iota) \\ &= (i_\Delta)^*(L) = \ell \quad \text{in } A^2(X). \end{aligned}$$

Here, the second equality is, by virtue of the commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{i_\Delta} & X \times X \\ \iota \downarrow & & \downarrow \iota\times\iota \\ X & \xrightarrow{i_\Delta} & X \times X. \end{array}$$

The third equality is, again, Lieberman's lemma, plus the fact that ${}^t\Gamma_\iota = \Gamma_\iota$. The last equality is (3.3). $\square$

It only remains to prove Theorem 3.5 for $(i,j)=(4,0)$. This follows from the fact that $A^4_{(0)}(X)$ is generated by ℓ^2 **[22]**, plus the fact that ℓ is ι-invariant (Proposition 3.6). Theorem 3.5 is now proven. $\square$

For later use, we remark that the argument of Proposition 3.6 also proves the following compatibility statement:

Lemma 3.7. *Let X be the variety of lines on a smooth cubic fourfold $Y\subset\mathbb{P}^5(\mathbb{C})$, and let $\iota\in\mathrm{Aut}(X)$ be an involution induced by an involution $\iota_Y\in\mathrm{Aut}(Y)$. Then,*

$$\iota^* A^i_{(j)}(X)\subset A^i_{(j)}(X)\quad \textit{for all } i,j.$$

Proof. Let $L\in A^2(X\times X)$ be the Shen-Vial class as above. We observe that equality (3.3) also implies

$$(\iota\times\iota)^*(L^r)=\big((\iota\times\iota)^*(L)\big)^r=L^r\quad\text{in } A^4(X\times X) \text{ for all } r\in\mathbb{N}.$$

Using Lieberman's lemma, this is equivalent to the commutativity relation

$$\Gamma_\iota\circ L^r=L^r\circ\Gamma_\iota\quad\text{in } A^{2r}(X\times X).$$

Since the Shen-Vial Fourier transform

$$\mathcal{F}\colon A^*(X)\longrightarrow A^*(X)$$

is defined by a polynomial in L **[22,** Section C.1], we find that

$$\mathcal{F}(\iota^*(a))=\iota^*\mathcal{F}(a)\quad\text{for all } a\in A^i(X).$$

This proves the lemma, for the decomposition of **[22]** is defined as

$$A^i_{(j)}(X):=\big\{a\in A^i(X)\mid \mathcal{F}(a)\in A^{4-i+j}(X)\big\}.\qquad\square$$

4. Corollaries. In this last section, we consider the quotient $Z:=X/\iota$, for (X,ι) as in Theorem 3.5. The variety Z is a slightly singular Calabi-Yau variety. As is well known, Chow groups with $\mathbb{Q}$-coefficients of quotient varieties such as Z still have a ring structure **[12,** Examples 8.3.12, 17.4.10]. For this reason, we will write $A^i(Z)$ for the Chow group of codimension i cycles on Z (just as in the smooth case).

Corollary 4.1. *Let* (X, ι) *be as in Theorem* 3.5*. Let* $Z := X/\iota$ *be the quotient. Then, the image of the intersection product map*

$$A^2(Z) \otimes A^2(Z) \longrightarrow A^4(Z)$$

has dimension 1*.*

Proof. We first establish a lemma:

Lemma 4.2. *Let* (X, ι) *be as in Theorem* 3.5*. Then,*

$$A^2(X)^\iota \subset A^2_{(0)}(X).$$

Proof. Let $c \in A^2(X)^\iota$, and suppose that

$$c = c_0 + c_2 \quad \text{in } A^2_{(0)}(X) \oplus A^2_{(2)}(X),$$

where $c_j \in A^2_{(j)}(X)$. Since c is ι-invariant, we also have

$$c = \iota^*(c) = \iota^*(c_0) + \iota^*(c_2) = \iota^*(c_0) - c_2 \ A^2(X)$$

(where we have used Theorem 3.5 to conclude that $\iota^*(c_2) = -c_2$. However, $\iota^*(c_0) \in A^2_{(0)}(X)$ (Lemma 3.7), and so (by unicity of the decomposition $c = c_0 + c_2$), we must have

