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UNIQUENESS OF POSITIVE SOLUTIONS FOR A CLASS OF SCHRÖDINGER SYSTEMS WITH SATURABLE NONLINEARITY

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ABSTRACT. This paper is devoted to the study of the nonexistence and the uniqueness of positive solutions for a class of the following nonlinear coupled Schrödinger systems with saturable nonlinearity

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \dfrac{u_1(\mu_1 u_1^2 + \beta u_2^2)}{1 + s(\mu_1 u_1^2 + \beta u_2^2)} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \lambda_2 u_2 = \dfrac{u_2(\mu_2 u_2^2 + \beta u_1^2)}{1 + s(\mu_2 u_2^2 + \beta u_1^2)} & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N),\ u_1 > 0,\ u_2 > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $\lambda_j, \mu_j, j = 1,2$, are positive constants, s is a positive parameter and β is a positive coupling parameter. Moreover, we will show that any positive solution is a priori bounded.

1. Introduction and main results. Much attention has been focused on nonlinear optics which provide good knowledge of the transmission of light at high velocity. The propagation of a beam with two mutually incoherent components in a bulk saturable medium in the isotropic approximation can be described by the following, time dependent two-component coupled nonlinear Schrödinger system with saturable nonlinearity

$$(1.1)\quad \begin{cases} -i\frac{\partial}{\partial t}\Phi = \frac{\alpha(|\Phi|^2+\Psi|^2)}{1+(|\Phi|^2+\Psi|^2)/I_0}\Phi & \text{for } t>0,\ x\in\mathbb{R}^N, \\ -i\frac{\partial}{\partial t}\Psi = \frac{\alpha(|\Phi|^2+\Psi|^2)}{1+(|\Phi|^2+\Psi|^2)/I_0}\Psi & \text{for } t>0,\ x\in\mathbb{R}^N, \\ \Phi = \Phi(t,x)\in\mathbb{C}, \Psi = \Psi(t,x)\in\mathbb{C} & \text{for } t>0,\ x\in\mathbb{R}^N, \\ \Phi(t,x)\longrightarrow 0,\ \Psi(t,x)\longrightarrow 0 & \text{as } |x|\to+\infty,\ t>0, \end{cases}$$

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where i is the imaginary unit. Physically, the solutions Φ and Ψ denote the amplitude of the first and second components of the beam in photorefractive crystals, see **[14, 16]**, α is the strength of the nonlinearity, I_0 is the saturation parameter and $(|\Phi|^2+|\Psi|^2)$ is the total intensity created by all incoherent components of the light beam. In order to obtain solitary wave solutions of the forms $\Phi = \sqrt{\alpha}e^{i\lambda_1 t}u_1(x)$, $\Psi = \sqrt{\alpha}e^{i\lambda_2 t}u_2(x)$ with u_1 and u_2 real-valued functions and λ_1, λ_2 the propagation constants associated to the mode profile, system (1.1) is transformed into the following system of two weakly coupled elliptic equations:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \dfrac{u_1(u_1^2+u_2^2)}{1+s(u_1^2+u_2^2)} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \lambda_2 u_2 = \dfrac{u_2(u_1^2+u_2^2)}{1+s(u_1^2+u_2^2)} & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $s = \alpha/I_0$. If I_0 tends to ∞ in system (1.1), i.e., s tends to 0 in system (1.2), system (1.1) reduces to the Manakov model, where the solutions Φ and Ψ denote the first and second components of the beam in Kerr-like photorefractive media, see **[1, 4, 17]**, which also appears in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states **[9, 19]**. System (1.2) with $s = 0$, i.e.,

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = u_1(u_1^2+u_2^2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \lambda_2 u_2 = u_2(u_1^2+u_2^2) & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N) \end{cases}$$

has been extensively studied, see, e.g., **[3, 5, 6, 12, 20, 24]** and the references therein. Recently, more physicists and mathematicians have become interested in the model (1.1), i.e., $s \neq 0$ in system (1.2).

