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GLOBAL STRUCTURE OF POSITIVE SOLUTIONS FOR PROBLEM WITH MEAN CURVATURE OPERATOR ON AN ANNULAR DOMAIN

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ABSTRACT. We study the global structure of positive solutions of the following mean curvature equation in the Minkowski space

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=\lambda f(x,u),$$

on an annular domain with the Robin boundary condition. According to the behavior of f near 0, we obtain the existence and multiplicity of positive solutions for this problem.

1. Introduction and main results. The study of hypersurfaces in the Minkowski space with coordinates $(x,t)\in\mathbb{R}^{N+1}$ and metric $(dx)^2-(dt)^2$ leads to the following mean curvature equation:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=H(x,u),\quad x\in\Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$ with $N\geq 1$ and $H:\Omega\times\mathbb{R}\to\mathbb{R}$ is the prescribed mean curvature of the hypersurface.

The hypersurface is maximal if the mean curvature is zero. If $H\equiv 0$ and $\Omega=\mathbb{R}^N$, Calabi [**4**] proved that equation (1.1) has only linear entire solutions for $N\leq 4$. Later, Cheng and Yao [**5**] proved that the only entire solution for equation (1.1) with $H\equiv 0$ and $\Omega=\mathbb{R}^N$ is linear for all N. When $\Omega=\mathbb{R}^N$ and $H\equiv c>0$, some renowned results for equation (1.1) were obtained by Treibergs [**12**]. If Ω is a bounded $C^{2,\alpha}$

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domain with some $\alpha > 0$, and $H = H(x, u) \in C^{0,\alpha}(\Omega \times \mathbb{R})$ is bounded, Bartnik and Simon [**1**] proved that equation (1.1) with $u = \varphi$ on $\partial\Omega$ has a strictly spacelike solution $u \in C^{2,\alpha}(\overline{\Omega})$, where φ is bounded and has an extension $\overline{\varphi} \in C^{2,\alpha}(\overline{\Omega})$ satisfying $|\nabla\overline{\varphi}| \leq 1 - \theta$ in $\overline{\Omega}$ for some $\theta > 0$. When $\Omega = B_R = B_R(0) := \{x \in \mathbb{R}^N : |x| < R\}$ with $R > 0$, Bereanu, Jebelean and Torres [**2, 3**] obtained some existence results for positive radial solutions of equation (1.1) with $u = 0$ on $\partial\Omega$. Recently, the first author [**6**] studied the nonexistence, existence and multiplicity of positive radial solutions of equation (1.1) on the unit ball with $u = 0$ on ∂B_R and $H = -\lambda f(x, s)$ via the bifurcation method.

The aim of this paper is to study the existence and multiplicity of positive radially symmetric solutions for equation (1.1) on an annular domain with the Robin boundary condition, which is mainly achieved by the bifurcation method.

Let $R_1, R_2 \in \mathbb{R}$ with $0 < R_1 < R_2$ and $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$. Consider the following problem with the Robin boundary condition

$$(1.2)\qquad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda f(x,u) & \text{in } \mathcal{A}, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_{R_1} & u = 0 \quad \text{on } \partial B_{R_2}, \end{cases}$$

where λ is a real parameter, $f : \overline{\mathcal{A}} \times [R_1, R_2] \to \mathbb{R}_+$ with $\mathbb{R}_+ = [0, +\infty)$ is a continuous function and is radially symmetric with respect to x, and $\partial v/\partial\nu$ is the outward normal derivative of v. Passing to polar coordinates, the problem (1.2) is reduced to the following problem

$$(1.3)\qquad \begin{cases} -(r^{N-1}\phi(v'))' = \lambda r^{N-1} f(r, v) \quad r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0, \end{cases}$$

where $r = |x|$, $v = u(|x|)$, and $\phi(s) = s/\sqrt{1-s^2}$. The solution of problem (1.3) may be understood in the classical sense.

Let λ_1 be the first eigenvalue of

$$(1.4)\qquad \begin{cases} -(r^{N-1}v')' = \lambda r^{N-1} v \quad r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

It is well known that λ_1 is simple, isolated and the associated eigenfunctions have one sign in $[R_1, R_2)$, see [**9, 13**]. Let $X = \{v \in C^1[R_1, R_2] :$

$v'(R_1) = v(R_2) = 0\}$ with the norm $\|v\| = \|v'\|_\infty$. From the fact that $\|v\|_\infty \le \|v'\|_\infty R$, it is easy to verify that the norm $\|v\|$ is equivalent to the usual norm $\|v\|_\infty + \|v'\|_\infty$.

