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EIGENVALUE PROBLEM ASSOCIATED WITH THE FOURTH ORDER DIFFERENTIAL-OPERATOR EQUATION

NIGAR M. ASLANOVA, MAMED BAYRAMOGLU
AND KHALIG M. ASLANOV

ABSTRACT. In this paper, we investigate the boundary value problem for fourth order differential operator equations with unbounded operator coefficients and one λ-dependent boundary condition. We obtain an asymptotic formula for eigenvalues and a trace formula for the corresponding self-adjoint operator.

1. Introduction. In this paper, we investigate the boundary value problem for fourth order differential equations with unbounded operator coefficients and one λ-dependent boundary condition. The main questions to be studied are the following:

(A) to establish the asymptotics of eigenvalue distribution (asymptotics of the distribution function $N(\lambda)$);

(B) to derive the regularized trace formula.

Eigenvalue distribution, which arises in quantum mechanics, for the Sturm-Lioville operator equation with an unbounded operator-valued coefficient having compact inverse and boundary conditions without the λ-parameter is established in [**10**].

The asymptotics of $N(\lambda)$ for the second and nth order differential operator equations with unbounded operator coefficients are treated, for example, in [**1, 3, 4, 6, 14, 15, 21**].

Due to the appearance of the eigenvalue parameter in the boundary condition, the problem considered herein is not self-adjoint. By introducing the direct sum of Hilbert spaces with a new scalar product

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defined in it, we consider that problem as the eigenvalue problem of a selfadjoint operator denoted L. Selfadjointness is essentially used for investigating the nature of the spectrum as well as for deriving the trace formula. First, we prove positive definiteness of L. Next, an application of the Rellich theorem proves compactness of the resolvent and, as a result, discreteness of spectrum. Finally, we find an asymptotic formula for eigenvalues.

Note that, in the case of the second order differential operator equation with λ containing boundary conditions, some roots of characteristic equations may be imaginary and form unbounded sequences, as in [**1, 3, 4, 21**], resulting in three different asymptotic behaviors of eigenvalues. In addition, note that eigenvalues of the problem are sums of squares of the roots and eigenvalues of the unbounded operator coefficient.

However, if, in the boundary condition, a linear function of λ appears as a coefficient before an unknown function as well as before its first derivative, some imaginary roots form a bounded sequence, and consequently, the asymptotic behavior of eigenvalues is given by one formula. In the scalar case for boundary value problems with eigenvalue dependent boundary conditions, the reader is referred to [**8, 9, 12, 20**], and the references therein.

The regularized trace formula for the scalar Sturm-Lioville operator is first established in [**13**]. The case of the differential-operator equation with unbounded operator coefficient is defined in [**18**]. Traces of abstract discrete operators with given eigenvalue distribution are studied in [**11, 22**]. Corresponding questions for operators generated by regular and singular differential expressions with unbounded operator coefficients are treated in [**2, 3, 5, 6, 7**].

In the present paper, we consider the space $L_2((0,1),H)$ (H is an abstract, separable Hilbert space) of the boundary value problem

$$ly(t) := y^{\mathrm{IV}}(t) + Ay(t) + q(t)y(t) = \lambda y(t), \tag{1.1}$$

$$y(0) = y''(0) = y'''(1) = 0, \tag{1.2}$$

$$y''(1) - \lambda y'(1) = 0, \tag{1.3}$$

where $A = A^* > \mathrm{I}$ (I is the identity operator) is an operator in H satisfying $A^{-1} \in \sigma_\infty$, $q(t)$ is an operator-valued function for each t defined in H and $\|q(t)\| \leq \text{const}$ for $t \in [0,1]$.

Problems with eigenvalue dependent boundary conditions arise in a variety of physical problems: vibration involving loads, heat conduction and electric circuit problems involving long cables.

Under the above-stated conditions, A is a discrete operator. We denote its eigenvalues by $\gamma_1 \leq \gamma_2 \leq \cdots$ and eigenvectors by $\varphi_1, \varphi_2, \ldots$. In addition, we assume here that:

(1) $q^*(t) = q(t)$ for all $t[0,1]$;

(2) $\int_0^1 (q(t)\varphi_j, \varphi_j)\, dt = 0$, $j = 1, 2, \ldots$;

(3) $q^{(l)}(t) \in \sigma_1$, $[q^{(l)}(t)]^* = q^{(l)}(t)$, $\|q^{(l)}\|_{\sigma_1} < \text{const}$ for $l = 0, 1, 2$.