As far as is known, the only results concerning system (1.2) are those in **[2, 7, 14, 15]**. de Almeida Maia, et al. **[7]** stated that the vector solutions $U = (u_1, u_2)$ with different L^2 weights on the components can be described by the general problem:

$$
\begin{cases}
-\Delta u_1 + \lambda_1 u_1 = \dfrac{u_1(\mu_1 u_1^2 + b u_2^2)}{1 + s(c u_1^2 + d u_2^2)} & \text{in } \mathbb{R}^N, \\
-\Delta u_2 + \lambda_2 u_2 = \dfrac{u_2(b u_1^2 + \mu_2 u_2^2)}{1 + s(c u_1^2 + d u_2^2)} & \text{in } \mathbb{R}^N, \\
u_1, u_2 \in H^1(\mathbb{R}^N),
\end{cases}
\tag{1.3}
$$

where s is a positive parameter, and b, c, d, λ_j, μ_j, $j = 1, 2$, are positive constants. In order to use the variational method to study the problem, the authors **[7]** were led to study the following, two-component coupled nonlinear Schrödinger system:

$$
\begin{cases}
-\Delta u_1 + \lambda_1 u_1 = \dfrac{\alpha u_1(\alpha u_1^2 + \beta u_2^2)}{1 + s(\alpha u_1^2 + \beta u_2^2)} & \text{in } \mathbb{R}^N, \\
-\Delta u_2 + \lambda_2 u_2 = \dfrac{\beta u_2(\alpha u_1^2 + \beta u_2^2)}{1 + s(\alpha u_1^2 + \beta u_2^2)} & \text{in } \mathbb{R}^N, \\
u_1, u_2 \in H^1(\mathbb{R}^N),
\end{cases}
\tag{1.4}
$$

which is variational, where s is a positive parameter, and $\lambda_1, \lambda_2, \alpha, \beta$ are positive constants, $N \geq 2$. The authors of **[7]** first considered the special case with $\alpha = \beta = 1$ and $\lambda_1 = \lambda_2 := \lambda$, then proved that, if $s \geq 1/\lambda$, then system (1.4) has no nontrivial solutions, and, if $s \in (0, 1/\lambda)$, then the totality of ground state solutions of system (1.4) can be given in a somewhat explicit manner (see **[7,** Theorem 2.1]). For the case where $\alpha > \beta$ and $\lambda_1 > \lambda_2$, they proved that system (1.4) has a semi-trivial ground state solution if $s \in ((\alpha - \beta)/(\lambda_1 - \lambda_2), \max\{\alpha/\lambda_1, \beta/\lambda_2\})$, and no nontrivial solution if $s > \max\{\alpha/\lambda_1, \beta/\lambda_2\}$. However, if $s \in (0, (\alpha-\beta)/(\lambda_1-\lambda_2))$, they conjectured only that system (1.4) should have semi-trivial ground state solutions. This conjecture was later settled by Mandel **[15]**. Moreover, the author of **[15]** also applied bifurcation theory to find positive and seminodal solutions for system (1.4). The authors of **[14]** considered L^2-normalized solutions for system (1.4) under the constraint condition

$$
\int_{\mathbb{R}^2} (u_1^2 + u_2^2)\, dx = 1
$$

in $\mathbb{R}^2$ with $\alpha = \beta$, $\lambda_1 = \lambda_2$ and $s = 1$, where λ_1 and λ_2 correspond to the corresponding Lagrange multipliers. Recently, the authors of **[2]** considered the prescribed L^2-norm solutions for system (1.4) with

$$\int_{\mathbb{R}^2} u_i^2\,dx = C_i > 0, \quad i = 1,2,$$

in $\mathbb{R}^2$, where $\alpha \neq \beta$ and λ_1, λ_2 correspond to the corresponding Lagrange multipliers.

Motivated by the above work, we study the nonexistence and the uniqueness of positive solutions for the following form of the Schrödinger system with saturable nonlinearity

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \dfrac{u_1(\mu_1 u_1^2 + \beta u_2^2)}{1 + s(\mu_1 u_1^2 + \beta u_2^2)} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \lambda_2 u_2 = \dfrac{u_2(\mu_2 u_2^2 + \beta u_1^2)}{1 + s(\mu_2 u_2^2 + \beta u_1^2)} & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N),\ u_1 > 0,\ u_2 > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where λ_j, μ_j, $j = 1,2$, are positive constants, s is a positive parameter and β is a positive coupling parameter. Physically, the positive constant μ_j exists for self-focusing in the jth component of the beam. The coupling constant β refers to the interaction between the two components of the beam. This interaction is attractive if $\beta > 0$ and repulsive if $\beta < 0$.