The following theorem comprises our main results.

Theorem 1.1. *Assume that* $f(r,s) > 0$ *for any* $(r,s) \in [R_1,R_2] \times (0, R_2 - R_1]$, *and that there exists an* $f_0 \in [0,+\infty]$ *such that*

$$\lim_{s\to 0^+} \frac{f(r,s)}{s} = f_0$$

uniformly for $r \in (R_1,R_2)$. *Then,*

(a) *if* $f_0 = 1$, *there is an unbounded component* $\mathscr{C}$ *of the set of positive solutions of problem* (1.3), *bifurcating from* $(\lambda_1, 0)$ *such that*

$$\mathscr{C} \subseteq ((\mathbb{R}_+ \times X) \cup \{(\lambda_1, 0)\})$$

is infinite in the direction of λ *and* $\lim_{\lambda\to+\infty} \|v_\lambda\| = 1$ *for* $(\lambda, v_\lambda) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$;

(b) *if* $f_0 = +\infty$, *there is an unbounded component* $\mathscr{C}$ *of the set of positive solutions of problem* (1.3), *emanating from* $(0,0)$ *such that*

$$\mathscr{C} \subseteq ((\mathbb{R}_+ \times X) \cup \{(0,0)\}),$$

which joins to $(+\infty, 1)$;

(c) *if* $f_0 = 0$, *there is an unbounded component* $\mathscr{C}$ *of the set of positive solutions of problem* (1.3) *in* $\mathbb{R}_+ \times X$ *which joins* $(+\infty, 1)$ *to* $(+\infty, 0)$.

From now on, we add the point ∞ to our space $\mathbb{R} \times X$ so that $(+\infty, 1)$ and $(+\infty, 0)$ are elements of $\mathscr{C}$. Figure 1 illustrates the global bifurcation branches of Theorem 1.1. It follows from Theorem 1.1 that problem (1.3) possesses at least one positive solution for any $\lambda \in (\lambda_1, +\infty)$ if $f_0 = 1$ and has at least one positive solution for any $\lambda \in (0, +\infty)$ if $f_0 = +\infty$, see Figure 1 (A) and (B). When $f_0 = 0$, there exists a $\lambda_* > 0$ such that problem (1.3) has at least two positive solutions for any $\lambda \in (\lambda_*, +\infty)$, see Figure 1 (C). Clearly, Theorem 1.1 improves the corresponding ones of [**10,** Theorems 1.1–1.3].

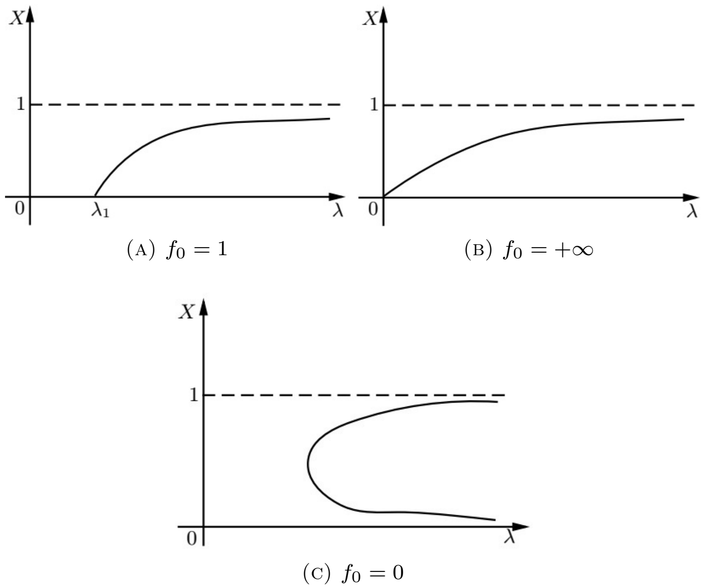
The remainder of this paper is arranged as follows. In Section 2, we show some necessary preliminary results. The proof of Theorem 1.1 is given in the last section.

2. Preliminaries. In order to study the bifurcation phenomenon of problem (1.3), we consider the following auxiliary problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = r^{N-1}g(r) & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0, \end{cases} \tag{2.1}$$

for any given $g \in Y$, where Y denotes the Banach space of continuous functions on $[R_1, R_2]$ endowed with the uniform norm $\|\cdot\|_\infty$. Define the continuous linear operator

$$H : Y \longrightarrow C^1[R_1, R_2]$$



(A) $f_0 = 1$ (B) $f_0 = +\infty$

(C) $f_0 = 0$

FIGURE 1. Bifurcation diagrams of Theorem 1.1.

by

$$Hu(r) = r^{1-N} \int_{R_1}^{r} s^{N-1} u(s)\, ds$$

for any $r \in [R_1, R_2]$.