$$\iota^*(c_0) = c_0, \quad -c_2 = c_2. \qquad \square$$

Let $p \colon X \to Z$ be the quotient morphism. Lemma 4.2 states that

$$p^* A^2(Z) \subset A^2_{(0)}(X).$$

It follows that

$$\begin{aligned} p^* \operatorname{Im}\big(A^2(Z) \otimes A^2(Z) \longrightarrow A^4(Z)\big) & \\ \subset \operatorname{Im}\big(p^* A^2(Z) \otimes p^* A^2(Z) &\longrightarrow A^4(X)\big) \\ \subset \operatorname{Im}\big(A^2_{(0)}(X) \otimes A^2_{(0)}(X) &\longrightarrow A^4(X)\big) \\ \subset A^4_{(0)}(X) \oplus A^4_{(2)}(X). & \end{aligned}$$

Here, for the last inclusion, we have used [**22,** Proposition 22.8].

On the other hand, we have

$$p^* \operatorname{Im}\big(A^2(Z) \otimes A^2(Z) \longrightarrow A^4(Z)\big) \subset A^4(X)^\iota,$$

and thus (by combining with the above inclusion), we find that

$$p^* \operatorname{Im}\big(A^2(Z) \otimes A^2(Z) \longrightarrow A^4(Z)\big) \subset \big(A^4_{(0)}(X) \oplus A^4_{(2)}(X)\big) \cap A^4(X)^\iota.$$

However, we have seen (Lemma 3.7) that ι respects the Fourier decomposition, and thus,

$$\big(A^4_{(0)}(X) \oplus A^4_{(2)}(X)\big) \cap A^4(X)^\iota = A^4_{(0)}(X)^\iota \oplus A^4_{(2)}(X)^\iota.$$

But, $A^4_{(2)}(X)^\iota = 0$ (Theorem 3.5), and so,

$$A^4(X)^\iota = A^4_{(0)}(X)^\iota.$$

Since ℓ^2 generates $A^4_{(0)}(X)$ and is ι-invariant (Proposition 3.6), we conclude that

$$A^4(X)^\iota = A^4_{(0)}(X)^\iota = A^4_{(0)})X) \cong \mathbb{Q}. \qquad \square$$

Corollary 4.3. *Let* (X, ι) *be as in Theorem* 3.5*. Let* $Z := X/\iota$ *be the quotient. Then, the image of the intersection product map*

$$\operatorname{Im}\big(A^3(Z) \otimes A^1(Z) \longrightarrow A^4(Z)\big)$$

has dimension 1*.*

Proof. As above, let $p\colon X \to Z$ denote the quotient morphism. It will suffice to show that

$$\operatorname{Im}\big(A^3(X)^\iota \otimes A^1(X)^\iota \longrightarrow A^4(X)^\iota\big)$$

is of dimension 1.

There is a decomposition

$$A^3(X)^\iota = A^3_{(0)}(X)^\iota \oplus A^3_{(2)}(X)^\iota$$

(this follows from Lemma 3.7). Moreover, it is known that

$$A^3_{(2)}(X) \cdot A^1(X) \subset A^4_{(2)}(X)$$

[**22,** Proposition 22.6]. However, we know from Theorem 3.5 that $A^4_{(2)}(X)^\iota = 0$, and so,

$$A^3_{(2)}(X)^\iota \otimes A^1(X)^\iota \longrightarrow A^4(X)^\iota$$

is the zero map. It follows that

$$\begin{aligned}\operatorname{Im}\bigl(A^3(X)^\iota \otimes A^1(X)^\iota \longrightarrow A^4(X)^\iota\bigr)\\ = \operatorname{Im}\bigl(A^3_{(0)}(X)^\iota \otimes A^1(X)^\iota \longrightarrow A^4(X)^\iota\bigr).\end{aligned}$$