In the present paper, we address another situation concerning system (1.3) which is substantially different from systems (1.2) and (1.4). Obviously, system (1.5) is another case of the general problem (1.3), which may have no variational structure. We prove the nonexistence and the uniqueness of positive solutions with different ranges of parameters β and s. By the strong maximum principle [**11,** Theorem 3.5] and the form of system (1.5), we can see that any nontrivial, nonnegative solution (u_1, u_2) of system (1.5) must be positive. Moreover, we can verify that

$$\frac{\partial}{\partial u_2} \frac{u_1(\mu_1 u_1^2 + \beta u_2^2)}{1 + s(\mu_1 u_1^2 + \beta u_2^2)} = \frac{2\beta u_1 u_2}{(1 + s(\mu_1 u_1^2 + \beta u_2^2))^2} \geq 0$$

and

$$\frac{\partial}{\partial u_1} \frac{u_2(\mu_2 u_2^2 + \beta u_1^2)}{1 + s(\mu_2 u_2^2 + \beta u_1^2)} = \frac{2\beta u_1 u_2}{(1 + s(\mu_2 u_2^2 + \beta u_1^2))^2} \geq 0.$$

Then, from [**8,** Theorem 2.1], [**23,** Theorem 1], we can see that the positive solutions of system (1.5) are radial and decreasing.

When $\beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, it is not difficult to verify that system (1.5) with $\lambda_1 = \lambda_2 = \lambda$ and $0 < s < 1/\lambda$ admits a positive solution of the form

$$u_0 = \left(\frac{\beta - \mu_2}{\beta^2 - \mu_1\mu_2}\right)^{1/2} w, \qquad v_0 = \left(\frac{\beta - \mu_1}{\beta^2 - \mu_1\mu_2}\right)^{1/2} w, \tag{1.6}$$

where w is the uniqueness (see [**22**] for existence and [**13, 18**] for uniqueness) of radial positive solution

$$\begin{cases} -\Delta w + \lambda w = \dfrac{w^3}{1 + sw^2} & \text{in } \mathbb{R}^N, \\ w(0) = \max_{x \in \mathbb{R}^N} w(x), w(x) \longrightarrow 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.7}$$

with $0 < s < 1/\lambda$. In general, in the following, we partially generalize the previous results of nonexistence and uniqueness of the positive solutions for $s = 0$ [**3, 12, 20, 24**] to the case of $s \neq 0$.

Theorem 1.1. *Assume that* $\lambda_1 = \lambda_2 =: \lambda$, $\mu_1 \neq \mu_2$, $s > 0$ *and* $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$. *Then, system* (1.5) *does not admit any positive solution for* $N \geq 1$.

Theorem 1.2. *Suppose that* $\lambda_1 = \lambda_2 =: \lambda$.

(i) *If* $\beta > \max\{\mu_1, \mu_2\}$, $0 < s < 1/\lambda$ *and* $N \geq 1$, *then* (u_0, v_0), *given by* (1.6), *is the unique, up-to-translation positive solution to system* (1.5).

(ii) *If* $0 < \beta < \min\{\mu_1, \mu_2\}$, *then there exists a* $\delta > 0$ *such that, for* $0 < s < \min\{1/\lambda, \delta\}$ *and* $N = 1$, *system* (1.5) *has a unique, up-to-translation positive solution* (u_0, v_0), *given by* (1.6).

Remark 1.3 ([**7,** Theorem 2.1])**.** In system (1.5), if $\lambda_1 = \lambda_2 =: \lambda$, $\mu_1 = \mu_2 = \beta = 1$ and $N \geq 2$, then, for $0 < s < 1/\lambda$, all positive solutions of system (1.5) have the following form:

$$(u_1(x), u_2(x)) = (w \cos\theta, w \sin\theta), \quad \theta \in (0, \pi/2),$$

where w is given in (1.7); for $s \geq 1/\lambda$, by the Pohozaev identity, there are no nontrivial solutions of system (1.5).