Lemma 2.1. *For each $g \in Y$, problem* (2.1) *has a unique solution given by*

$$v = \int_{r}^{R_2} \phi^{-1} \circ H(g)\, ds := \Psi(g)$$

for any $r \in [R_1, R_2]$. Moreover, the operator $\Psi : Y \to X$ is continuous and sends bounded sets in Y into relatively compact sets in X.

Proof. Integrating the first equation of problem (2.1) from R_1 to $r \in [R_1, R_2]$, we have that

$$-\phi(v') = H(g).$$

Note that $\phi : (-1, 1) \to \mathbb{R}$ is an increasing diffeomorphism satisfying $\phi(0) = 0$. It follows that

$$v' = -\phi^{-1} \circ H(g).$$

Integrating the last equation from R_2 to r, in view of $v(R_2) = 0$, we arrive at

$$v = \int_{r}^{R_2} \phi^{-1} \circ H(g)\, ds := \Psi(g),$$

which shows the existence.

Clearly, we have that

$$|\phi(v')| = \left| r^{1-N} \int_{R_1}^{r} s^{N-1} g(s)\, ds \right| \le (R_2 - R_1)\|g\|_\infty := M$$

for any $r \in [R_1, R_2]$. It follows that

$$|v'| \le \phi^{-1}(M) < 1,$$

where ϕ^{-1} denotes the inverse function of ϕ. Thus, Ψ maps Y into X.

The continuity of Ψ is obvious. It suffices to prove that, if $\{g_n\}$ is a bounded subsequence in Y with $\|g_n\|_\infty \le M$ for some positive constant M and any $n \in \mathbb{N}$, then $v_n = \Psi(g_n)$ contains a convergent subsequence in X. Clearly, we have that $\{H(g_n)\}$ is uniformly bounded. For any $r \in [R_1, R_2]$, with calculations, we can show that

$$\begin{aligned}|(H(g_n))'(r)| &= \left| g_n(r) - (N-1)\frac{\int_{R_1}^r s^{N-1} g_n(s)\,ds}{r^N} \right| \\ &\le M\left(1 + (N-1)\frac{\int_{R_1}^r s^{N-1}\,ds}{r^N}\right) \\ &= M\left(1 + \frac{(N-1)}{N}\frac{r^N - R_1^N}{r^N}\right) \\ &\le \frac{M(2N-1)}{N}.\end{aligned}$$

For any $r, r' \in [R_1, R_2]$, it follows from the Lagrange mean theorem that

$$|(H(g_n))'(r) - (H(g_n))'(r')| \le \frac{M(2N-1)}{N}|r - r'|.$$

Thus, the sequence $\{H(g_n)\}$ is also equicontinuous. From the Arzelà-Ascoli theorem, up to a subsequence, $H(g_n)$ is convergent in Y. Since $\phi^{-1} : Y \to Y$ is continuous, it follows that $v_n' = -\phi^{-1}(H(g_n))$ converges to $-\phi^{-1}(g_0) := v_0$ in Y, where g_0 is the limit of $H(g_n)$. Reasoning as in the above paragraph, we have that $\|v_0\|_\infty < 1$. Therefore, we have that v_n converges to

$$\int_0^r v_0(s)\,ds := v \quad \text{in } X. \qquad \square$$

We also must consider the following auxiliary problem

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}h(r) & \text{in } (R_1, R_2), \\ u'(R_1) = u(R_2) = 0, \end{cases} \tag{2.2}$$

for a given $h \in Y$. Analogously to that of Lemma 2.1, we can show that problem (2.2) has a unique solution, which is denoted by $\Phi(h)$, and

$$\Phi : Y \longrightarrow X$$

is continuous, compact and linear. Further, we consider the following problem with a parameter

$$\begin{cases} -\left(r^{N-1}\frac{v'}{\sqrt{1-t^2v'^2}}\right)' = r^{N-1}g(r), & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0, \end{cases} \tag{2.3}$$

for any $t \in (0, 1]$ and any given $g \in Y$. Letting $w = tv$, problem (2.3) is equivalent to

$$\begin{cases} -(r^{N-1}\phi(w'))' = tr^{N-1}g(r), & r \in (R_1, R_2), \\ w'(R_1) = w(R_2) = 0. \end{cases} \tag{2.4}$$