In order to analyze the right-hand side, we observe that

$$A^3_{(0)}(X) = A^1(X)_{\text{prim}} \cdot A^2_{(0)}(X)$$

(the inclusion "$\supset$" is proven in [**22,** Proposition 22.7]; the inclusion "$\subset$" follows from [**22,** Remark 4.7]). It follows that

$$\begin{aligned}\operatorname{Im}\bigl(A^3_{(0)}(X)^\iota \otimes A^1(X)^\iota \longrightarrow A^4(X)^\iota\bigr)\\ &\subset \bigl(A^1(X)_{\text{prim}} \cdot A^2_{(0)}(X) \cdot A^1(X)\bigr) \cap A^4(X)^\iota\\ &\subset \bigl(A^4_{(0)}(X) \oplus A^4_{(2)}(X)\bigr) \cap A^4(X)^\iota\\ &= A^4_{(0)}(X).\end{aligned}$$

Here, the second equality is [**22,** equation (118)], and the third equality is Theorem 3.5 combined with Lemma 3.7. $\square$

We can also say something about 1-cycles on the quotient:

Corollary 4.4. *Let (X, ι) be as in Theorem 3.5. Let $Z := X/\iota$ be the quotient. Then, the image of the intersection product map*

$$\operatorname{Im}\bigl(A^2(Z) \otimes A^1(Z) \longrightarrow A^3(Z)\bigr)$$

is a finite-dimensional $\mathbb{Q}$-vector space.

Proof. As above, let $p\colon X \to Z$ denote the quotient morphism. We have seen (Lemma 4.2) that

$$p^* A^2(Z) \ \subset \ A^2_{(0)}(X),$$

and thus,

$$\begin{aligned}p^* \operatorname{Im}\bigl(A^2(Z) \otimes A^1(Z) \longrightarrow A^3(Z)\bigr)\\ \subset \operatorname{Im}\bigl(A^2_{(0)}(X) \otimes A^1(X) \longrightarrow A^3(X)\bigr).\end{aligned}$$

Let $N^2H^4(X) \subset H^4(X)$ denote the $\mathbb{Q}$-vector space of cycle classes. There is an exact sequence

$$0 \longrightarrow A^2_{(0),\mathrm{hom}}(X) \longrightarrow A^2_{(0)}(X) \longrightarrow N^2H^4(X) \longrightarrow 0.$$

Let $b_1, \ldots, b_r$ be (non-canonical) lifts of a basis of the finite-dimensional $\mathbb{Q}$-vector space $N^2H^4(X)$ to $A^2_{(0)}(X)$, and let $B \subset A^2_{(0)}(X)$ be the finite-dimensional sub-vector space spanned by the b_i.

It is known that the intersection product map

$$A^2_{(0),\mathrm{hom}}(X) \otimes A^1(X) \longrightarrow A^3(X)$$

is the zero-map [**22,** Proposition 22.4]. It follows that

$$\mathrm{Im}\big(A^2_{(0)}(X) \otimes A^1(X) \longrightarrow A^3(X)\big) = \mathrm{Im}\big(B \otimes A^1(X) \longrightarrow A^3(X)\big),$$

which is finite-dimensional. Since p^* is injective, this proves the corollary. $\square$

Remark 4.5. Let X and Z be as in Corollary 4.4. The statement obtained in Corollary 4.4 is presumably less than optimal, in the following sense. It should be the case that the cycle class map induces an injection

$$\mathrm{Im}\big(A^2(Z) \otimes A^1(Z) \longrightarrow A^3(Z)\big) \overset{??}{\hookrightarrow} H^6(Z). \tag{4.1}$$

Indeed, as we have seen, $p^*A^2(Z) \subset A^2_{(0)}(X)$. Now, *if* we knew that

$$\mathrm{Im}\big(A^2_{(0)}(X) \otimes A^1(X) \longrightarrow A^3(X)\big) \subset A^3_{(0)}(X), \tag{4.2}$$

then one could conclude that (4.1) is true.

Unfortunately, the inclusion (4.2) is known only for Fano varieties of *very general* cubic fourfolds. The problem is thus the absence of an MCK decomposition for Fano varieties of arbitrary smooth cubic fourfolds.

Acknowledgments. Thanks to the referee for helpful comments. Thanks to Kai and Len and Yoyo for daily moral support.

ENDNOTES

1. This was suggested by the referee.

REFERENCES

1. A. Beauville, *Some remarks on Kähler manifolds with* $c_1 = 0$, in: *Classification of algebraic and analytic manifolds*, Birkhäuser, Boston, 1983.