The remainder of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we obtain an a priori estimate, which plays a crucial role in the proof of Theorem 1.2. The proof of Theorem 1.2 is given in Section 4.

2. Proof of Theorem 1.1. We first provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Multiplying the equation for u_1 in system (1.5) by u_2, and then integrating over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N}(\nabla u_1\nabla u_2+\lambda u_1u_2)=\int_{\mathbb{R}^N}\frac{u_1u_2(\mu_1u_1^2+\beta u_2^2)}{1+s(\mu_1u_1^2+\beta u_2^2)}. \tag{2.1}$$

Similarly, multiplying the equation for u_2 in system (1.5) by u_1, and then integrating over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N}(\nabla u_1\nabla u_2+\lambda u_1u_2)=\int_{\mathbb{R}^N}\frac{u_1u_2(\mu_2u_2^2+\beta u_1^2)}{1+s(\mu_2u_2^2+\beta u_1^2)}. \tag{2.2}$$

Subtracting (2.1) by (2.2) gives

$$\int_{\mathbb{R}^N}u_1u_2\frac{(\mu_1-\beta)u_1^2+(\beta-\mu_2)u_2^2}{(1+s(\mu_1u_1^2+\beta u_2^2))(1+s(\mu_2u_2^2+\beta u_1^2))}=0, \tag{2.3}$$

from which we obtain the result of Theorem 1.1. □

3. A priori bounds. In order to avoid technicalities, and without loss of generality, we assume that $\lambda_1=\lambda_2:=\lambda=1$. As pointed out in the introduction, any nontrivial positive solution of (1.5) is radially symmetric. Then, any positive solution (u_1,u_2) of system (1.5) satisfies the following system with $N\geq 1$:

(3.1)
$$\begin{cases}-(r^{N-1}u_1')'+r^{N-1}u_1=r^{N-1}\dfrac{u_1(\mu_1u_1^2+\beta u_2^2)}{1+s(\mu_1u_1^2+\beta u_2^2)} & \text{in } (0,+\infty),\\ -(r^{N-1}u_2')'+r^{N-1}u_2=r^{N-1}\dfrac{u_2(\mu_2u_2^2+\beta u_1^2)}{1+s(\mu_2u_2^2+\beta u_1^2)} & \text{in } (0,+\infty),\\ u_1(r),u_2(r)>0 & \text{in } (0,+\infty),\\ u_1'(0)=u_2'(0)=0 \quad\text{and}\quad u_1(r),u_2(r)\longrightarrow 0 & \text{as } r\to+\infty.\end{cases}$$

The next lemma is useful in deriving the rate of decay of u_1,u_2.

Lemma 3.1. *There exists an* $M>0$ *such that* $u_1, u_2 \le M r^{(1-N)/2} e^{-r/2}$ *for* r *sufficiently large.*

Proof. Here, we only prove the exponential decay of u_1. Then, the exponential decay of u_2 can be similarly proven. Let

$$q(r) = 1 - \frac{\mu_1 u_1^2 + \beta u_2^2}{1 + s(\mu_1 u_1^2 + \beta u_2^2)} \quad \text{and} \quad \widetilde{u}_1 = r^{(N-1)/2} u_1.$$

By using simple calculations, we can show that $\widetilde{u}_1$ satisfies the following equation:

$$\widetilde{u}_1'' = (q(r) + \frac{(N-1)(N-3)}{4r^2})\widetilde{u}_1.$$

It follows that

$$\left(\frac{1}{2}\widetilde{u}_1^2\right)'' = (\widetilde{u}_1')^2 + (q(r) + p(r))\widetilde{u}_1^2,$$

where

$$p(r) = \frac{(N-1)(N-3)}{4r^2}.$$

Since $u_1(r), u_2(r), p(r) \to 0$ as $r \to +\infty$, we have that $q(r)+p(r) \ge 1/2$ for r large enough. Let $\psi = \widetilde{u}_1^2$. Then, by a similar argument to that of [**21,** Theorem 3], we can show that $\psi = O(e^{-r})$. Therefore, there exists an $M > 0$ such that $u_1 \le M r^{(1-N)/2} e^{-r/2}$. □

Next, we prove that any radial positive solution of system (1.5) is L^∞-bounded.