By Lemma 2.1, problem (2.4) has a unique solution $w = \Psi(tg)$. Thus, $v = \Psi(tg)/t$ is the unique solution of problem (2.3). For any $g \in Y$, define

$$G(t, g) = \begin{cases} \frac{\Psi(tg)}{t} & \text{if } t \in (0, 1], \\ \Phi(g) & \text{if } t = 0. \end{cases} \tag{2.5}$$

Then, we can show that:

Lemma 2.2. *$G : [0, 1] \times Y \to X$ is completely continuous.*

Proof. We first prove the *continuity* of G. For any $g_n, g \in Y$ and $t_n, t \in [0, 1]$ with $g_n \to g$ in Y and $t_n \to t$ in $[0, 1]$ as $n \to +\infty$, it is sufficient to show that $G(t_n, g_n) := v_n \to G(t, g) := v$ in X. If $t > 0$, without loss of generality, we can assume that $t_n > 0$ for any $n \in \mathbb{N}$. It follows from Lemma 2.1 that $v_n \to v$ in X as $n \to +\infty$. If $t = 0$ and there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} = 0$, then

$$v_{n_i} = G(t_{n_i}, g_{n_i}) = \Phi(g_{n_i}) \longrightarrow \Phi(g) = v$$

in X as $i \to +\infty$. Thus, next, we assume that $t = 0$ and $t_n > 0$ for any $n \in \mathbb{N}$. From Lemma 2.1, we know that problem (2.4) has only a trivial solution when $t = 0$. In addition, by Lemma 2.1, we have that $w_n \to 0$ in X as $n \to +\infty$.

Note that v_n satisfies

$$\begin{cases} -\frac{v''}{(1-w_n'^2)^{3/2}} - (N-1)\frac{v'}{r\sqrt{1-w_n'^2}} = g_n(r), & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

It follows that there exists an $N_0 > 0$ such that $\|v_n\|_{C^2[0,R]} \leq C$ for any $n \geq N_0$ and some positive constant C, which depends only upon g and R_1. Hence, there exists a $v \in X$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v$ in X as $k \to +\infty$. Note that

$$v''_{n_k} = -(N-1)\frac{v'_{n_k}(1-w'^2_{n_k})}{r} - (1-w'^2_{n_k})^{3/2} g_{n_k}(r), \quad r \in (R_1, R_2).$$

Integrating the above equation from R_1 to $r \in (R_1, R_2)$, we obtain that

$$v'_{n_k}(r) = \int_{R_1}^{r} \left(-(N-1)\frac{v'_{n_k}(1-w'^2_{n_k})}{s} - (1-w'^2_{n_k})^{3/2} g_{n_k}(s) \right) ds.$$

By the Lebesgue dominated convergence theorem, we have that

$$v'(r) = \int_{R_1}^{r} \left(-(N-1)\frac{v'}{s} - g(s) \right) ds.$$

It follows that

$$\begin{cases} -(r^{N-1}v')' = r^{N-1}g(r), & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

Hence, we have that $v = \Phi(g) = G(0, g)$. We claim that $v_n \to v$ in X. Otherwise, there would exist a subsequence $\{v_{m_j}\}$ of $\{v_n\}$ in X and $\epsilon_0 > 0$ such that, for any $j \in \mathbb{N}$, we have $\|v_{m_j} - v\| \geq \epsilon_0$. However, reasoning as above, $\{v_{m_j}\}$ would contain an additional subsequence $v_{m_{j_l}} \to v$ in X as $l \to +\infty$, which contradicts $\|v_{m_{j_l}} - v\| \geq \epsilon_0$. Therefore, $v_n \to v$ in X.

Next, we show the *compactness* of G. Clearly, $G(t, \cdot)$ is compact for any fixed $t \in [0,1]$. We claim that the *continuity of G with respect to t at any $t_0 \in [0,1]$ is uniform for $g \in Y$*, that is to say, for any $\epsilon > 0$ and $g \in Y$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|G(t,g) - G(t_0,g)\| < \epsilon$ when $|t - t_0| < \delta$ with $t \in [0,1]$. Suppose, by contradiction, that there exist $\epsilon_0 > 0$, $g_0 \in Y$, such that, for any $n \in \mathbb{N}$, existing $t_n \in [0,1]$ with $|t_n - t_0| < 1/n$ such that

$$\|G(t_n, g_0) - G(t_0, g_0)\| \geq \epsilon_0. \tag{2.6}$$

Up to a subsequence, we have $t_n \to t_0 \in [0,1]$ as $n \to +\infty$. Letting $n \to +\infty$ in (2.6), we have that

$$0 = \lim_{n \to +\infty} \|G(t_n, g_0) - G(t_0, g_0)\| \geq \epsilon_0,$$

which is a contradiction.