Recall here that σ_1 is a trace class (class of compact operators whose singular values form convergent series, [**11,** page 521].

We shall give an operator-theoretic formulation of problem (1.1)–(1.3), associating with it a self-adjoint operator. The asymptotics of the eigenvalues of that operator will be investigated. Also, employing perturbation theory and residue calculus, we shall calculate the regularized trace.

2. A self-adjoint operator.

We introduce the space $H_1 = L_2((0,1), H) \oplus H$ of two component vectors and define in it an inner product by

$$(Y, Z)_{H_1} = \int_0^1 (y(t), z(t))\, dt + (y_1, z_1) \tag{2.1}$$

for

$$Y = (y(t), y_1), \quad Z = \{z(t), z_1\}$$
$$y(t), z(t) \in L_2((0,1), H), \quad y_1, z_1 \in H.$$

It is assumed that $(\cdot,\cdot)$ is a scalar product and $\|\cdot\|$ a norm in H.

The operator-theoretic formulation of (1.1)–(1.3) with $q(t) \equiv 0$ is

$$L_0 Y = \{y^4(t) + Ay(t),\ y''(1)\},$$

$$D(L_0) = \{Y = \{y(t), y_1\} \in L_2/y'''(t)$$

is absolutely continuous in norm $\|\cdot\|$,

$$ly \in L_2((0,1), H),$$

$$y(0) = y''(0) = y'''(1) = 0$$

and

$$y_1 = y'(1).$$

It follows that L_0 is densely defined and symmetric with respect to the scalar product in (2.1), as well as self-adjoint. Symmetry follows from the relation

$$\begin{aligned}(L_0 Y, Y)_{H_1} &= \int_0^1 (y^{\mathrm{IV}}(t) + Ay(t), y(t))\,dt \\ &\quad + (y''(1), y'(1)) = (y'''(1), y(1)) - (y'''(0), y(0)) \\ &\quad - (y''(1), y'(1)) + (y''(0), y'(0)) + (y''(1), y'(1)) \\ &\quad - (y'(0), y''(0)) - (y(1), y'''(1)) + (y(0), y'''(0)) \\ &\quad + (y'(1), y''(1)) + \int_0^1 (y(t), Ay(t) + y^{\mathrm{IV}}(t))\,dt \\ &= (Y, L_0 Y)_{H_1}.\end{aligned}$$

L_0 is positive definite:

$$\begin{aligned}(L_0, Y, Y)_{H_1} &= \int_0^1 (y^{\mathrm{IV}}(t) + Ay(t), y(t))\,dt + (y''(1), y'(1)) \\ &= (y'''(1), y(1)) + (y''(1), y'(1)) \\ &\quad - \int_0^1 (y'''(t), y'(t))\,dt + \int_0^1 (Ay(t), y(t))\,dt \\ &= (y''(1), y'(1)) - (y''(t), y'(t))|_0^1 \\ &\quad + \int_0^1 \|y''(t)\|^2 dt + \int_0^1 (Ay(t), y(t))\,dt \\ &= \int_0^1 \|y''(t)\|^2 dt + \int_0^1 (Ay(t), y(t))\,dt \\ &\geq \int_0^1 \|y''(t)\|^2 dt + \int_0^1 \|y(t)\|^2 dt\end{aligned}$$

since $W_2^2((0,1), H) \subset C([0,1], H)$ is continuous, [**17,** Theorem 3.1] and

$\|y'(1)\|_H \leq C\|y(t)\|_{W_2^2([0,1],H)}$. Therefore,

$$(L_0Y, Y)_{H_1} \geq C\bigg(\int_0^1 \|y(t)^2 dt + \|y'(1)\|^2\bigg) \geq C\|Y\|_{H_1}^2.$$

Under conditions shown by using the Rellich theorem **[19]** that L_0^{-1} is compact, the spectrum of L_0 is discrete. Take $L = L_0 + Q$ and $QY = \{q(t)y(t), 0\}$. Q is bounded in H_1 since $q(t)$ is bounded for each t in H. Due to the boundedness of the Q spectrum of L, it is also discrete.

We need the fact that the eigenvalues of L and L_0 are denoted by $\mu_1 \leq \mu_2 \leq \cdots$ and $\lambda_1 \leq \lambda_2 \leq \cdots$, counting multiplicities, in what follows.