Lemma 3.2. *Given* $0 < \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, *there exists a constant* $C = C(\beta)$ *such that, for any radial positive solution* (u_1, u_2) *of system* (1.5), *we have that*

$$\|u_1\|_{L^\infty(\mathbb{R}^N)}, \qquad \|u_2\|_{L^\infty(\mathbb{R}^N)} \le C.$$

Proof. We proceed by contradiction, assuming that there is a sequence of solutions (u_n, v_n) to system (3.1) with

$$\max_{r\in(0,+\infty)} u_n(r) + \max_{r\in(0,+\infty)} v_n(r) \longrightarrow +\infty \quad \text{as } n \to +\infty.$$

We follow a blow-up procedure introduced by Gidas and Spruck [**10**] for scalar equations which has already been generalized to a class of nonlinear Schrödinger systems [**5, 6**]. Without loss of generality, we may assume that

$$M_n := u_n(0) = \max_{r\in(0,+\infty)} u_n(r) \geq v_n(0) = \max_{r\in(0,+\infty)} v_n(r).$$

Set $r = \overline{r}/M_n$, and define U_n, V_n by

$$U_n(\overline{r}) = \frac{u_n(\overline{r}/M_n)}{M_n}, \qquad V_n(\overline{r}) = \frac{v_n(\overline{r}/M_n)}{M_n}.$$

Then,

$$\max_{\overline{r}\in(0,+\infty)} V_n(\overline{r}) \leq \max_{\overline{r}\in(0,+\infty)} U_n(\overline{r}) = 1,$$

and (U_n, V_n) solves the rescaled problem

$$\begin{cases} -(r^{N-1}U_n')' + r^{N-1}\dfrac{U_n}{M_n^2} = r^{N-1}\dfrac{U_n(\mu_1 U_n^2 + \beta V_n^2)}{M_n^2(M_n^{-2} + s(\mu_1 U_n^2 + \beta V_n^2))}, \\ -(r^{N-1}V_n')' + r^{N-1}\dfrac{V_n}{M_n^2} = r^{N-1}\dfrac{V_n(\mu_2 V_n^2 + \beta U_n^2)}{M_n^2(M_n^{-2} + s(\mu_2 V_n^2 + \beta U_n^2))}. \end{cases}$$

Passing to a subsequence, if necessary, we see that $(U_n, V_n) \to (U_0, V_0)$ locally uniformly as $n \to +\infty$, and (U_0, V_0) is a nontrivial and nonnegative bounded radial solution of

$$-(r^{N-1}u_1')' = 0, \qquad -(r^{N-1}u_2')' = 0.$$

Integrating over $(0, r)$, we have $U_0'(r) = 0$; thus, $U_0(r)$ is a constant. By $U_n(\overline{r}) \to 0$ as $\overline{r} \to +\infty$ and $U_n(\overline{r}) \to U_0(\overline{r})$ as $n \to +\infty$, we can see that $U_0(0) = 0$, which contradicts $U_0(0) = 1$. □

4. Proof of Theorem 1.2. Now we are in a position to prove Theorem 1.2 by virtue of Lemmas 3.1 and 3.2.

Proof of Theorem 1.2. Let $(u_1, \widetilde{u}_2)$ be a positive solution of system (1.5) with $\lambda_1 = \lambda_2 := \lambda$ and $s < 1/\lambda$. By the assumptions of Theorem 1.2, we can define $\gamma = ((\beta - \mu_1)/(\beta - \mu_2))^{1/2}$ and $u_2 = \widetilde{u}_2/\gamma$. In order to prove the uniqueness of the positive solution of system (1.5), it is sufficient to prove that $u_2 = u_1$ for all $r \geq 0$ for the uniqueness result of the single scalar equation (1.7). Then, u_1, u_2 satisfies the following system:

(4.1)
$$\begin{cases} -(r^{N-1}u_1')' + \lambda r^{N-1}u_1 = r^{N-1}\dfrac{u_1(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)}{1 + s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)}, \\ -(r^{N-1}u_2')' + \lambda r^{N-1}u_2 = r^{N-1}\dfrac{u_2(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}{1 + s(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}, \\ u_1(r), u_2(r) > 0 & \text{in } (0, +\infty), \\ u_1'(0) = u_2'(0) = 0 \quad \text{and} \quad u_1(r), u_2(r) \longrightarrow 0 & \text{as } r \to +\infty. \end{cases}$$

Multiplying the equation for u_1 in system (4.1) by u_2, we can deduce that

(4.2)
$$-(r^{N-1}u_1'u_2)' + r^{N-1}u_1'u_2' + \lambda r^{N-1}u_1u_2 = r^{N-1}\frac{u_1u_2(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)}{1 + s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)}.$$

Similarly, multiplying the equation for u_2 in system (4.1) by u_1, we can deduce that

(4.3)
$$-(r^{N-1}u_1u_2')' + r^{N-1}u_1'u_2' + \lambda r^{N-1}u_1u_2 = r^{N-1}\frac{u_1u_2(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}{1 + s(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}.$$

Subtracting (4.2) by (4.3) gives

$$\begin{aligned} &-(r^{N-1}(u_1'u_2 - u_1u_2'))' \\ &\quad = r^{N-1}u_1u_2\left(\frac{\mu_1 u_1^2 + \beta\gamma^2 u_2^2}{1+s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)} - \frac{\mu_2\gamma^2 u_2^2 + \beta u_1^2}{1+s(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}\right). \end{aligned} \tag{4.4}$$

Integrating (4.4) over $(0, +\infty)$, and, by the definition of γ, we have

$$\begin{aligned} &-r^{N-1}(u_1'u_2 - u_1u_2')|_0^{+\infty} \\ &= (\mu_1 - \beta)\int_0^{+\infty} r^{N-1}\frac{u_1u_2(u_1^2 - u_2^2)}{(1+s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2))(1+s(\mu_2\gamma^2 u_2^2 + \beta u_1^2))}. \end{aligned} \tag{4.5}$$

Integrating the equation for u_1 in system (3.1) over $(0, r)$, we have

$$r^{N-1}u_1'(r) = \int_0^r t^{N-1}u_1(t)q(t)\,dt,$$

where $q(t)$ is defined in the proof of Lemma 3.1. Then, from Lemma 3.1, we can deduce that

$$r^{N-1}u_1'u_2 \longrightarrow 0 \quad \text{as } r \to +\infty. \tag{4.6}$$

Similarly, $r^{N-1}u_1u_2' \to 0$ as $r \to +\infty$. Combining with Lemma 3.2 and $u_1'(0) = u_2'(0) = 0$, we can deduce that the left-hand side of (4.5) is equal to zero. Then,

$$(\mu_1 - \beta)\int_0^{+\infty} r^{N-1}\frac{u_1u_2(u_1^2-u_2^2)}{(1+s(\mu_1u_1^2+\beta\gamma^2u_2^2))(1+s(\mu_2\gamma^2u_2^2+\beta u_1^2))} = 0. \tag{4.7}$$

Since $\beta \neq \mu_1$, it is sufficient to prove that $u_1 \geq u_2$ or $u_1 \leq u_2$.

In the following, we prove that $u_1(r) \geq u_2(r)$ or $u_1(r) \leq u_2(r)$ for all r. Suppose, to the contrary, that $(u_1 - u_2)$ changes sign. Subtracting the first equation of (4.1) by the second, we can see that $(u_1 - u_2)$ satisfies

$$-(u_1-u_2)'' - \frac{N-1}{r}(u_1'-u_2') + \lambda(u_1-u_2) = \varphi(r)(u_1-u_2) \quad \text{in } [0,+\infty), \tag{4.8}$$

where

$$\varphi(r) := \frac{\mu_1u_1^2+(\mu_1-\beta)u_1u_2+\mu_2\gamma^2u_2^2+s(\mu_1u_1^2+\beta\gamma^2u_2^2)(\mu_2\gamma^2u_2^2+\beta u_1^2)}{(1+s(\mu_1u_1^2+\beta\gamma^2u_2^2))(1+s(\mu_2\gamma^2u_2^2+\beta u_1^2))},$$