For any $(t_n, g_n) \in [0,1] \times Y$ with $\{g_n\}$ bounded in Y for any $n \in \mathbb{N}$, it suffices to show that $\{G(t_n, g_n)\}$ possesses a convergent subsequence. Without loss of generality, we assume that $t_n \to t_0 \in [0,1]$. We know that $\{G(t_1, g_n)\}$ has a convergent subsequence. Thus, there exists a subsequence $\{g_n^{(1)}\}$ of $\{g_n\}$ such that the diameter of $\{G(t_1, g_n^{(1)})\}$ is less than 1. Similarly, there exists a $\{g_n^{(2)}\} \subseteq \{g_n^{(1)}\}$ such that the diameter of $\{G(t_2, g_n^{(2)})\}$ is less than $1/2$. In general, there exists a $\{g_n^{(k)}\} \subseteq \{g_n^{(k-1)}\}$ such that the diameter of $\{G(t_k, g_n^{(k)})\}$ is less than $1/k$, $k \geq 3$. We claim that $\{G(t_n, g_n^{(n)})\}$ is convergent. We have shown that, for any $\epsilon > 0$ and $g \in Y$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|G(t,g) - G(t_0, g)\| < \epsilon/3$ when $|t - t_0| < \delta$ with $t \in [0,1]$. Take $N_1 > 3/\epsilon$ such that $|t_n - t_0| < \delta$ for any $n > N_1$. Consequently, when $m > n > N_1$, we have that

$$\begin{aligned}\|G(t_m, g_m^{(m)}) - G(t_n, g_n^{(n)})\| &< \|G(t_m, g_m^{(m)}) - G(t_0, g_m^{(m)})\| \\ &\quad + \|G(t_0, g_m^{(m)}) - G(t_n, g_m^{(m)})\| \\ &\quad + \|G(t_n, g_m^{(m)}) - G(t_n, g_n^{(n)})\| \\ &< \quad \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{1}{n} < \epsilon.\end{aligned}$$

It follows that $\{G(t_n, g_n^{(n)})\}$ is the Cauchy sequence. Thus, we have that $G(t_n, g_n^{(n)}) \to v_0$ for some $v_0 \in X$.

Finally, we show that $G(t_n^{(n)}, g_n^{(n)}) \to v_0$ as $n \to +\infty$. Clearly, there exists an $N_2 > 0$ such that $|t_n - t_0| < \delta$, $|t_n^{(n)} - t_0| < \delta$ and $\|G(t_n, g_n^{(n)}) - v_0\| < \epsilon/3$ any $n > N_2$. Hence, when $n > N_2$, we obtain that

$$\begin{aligned}\|G(t_n^{(n)}, g_n^{(n)}) - v_0\| &< \|G(t_n^{(n)}, g_n^{(n)}) - G(t_0, g_n^{(n)})\| \\ &\quad + \|G(t_0, g_n^{(n)}) - G(t_n, g_n^{(n)})\| \\ &\quad + \|G(t_n, g_n^{(n)}) - v_0\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.\end{aligned}$$

Therefore, we obtain that $G(t_n^{(n)}, g_n^{(n)}) \to v_0$ in X as $n \to +\infty$. $\square$

Consider the following problem

$$(2.7)\qquad \begin{cases} -(r^{N-1}\phi(v'))' = \lambda r^{N-1} v, & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

Obviously, problem (2.7) is equivalent to the operator equation $v = \Psi(\lambda v) := \Psi_\lambda(v)$. By Lemma 2.1, we see that $\Psi_\lambda : X \to X$ is complete and continuous. Furthermore, we have the following topological degree jumping result.

Lemma 2.3. *For any $r \in (0,1)$, we have that*

$$(2.8)\qquad \deg(I - \Psi_\lambda, \mathcal{B}_r(0), 0) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta) \end{cases}$$

for some $\delta > 0$, where $\mathcal{B}_r(0) = \{u \in X : \|u\| < r\}$.