3. Asymptotics of eigenvalues. By using the eigenfunction expansion for A, the next problem naturally arises:

$$y_k^{\mathrm{IV}}(t) + \gamma_k y_k(t) = \lambda y_k(t), \tag{3.1}$$

$$y_k(0) = y_k''(0) = 0, \tag{3.2}$$

$$y_k'''(1) = 0, \tag{3.3}$$

$$y_k''(1) - \lambda y_k'(1) = 0; \tag{3.4}$$

here, $y_k(t) = (y(t), \varphi_k), k = \overline{1, \infty}$.

The solution of equation (3.1) from $L_2(0,1)$ in order to satisfy condition (3.2) is

$$y_k(t) = c_1 \sin \sqrt[4]{\lambda - \gamma_k} t + c_2 \operatorname{sh} \sqrt[4]{\lambda - \gamma_k} t. \tag{3.5}$$

In order for the solution to satisfy boundary conditions (3.3) and (3.4), we obtain the following equations:

$$-c_1 \cos \sqrt[4]{\lambda - \gamma_k} + c_2 \operatorname{ch} \sqrt[4]{\lambda - \gamma_k} = 0,$$

$$\begin{aligned} - c_1 \sin \sqrt[4]{\lambda - \gamma_k} + c_2 \operatorname{sh} \sqrt[4]{\lambda - \gamma_k} - \sqrt[4]{\lambda - \gamma_k}^3 c_1 \cos \sqrt[4]{\lambda - \gamma_k} \\ - \sqrt[4]{\lambda - \gamma_k}^3 c_2 \operatorname{ch} \sqrt[4]{\lambda - \gamma_k} = 0. \end{aligned}$$

Denote $\sqrt[4]{\lambda-\gamma_k}=z$ so that

(3.6)
$$\begin{cases} -c_1\cos z + c_2 \operatorname{ch} z = 0, \\ -c_1 \sin z + c_2 \operatorname{sh} z - (z^4+\gamma_k)c_1\cos z - (z^4+\gamma_k)c_2 \operatorname{ch} z = 0. \end{cases}$$

The system of equations (3.6) has a nontrivial solution if and only if

$$\begin{vmatrix} -\cos z & \operatorname{ch} z \\ -z\sin z - (z^4+\gamma_k)\cos z & z \operatorname{sh} z - (z^4+\gamma_k)\operatorname{ch} z \end{vmatrix} = 0,$$

or

$$-z\cos z \operatorname{sh} z + 2\operatorname{ch} z \cos(z^4+\gamma_k) + z\operatorname{ch} z \sin z = 0. \tag{3.7}$$

By using condition (3.3), we obtain that

$$c_2 = \frac{\cos z}{\operatorname{ch} z} c_1,$$

and therefore,

$$y_k(t) = c_1\left(\sin zt + \frac{\cos z}{\operatorname{ch} z}\operatorname{sh} zt\right). \tag{3.8}$$

Thus, the eigenvectors of operator L_0 are

$$Y = c\left\{\left[\sin zt + \frac{\cos z}{\operatorname{ch} z}\operatorname{sh} zt\right]\varphi_k, (2z\cos z)\varphi_k\right\}.$$

We must find the norms of the vectors. Obviously,

$$\begin{aligned}
\|Y\|^2_{H_1} &= (Y,Y)_{H_1} \\
&= c^2\int_0^1 \left[\sin^2 zt + \frac{2\cos z}{\operatorname{ch} z}\sin zt \operatorname{sh} zt + \frac{\cos^2 z}{\operatorname{ch}^2 z}\operatorname{sh}^2 zt\right] dt + 4z^2\cos^2 z \\
&= c^2\left[\frac{1}{2} - \frac{\sin 2z}{4z} + \frac{\cos z}{z\operatorname{ch} z}(\operatorname{ch} z\sin z - \operatorname{sh} z\cos z)\right. \\
&\qquad\left. + \frac{\cos^2 z}{\operatorname{ch}^2 z}\frac{\operatorname{sh}^2 z}{4z} - \frac{1}{2}\frac{\cos^2 z}{\operatorname{ch}^2 z} + 4z^2\cos^2 z\right] \\
&= c^2\left[\frac{2z\operatorname{ch}^2 z - \sin 2z \operatorname{ch}^2 z + 8z^3\cos^2 z \operatorname{ch}^2 z}{4z\operatorname{ch}^2 z}\right. \\
&\qquad\left. + \frac{\cos^2 z\operatorname{sh}^2 z - 2z\cos^2 z - 8(\gamma_k/z)\cos^2 z\operatorname{ch}^2 z}{4z\operatorname{ch}^2 z}\right].
\end{aligned}$$