2. ———, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differ. Geom. **18** (1983), 755–782.

3. ———, *On the splitting of the Bloch-Beilinson filtration*, in: *Algebraic cycles and motives*, J. Nagel and C. Peters, eds., Cambridge University Press, Cambridge, 2007.

4. A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C.R. Acad. Sci. Paris **301** (1985), 703–706.

5. C. Camere, *Symplectic involutions of holomorphic symplectic fourfolds*, Bull. Lond. Math. Soc. **44** (2012), 687–702.

6. C. Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. France **103** (1975), 203–223.

7. L. Fu, *Decomposition of small diagonals and Chow rings of hypersurfaces and Calabi-Yau complete intersections*, Adv. Math. **244** (2013), 894–924.

8. ———, *On the action of symplectic automorphisms on the* CH_0*-groups of some hyper-Kähler fourfolds*, Math. Z. **280** (2015), 307–334.

9. ———, *Classification of polarized symplectic automorphisms of Fano varieties of cubic fourfolds*, Glasgow Math. J. **58** (2016), 17–37.

10. L. Fu, R. Laterveer and C. Vial, *The generalized Franchetta conjecture for some hyper–Kähler varieties*, `arXiv:1708.02919`.

11. L. Fu, Z. Tian and C. Vial, *Motivic hyperkähler resolution conjecture for generalized Kummer varieties*, `arXiv:1608.04968`.

12. W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

13. U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, in *Motives*, U. Jannsen, et al., eds., Proc. Symp. Pure Math. **55** (1994).

14. R. Laterveer, *Algebraic cycles on a very special* EPW *sextic*, Rend. Sem. Mat. Univ. Padova, to appear.

15. ———, *About Chow groups of certain hyperkähler varieties with non-symplectic automorphisms*, Vietnam J. Math. **46** (2018), 453–470.

16. ———, *On the Chow groups of some hyperkähler fourfolds with a non-symplectic involution*, Inter. J. Math. **28** (2017), 1–18.

17. ———, *On the Chow groups of certain* EPW *sextics*, submitted.

18. J. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety*, Parts I and II, Indag. Math. **4** (1993), 177–201.

19. J. Murre, J. Nagel and C. Peters, *Lectures on the theory of pure motives*, Amer. Math. Soc. Univ. Lect. Ser. **61**, Providence, 2013.

20. K. Paranjape, *Cohomological and cycle-theoretic connectivity*, Ann. Math. **139** (1994), 641–660.

21. T. Scholl, *Classical motives*, in *Motives*, U. Jannsen, et al., eds., Proc. Symp. Pure Math. **55** (1994).

22. M. Shen and C. Vial, *The Fourier transform for certain hyperKähler fourfolds*, Mem. Amer. Math. Soc. **240** (2016).

23. ______, *The motive of the Hilbert cube $X^{[3]}$*, Forum Math. Sigma **4** (2016).

24. C. Vial, *Remarks on motives of abelian type*, Tohoku Math. J. **69** (2017), 195–220.

25. ______, *On the motive of some hyperkähler varieties*, J. reine angew. Math. **725** (2017), 235–247.

26. C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spec. Soc. Math. France, Paris, 2002.

27. ______, *Chow rings and decomposition theorems for $K3$ surfaces and Calabi-Yau hypersurfaces*, Geom. Topol. **16** (2012), 433–473.

28. ______, *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections*, Ann. Sci. Ecole Norm. Sup. **46** (2013), 449–475.

29. ______, *Bloch's conjecture for Catanese and Barlow surfaces*, J. Differ. Geom. **97** (2014), 149–175.

30. ______, *Chow rings, Decomposition of the diagonal, and the topology of families*, Princeton University Press, Princeton, 2014.

31. ______, *The generalized Hodge and Bloch conjectures are equivalent for general complete intersections*, II, J. Math. Sci. Univ. Tokyo **22** (2015), 491–517.

Institut de Recherche Mathématique Avancée, CNRS–Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg CEDEX, France
Email address: **robert.laterveer@math.unistra.fr**