Proof. Since λ_1 is isolated, we can choose $\delta > 0$ such that problem (1.4) has no eigenvalue in $(\lambda_1, \lambda_1 + \delta)$. We claim that the Leray-Schauder degree

$$\deg(I - G(t, \lambda\cdot), \mathcal{B}_r(0), 0)$$

is well defined for any $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$ and $t \in [0, 1]$. The claim is obvious for $t = 0$. Thus, in view of Lemma 2.2, it is sufficient to show that $v = G(t, \lambda v)$ has no solution with $\|v\| = r$ for r sufficiently small and any $t \in (0, 1]$. Otherwise, there exists a sequence $\{v_n\}$ such that $v_n = \Psi_\lambda(tv_n)/t$ and $\|v_n\| \to 0$ as $n \to +\infty$. Letting $\widetilde{w}_n = v_n/\|v_n\|$, we have that $\widetilde{w}_n$ satisfies

$$\begin{cases} -\frac{\widetilde{w}_n''}{(1-w_n'^2)^{3/2}} - (N-1)\frac{\widetilde{w}_n'}{r\sqrt{1-w_n'^2}} = \lambda \widetilde{w}_n, & r \in (R_1, R_2), \\ \widetilde{w}_n'(R_1) = \widetilde{w}_n(R_2) = 0. \end{cases}$$

Similar to the reasoning of Lemma 2.2, we can show that, for some convenient subsequence, $\widetilde{w}_n \to \widetilde{w}$ as $n \to +\infty$ and $\widetilde{w}$ verify problem (1.4) with $\|\widetilde{w}\| = 1$. This implies that λ is an eigenvalue of problem (1.4), a contradiction.

From the invariance of the degree under homotopies, we obtain that

$$\begin{aligned} \deg(I - \Psi_\lambda, \mathcal{B}_r(0), 0) &= \deg(I - G(1, \lambda\cdot), \mathcal{B}_r(0), 0) \\ &= \deg(I - G(0, \lambda\cdot), \mathcal{B}_r(0), 0) = \deg(I - \lambda\Phi, \mathcal{B}_r(0), 0). \end{aligned}$$

By [**8,** Theorem 8.10], we obtain that

$$\deg(I-\lambda\Phi,\mathcal{B}_r(0),0)=\begin{cases}1 & \text{if } \lambda\in(0,\lambda_1),\\ -1 & \text{if } \lambda\in(\lambda_1,\lambda_1+\delta).\end{cases}$$

Therefore, we have that

$$\deg(I-\Psi_\lambda,\mathcal{B}_r(0),0)=\begin{cases}1 & \text{if } \lambda\in(0,\lambda_1),\\ -1 & \text{if } \lambda\in(\lambda_1,\lambda_1+\delta),\end{cases}$$

which is the desired conclusion. □

Graphs which are solutions to problem (1.3) are strictly space-like in $\mathcal{A}$. The following lemma ensures a priori that each possible solution v of problem (1.3) is strictly space-like on the boundary of $\mathcal{A}$, as well.

Lemma 2.4. *Let v be any solution to problem* (1.3) *with any fixed λ. Then, $|v'|<1$ on $[R_1,R_2]$.*

Proof. It suffices to show that $|v'(R_2)|<1$. Suppose, by way of contradiction, that there exists a sequence $\{r_k\}\subset(R_1,R_2)$ such that

$$\lim_{k\to+\infty} r_k=R_2,$$
$$\lim_{k\to+\infty}|v'(r_k)|=|v'(R_2)|=1$$

and

$$\lim_{k\to+\infty}|\phi(v')(r_k)|=+\infty.$$

Clearly, we have that

$$\frac{(r^{N-1}\phi(v'))'}{r^{N-1}\phi(v')}=-\frac{\lambda f(r,v)}{\phi(v')}.$$

Obviously, there exists an $\overline{r}\in(R_1,R_2)$ such that $|v'|>1/2$ for all $r\in(\overline{r},R_2)$. Integrating the above equality from $\overline{r}$ to r_k, we obtain that

$$\log|(r_k+\varepsilon)^{N-1}\phi(v'(r_k))|-\log|(\overline{r}+\varepsilon)^{N-1}\phi(v'(\overline{r}))|=-\lambda\int_{\overline{r}}^{r_k}\frac{f(r,v)}{\phi(v')}\,dr.$$

Letting $k\to+\infty$, we see that the left member tends to infinity while the right one is bounded, which is a contradiction. □

3. Proof of Theorem 1.1. Now, we give the proof of our main result.

Proof of Theorem 1.1.