Here, we use (3.7). Thus, denoting the roots of (3.7) by α_m, we obtain that the orthonormal eigenvectors are:

$$\sqrt{\frac{4\alpha_m \operatorname{ch}^2 \alpha_m}{H_{k,m}}}\left\{\left(\sin \alpha_m t + \frac{\cos \alpha_m}{\operatorname{ch} \alpha_m} \operatorname{sh} \alpha_m t,\right)\varphi_k, 2\alpha_m \cos \alpha_m \varphi_k\right\}.$$

Denote

$$H_{k,m} = 2\alpha_m \operatorname{ch}^2 \alpha_m - \sin 2\alpha_m \operatorname{ch}^2 \alpha_m + 8\alpha_m^3 \cos^2 \alpha_m \operatorname{ch}^2 \alpha_m + \cos^2 \alpha_m \operatorname{sh}^2 \alpha_m - 2\alpha_m \cos^2 \alpha_m - 8\frac{\gamma_k}{\alpha_m} \cos^2 \alpha_m \operatorname{ch}^2 \alpha_m.$$

Now, we shall investigate the behavior of the roots of equation (3.7). We begin by rewriting it as

$$2(z^4 + \gamma_k) + z \operatorname{tg} z - z \operatorname{th} z = 0. \tag{3.9}$$

For real roots, we get that

$$\alpha_m = -\frac{\pi}{2} + \pi m + O\left(\frac{1}{m}\right) \quad \text{when } m \to \infty. \tag{3.10}$$

Then, the eigenvalues corresponding to the roots are $\alpha_m^4 + \gamma_k$.

In order to verify whether there is any imaginary root, take, in (3.7), $z = iy$, $y > 0$:

$$tgy = \frac{y - 2(y^4 + \gamma_k) \operatorname{cth} y}{y \operatorname{cth} y},$$

from which the roots of this equation behave like

$$\alpha_m = -\frac{\pi}{2} + \pi m + O\left(\frac{1}{m}\right). \tag{3.11}$$

Thus, imaginary roots are $\beta_m = i\alpha_m$.

Now, we look for roots of the form $y + iy$ $(y > 0)$ since the fourth degree of that number is real. Hence, taking $z = y + iy$ in (3.7), we have

$$-(y+iy)\cos(y+iy)\operatorname{sh}(y+iy) + 2\cos(y+iy)\operatorname{ch}(y+iy)(\gamma_k - 4y^4) + (y+iy)\operatorname{ch}(y+iy)\sin(y+iy) = 0,$$

or

$$
\begin{aligned}
-(y+iy)\frac{e^{iy-y}+e^{-iy+y}}{2}\frac{e^{y+iy}-e^{-y-iy}}{2}& \\
+2\frac{e^{iy-y}+e^{-iy+y}}{2}\frac{e^{y+iy}+e^{-y-iy}}{2}(\gamma_k-4y^4)& \\
+(y+iy)\frac{e^{y+iy}+e^{-y-iy}}{2}\frac{e^{iy-y}+e^{-iy+y}}{2i}&=0,
\end{aligned}
$$

which simplifies

$$
\begin{aligned}
-(y+iy)\frac{e^{2iy}-e^{-2iy}+e^{2y}-e^{-2y}}{4}& \\
+\frac{e^{2iy}+e^{-2iy}+e^{-2y}+e^{2y}}{2}(\gamma_k-4y^4)& \\
+(y+iy)\frac{e^{2iy}+e^{-2iy}-e^{2y}+e^{-2y}}{4i}&=0.
\end{aligned}
$$

From the last equation, we obtain

$$
\begin{aligned}
-(y+iy)\frac{1}{2}(i\sin 2y+\operatorname{sh}^2 y)(\gamma_k-4y^4)(\cos 2y+\operatorname{ch}^2 y)& \\
+(y+iy)\frac{1}{2}\left(\sin 2y-\frac{\operatorname{sh}^2 y}{i}\right)&=0.
\end{aligned}
$$

Thus, by opening the brackets, we finally obtain

$$
2y\sin 2y-2y\operatorname{sh}^2 y=2(4y^4-\gamma_k)(\cos 2y+\operatorname{ch}^2 y). \tag{3.12}
$$