and $u_1(r) - u_2(r) \to 0$ as $r \to +\infty$. Then, by the maximum principle, $u_1(r) - u_2(r)$ changes sign only in finite time. Without loss of generality, we can assume that $u_1(r) > u_2(r)$ for r large enough. Thus, there exists an $\widetilde{r} > 0$ such that

$$u_1(\widetilde{r}) - u_2(\widetilde{r}) = 0 \quad \text{and} \quad u_1(r) - u_2(r) > 0 \quad \text{for } r > \widetilde{r}, \tag{4.9}$$

which implies that

$$u_1'(\widetilde{r}) - u_2'(\widetilde{r}) \geq 0. \tag{4.10}$$

Integrating (4.4) over $(\widetilde{r}, +\infty)$, then using the definition of γ and (4.6), we have

$$\begin{aligned}&\widetilde{r}^{N-1}(u_1'u_2 - u_1u_2')(\widetilde{r}) \\ &\quad = (\mu_1-\beta)\int_{\widetilde{r}}^{+\infty} r^{N-1}\frac{u_1u_2(u_1^2-u_2^2)}{(1+s(\mu_1u_1^2+\beta\gamma^2u_2^2))(1+s(\mu_2\gamma^2u_2^2+\beta u_1^2))},\end{aligned} \tag{4.11}$$

which yields

$$u_1'(\widetilde{r}) - u_2'(\widetilde{r}) > 0. \tag{4.12}$$

Otherwise, if $u_1'(\widetilde{r}) - u_2'(\widetilde{r}) = 0$, from (4.9) and (4.11), we can obtain a contradiction for $\beta \neq \mu_1$.

If $\beta > \max\{\mu_1, \mu_2\}$, then, from (4.9) and (4.12), the value of the right-hand side of (4.11) is less than zero, and the value of the left-hand side of (4.11) is larger than zero. This is a contradiction.

If $0 < \beta < \min\{\mu_1, \mu_2\}$, when $N = 1$, we divide the situation into two cases:

Case 1. For all $r > \widetilde{r}$,

$$(u_1' u_2 - u_1 u_2')(r) > 0. \tag{4.13}$$

This is due to the fact that $(u_1' u_2 - u_1 u_2')$ is continuous in $[0, +\infty)$ and, from (4.9) and (4.12), $(u_1' u_2 - u_1 u_2')(\widetilde{r}) > 0$. Multiplying the equation for u_1 in (4.1) with $N = 1$ by u_1', we obtain

$$\begin{aligned} &-\frac{1}{2}((u_1')^2)' + \frac{1}{2}\lambda(u_1^2)' \\ &\quad= \frac{(1/4)\mu_1(u_1^4)' + (1/4)\beta\gamma^2(u_1^2 u_2^2)' + (1/2)\beta\gamma^2 u_1 u_2 (u_1' u_2 - u_1 u_2')}{1 + s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)}. \end{aligned} \tag{4.14}$$

Multiplying the equation for u_2 in (4.1) with $N = 1$ by u_2', we obtain

$$\begin{aligned} &-\frac{1}{2}((u_2')^2)' + \frac{1}{2}\lambda(u_2^2)' \\ &\quad= \frac{(1/4)\mu_2\gamma^2(u_2^4)' + (1/4)\beta(u_1^2 u_2^2)' + (1/2)\beta u_1 u_2 (u_1 u_2' - u_1' u_2)}{1 + s(\mu_2\gamma^2 u_2^2 + \beta u_1^2)}. \end{aligned} \tag{4.15}$$

Subtracting (4.14) by (4.15) and integrating over $(\widetilde{r}, +\infty)$, we can obtain that

$$\begin{aligned} &\frac{1}{2}(-(u_1')^2 + (u_2')^2 + \lambda u_1^2 - \lambda u_2^2)|_{\widetilde{r}}^{+\infty} \\ &= \int_{\widetilde{r}}^{+\infty} \Big(\frac{(1/4)\mu_1(u_1^4)' + (1/4)\beta\gamma^2(u_1^2 u_2^2)' + (1/2)\beta\gamma^2 u_1 u_2 (u_1' u_2 - u_1 u_2')}{1 + s(\mu_1 u_1^2 + \beta\gamma^2 u_2^2)} \\ &- \frac{(1/4)\mu_2\gamma^2(u_2^4)' + (1/4)\beta(u_1^2 u_2^2)' - (1/2)\beta u_1 u_2 (u_1' u_2 - u_1 u_2')}{1 + s(\mu_2\gamma^2 u_2^2 + \beta u_1^2)} \Big) := F(s). \end{aligned} \tag{4.16}$$