(a) Let $\xi(r,s)=f(r,s)-f_0s$. Then, we have that

$$\lim_{s\to 0^+}\frac{\xi(r,s)}{s}=0$$

uniformly for $r\in(R_1,R_2)$. Consider

$$\begin{cases}-(r^{N-1}\phi(v'))'=\lambda r^{N-1}(f_0v+\xi(r,v)),\\ v'(R_1)=v(R_2)=0\end{cases}$$

as a bifurcation problem from the trivial solution axis.

Define $F_\lambda(s,v):[0,1]\times X\to Y$

$$F_\lambda(s,v)=\lambda(f_0v+s\xi(r,v)).$$

Then, it is easy to see that F_λ is continuous and takes bounded sets into bounded sets. Consider the following problem

$$\begin{cases}-(r^{N-1}\phi(v'))'=r^{N-1}F_\lambda(s,v),\\ v'(R_1)=v(R_2)=0.\end{cases}\tag{3.1}$$

Then, problem (3.1) can be equivalently rewritten as

$$v=\Psi(F_\lambda(s,v)):=T_\lambda(s,v).\tag{3.2}$$

By Lemma 2.1, $T_\lambda:[0,1]\times X\to X$ is completely continuous. In particular, $H_\lambda:=T_\lambda(1,\cdot):X\to X$ is completely continuous. Similarly to that of **[6,** Theorem 1.1], we have that

$$\left|\frac{\xi(r,v)}{\|v\|}\right|\longrightarrow 0\quad\text{as }\|v\|\to 0,\tag{3.3}$$

uniformly in $r\in(R_1,R_2)$.

We claim that the Leray-Schauder degree $\deg(I-T_\lambda(s,\cdot),\mathcal{B}_r(0),0)$ is well defined for $\lambda\in(0,\lambda_1+\delta)\setminus\{\lambda_1\}$ and r small enough. Suppose, by contradiction, that there exists a sequence $\{v_n\}$ such that $v_n=T_\lambda(s,v_n)$ and $\|v_n\|\to 0$ as $n\to+\infty$. Letting $\widehat{w}_n=v_n/\|v_n\|$, we have that $\widehat{w}_n$ satisfies

$$\begin{cases} -\frac{\widehat{w}_n''}{(1-v_n'^2)^{3/2}} - (N-1)\frac{\widehat{w}_n'}{r\sqrt{1-v_n'^2}} = \lambda(s\frac{\xi(r,v_n)}{\|v_n\|} + f_0\widehat{w}_n), \\ \widehat{w}_n'(R_1) = \widehat{w}_n(R_2) = 0. \end{cases}$$

Then, by (3.3) and an argument similar to that of Lemma 2.2, we can show that $\widehat{w}_n \to \widehat{w}$ in X as $n \to +\infty$ and

$$\begin{cases} -(r^{N-1}\widehat{w}')' = \lambda r^{N-1}\widehat{w}, \quad r \in (R_1, R_2), \\ \widehat{w}'(R_1) = \widehat{w}(R_2) = 0. \end{cases}$$

Clearly, we have $\|\widehat{w}\| = 1$. Therefore, λ is an eigenvalue of problem (1.4), which is absurd.

By the invariance of the degree under homotopy, we obtain that

$$\begin{aligned} \deg(I - H_\lambda, \mathcal{B}_r(0), 0) &= \deg(I - T_\lambda(1, \cdot), \mathcal{B}_r(0), 0) \\ &= \deg(I - T_\lambda(0, \cdot), \mathcal{B}_r(0), 0) \\ &= \deg(I - \Psi_\lambda, \mathcal{B}_r(0), 0). \end{aligned}$$

By Lemma 2.3, we have that

$$\deg(I - H_\lambda, \mathcal{B}_r(0), 0) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}$$

By the Global bifurcation theorem **[11]**, there exists a continuum $\mathscr{C}$ of a nontrivial solution of problem (1.3), bifurcating from $(\lambda_1, 0)$, which is either unbounded or

$$\mathscr{C} \cap (\mathbb{R} \setminus \{\lambda_1\} \times \{0\}) \neq \emptyset.$$

Since $(0,0)$ is the only solution of problem (1.3) for $\lambda = 0$ and 0 is not an eigenvalue of problem (1.4), so $\mathscr{C} \cap (\{0\} \times X) = \emptyset$. Analogously to that of [**2,** Lemma 1], we can show that v is nonnegative on $[R_1, R_2)$ and decreasing for any $(\lambda, v) \in \mathscr{C}$.

We claim that $\mathscr{C} \cap (\mathbb{R} \setminus \{\lambda_1\} \times \{0\}) = \emptyset$. Otherwise, there exists a nontrivial, nonnegative solution sequence $(\lambda_n, v_n) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$ such that $\lambda_n \to \mu$ and $v_n \to 0$ as $n \to +\infty$. Let $w_n = v_n/\|v_n\|$. By (3.3) and an argument like that of Lemma 2.2, we can show that $w_n \to w$ as $n \to +\infty$ and w verifies problem (1.4) with $\|w\| = 1$. It follows that $\mu = \lambda_1$, a contradiction.

Therefore, $\mathscr{C}$ is unbounded in $(0, +\infty) \times X$, and v is nontrivially nonnegative for any $(\lambda, v) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$. Furthermore, as in **[2,** Lemma 1], we can show that v is positive and strictly decreasing for any $(\lambda, v) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$. By Lemma 2.4, we see that the projection of $\mathscr{C}$ on $\mathbb{R}_+$ is unbounded.

Finally, we investigate the asymptotic behavior of v_λ as $\lambda \to +\infty$ for $(\lambda, v_\lambda) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$. Suppose, by contradiction, that there exist a constant $\delta > 0$ and $(\lambda_n, v_n) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$ with $\lambda_n \to +\infty$ as $n \to +\infty$ such that $\|v_n\|^2 \leq 1 - \delta^2$ for any $n \in \mathbb{N}$. Note that (λ_n, v_n) satisfies the following problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = \lambda r^{N-1} a(r) v, & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0, \end{cases} \tag{3.4}$$

where

$$a(r) = \frac{f(r, v)}{v(r)}.$$

The assumptions of $f_0 = 1$ and $f(r, s) > 0$ for any $(r, s) \in [R_1, R_2] \times (0, R_2 - R_1]$ imply that there exists a positive constant $\rho > 0$ such that

$$a(r) \geq \rho$$

for any $r \in [R_1, R_2]$.

Let φ_1 be a positive eigenfunction associated to λ_1. It is easy to see that φ_1 is decreasing in $[R_1, R_2]$. Multiplying the first equation of problem (3.4) by φ_1, and obtaining, after integrations by parts, that

$$\begin{aligned} \frac{\lambda_1}{\delta} \int_{R_1}^{R_2} r^{N-1} v_n \varphi_1 \, dr &= \frac{1}{\delta} \int_{R_1}^{R_2} r^{N-1} v_n' \varphi_1' \, dr \\ &\geq \int_{R_1}^{R_2} r^{N-1} \frac{v_n' \varphi_1'}{\sqrt{1 - |v_n'|^2}} \, dr \\ &= \lambda_n \int_{R_1}^{R_2} r^{N-1} a(r) v_n \varphi_1 \, dr \\ &\geq \lambda_n \rho \int_{R_1}^{R_2} r^{N-1} v_n \varphi_1 \, dr. \end{aligned}$$

It follows that $\lambda_n \leq \lambda_1 / (\delta\rho)$, which is a contradiction.

(b) For any $n \in \mathbb{N}$, define

$$f^n(r,s) = \begin{cases} ns & s \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ n\left(f\left(r, \frac{2}{n}\right) - 1\right)\left(s - \frac{1}{n}\right) + 1 & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ -n\left(f\left(r, -\frac{2}{n}\right) + 1\right)\left(s + \frac{1}{n}\right) - 1 & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ f(r,s) & s \in \left(-\infty, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, +\infty\right). \end{cases}$$

Then, consider the following problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = \lambda r^{N-1} f^n(r,v), & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

By the conclusion of (a) and an argument similar to that of [**7,** Theorem 1.2], we can obtain the desired conclusion.

(c) For any $n \in \mathbb{N}$, define

$$f_n(r,s) = \begin{cases} \frac{1}{n}s & s \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ \left(f\left(r, \frac{2}{n}\right) - \frac{1}{n^2}\right) n\left(s - \frac{1}{n}\right) + \frac{1}{n^2} & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ -\left(f\left(r, -\frac{2}{n}\right) + \frac{1}{n^2}\right) n\left(s + \frac{1}{n}\right) - \frac{1}{n^2} & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ f(r,s) & s \in \left(-\infty, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, +\infty\right), \end{cases}$$

and consider the following problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = \lambda r^{N-1} f_n(r,v), & r \in (R_1, R_2), \\ v'(R_1) = v(R_2) = 0. \end{cases}$$

Then, by an argument similar to that of [**7,** Theorem 1.3] and the aid of (a), we can derive the desired conclusion. $\square$

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