Obviously, $F(s)$ is continuous with respect to $s > 0$. Moreover, since $s \to 0$,

$$
\begin{aligned}
F(s) \longrightarrow & \int_{\widetilde{r}}^{+\infty} \bigg(\frac{1}{4}\mu_1(u_1^4)' + \frac{1}{4}\beta\gamma^2(u_1^2u_2^2)' + \frac{1}{2}\beta\gamma^2 u_1u_2(u_1'u_2 - u_1u_2') \\
& \qquad - \Big(\frac{1}{4}\mu_2\gamma^2(u_2^4)' + \frac{1}{4}\beta(u_1^2u_2^2)' - \frac{1}{2}\beta u_1u_2(u_1'u_2 - u_1u_2')\Big)\bigg) \\
= & \frac{1}{4}(\mu_1u_1^4 + \beta\gamma^2u_1^2u_2^2 - \mu_2\gamma^2u_2^4 - \beta u_1^2u_2^2)|_{\widetilde{r}}^{+\infty} \\
& + \frac{1}{2}\beta\int_{\widetilde{r}}^{+\infty}(\gamma^2u_1u_2(u_1'u_2 - u_1u_2') + u_1u_2(u_1'u_2 - u_1u_2')) \\
= & \frac{1}{2}\beta\int_{\widetilde{r}}^{+\infty}(\gamma^2u_1u_2(u_1'u_2 - u_1u_2') + u_1u_2(u_1'u_2 - u_1u_2')) \geq 0,
\end{aligned}
$$

where the last equality follows from $u_1(\widetilde{r}) = u_2(\widetilde{r})$, $u_1(+\infty) = u_2(+\infty) = 0$ and the definition of γ, and the last inequality follows from (4.13). Thus, there exists a $\delta > 0$ such that the value of the right-hand side of (4.16) is not less than zero for $0 < s < \delta$. Since $u_1(\widetilde{r}) = u_2(\widetilde{r})$, $u_1(+\infty) = u_2(+\infty) = 0$ and $u_1'(+\infty) = u_2'(+\infty) = 0$, the value of the left-hand side of (4.16) is equal to $((u_1')^2 - (u_2')^2)(\widetilde{r})/2$. Then, by $0 > u_1'(\widetilde{r}) > u_2'(\widetilde{r})$, the value of the left-hand side of (4.16) is less than zero. This is a contradiction.

Case 2. There exists an $\overline{r} > \widetilde{r}$ such that

$$(u_1'u_2 - u_1u_2')(\overline{r}) = 0. \tag{4.17}$$

Integrating (4.4) with $N = 1$ over $(\overline{r}, +\infty)$, we obtain

$$
\begin{aligned}
0 &= (u_1'u_2 - u_1u_2')(\overline{r}) \\
&= (\mu_1 - \beta)\int_{\overline{r}}^{\infty} r^{N-1}\frac{u_1u_2(u_1^2 - u_2^2)}{(1+s(\mu_1u_1^2+\beta\gamma^2u_2^2))(1+s(\mu_2\gamma^2u_2^2+\beta u_1^2))} \\
&> 0,
\end{aligned}
$$

where the first equality follows from (4.6) with $N = 1$ and (4.17), the last inequality follows from (4.9) and $0 < \beta < \min\{\mu_1, \mu_2\}$. This is a contradiction.

Therefore, the proof of Theorem 1.2 is complete. □

Remark 4.1. If $0 < \beta < \min\{\mu_1, \mu_2\}$, in this paper we only prove the uniqueness of the positive solution for system (1.5) with $N = 1$. When $N > 1$, we cannot obtain similar equations to (4.14) and (4.15) since system (1.5) may have no variational structure.

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