Expanding the trigonometric functions in equation (3.12) into a power series, we have:

$$
\begin{aligned}
y\bigg(2y-\frac{(2y)^3}{3!}+\frac{(2y)^5}{5!}-\frac{(2y)^7}{7!}-\cdots-2y-\frac{(2y)^3}{3!}-\frac{(2y)^5}{5!}-\cdots\bigg)& \\
=(4y^4-\gamma_k)\bigg(1-\frac{(2y)^2}{2!}+\frac{(2y)^4}{4!}-\frac{(2y)^6}{6!}+\frac{(2y)^8}{8!}-\cdots& \\
+1+\frac{(2y)^2}{2!}+\frac{(2y)^4}{4!}+\cdots\bigg)&
\end{aligned}
$$

or

$$
y\left[\frac{(2y)^3}{3!}+\frac{(2y)^7}{7!}+\frac{(2y)^{11}}{11!}+\cdots\right]=(\gamma_k-4y^4)\left[\frac{(2y)^4}{4!}+\frac{(2y)^8}{8!}+\cdots\right].
$$

By using σ, we have

$$y\sum_{n=1}^{\infty}\frac{(2y)^{4n-1}}{(4n-1)!}=(\gamma_k-4y^4)\sum_{n=1}^{\infty}\frac{(2y)^{4n}}{(4n)!}, \tag{3.13}$$

or

$$\frac{y(4-2\gamma_k)}{2\cdot 4!} + \sum_{n=1}^{\infty}\frac{y^{4n+4}(4n+4-2\gamma_k+8(4n+1)(4n+2)(4n+3)(4n+4))}{2(4n+4)!}=0. \tag{3.13a}$$

Clearly, this series has one sign change of coefficients which turn positive after some n value. Thus, by Descarte's rule of signs, equation (3.13)–(3.13a) has exactly one positive root (by Descarte's rule that the number of positive roots and sign changes of coefficients is the same).

Find the asymptotics of the roots. Rewrite equation (3.12) as

$$\frac{\sin 2y-\operatorname{sh}^2 y}{\cos 2y+\operatorname{ch}^2 y}=\frac{4y^4-\gamma_k}{y}, \tag{3.14}$$

or

$$\frac{\sum_{n=0}^{\infty}(2y)^{4n+1}/(4n+1)!}{\sum_{n=0}^{\infty}(2y)^{4n}/(4n)!}=\frac{\gamma_k-4y^4}{y}. \tag{3.15}$$

Obviously, the root of (3.15) must satisfy $\gamma_k-4y^4>0,\ y<\sqrt[4]{\gamma_k/4}$, since the left hand side of the equation is positive. Hence,

$$2y\frac{\sum_{n=0}^{\infty}(2y)^{4n}/(4n+1)!}{\sum_{n=0}^{\infty}(2y)^{4n}/(4n)!}=\frac{\gamma_k-4y^4}{y}.$$

Denote

$$\frac{\sum_{n=0}^{\infty}(2y)^{4n+1}/(4n+1)!}{\sum_{n=0}^{\infty}(2y)^{4n}/(4n)!}=\alpha(y),$$

$2\alpha(y)y^2 = \gamma_k - 4y^4$, where $\alpha(y) < 1$ and $\alpha(y)$ is close to 1. For large k values

$$y^2 \sim \frac{\sqrt{16\gamma_k + 4\alpha^2(y)} - 2\alpha(y)}{8}.$$

Thus, for eigenvalues of L_0,

$$\lambda_k \sim \alpha(y)\gamma_k. \tag{3.16}$$

Taking in (3.7) $z = y - iy$, after simplifications, we come to equation (3.14). Thus, the eigenvalues corresponding to the roots $y + iy$ are the same as those corresponding to roots of the form $y - iy$.

Equation (3.7) cannot have other complex roots since, otherwise, the self-adjoint operator associated with scalar problem (3.1)–(3.4) would have complex eigenvalues.

Hence, we have proved the next theorem.

Theorem 3.1. *The multiplicity of eigenvalues of L_0 is two, and*

$$\lambda_{k,m} = \gamma_k + \left(\frac{\pi}{2} + \pi m + O\left(\frac{1}{m}\right)\right)^4, \quad m \to \infty, \tag{3.17}$$

$$\lambda_k \sim \alpha(y)\sqrt{\gamma_k} \text{ where } \alpha(y) < 1 \text{ and close to } 1. \tag{3.18}$$

Using Theorem 1.1, by the method of [**21**], the next lemma can be easily proven.

Lemma 3.2. *If the eigenvalues of operator A for large k values satisfy $\gamma_k \sim ak^\alpha (a > 0,\ \alpha > 0)$, then, for the eigenvalues of L_0 and L, we have*

$$\lambda_n \sim \mu_n \sim an^{4\alpha/(\alpha+4)}.$$

Note that, in [**21**], imaginary roots of the characteristic equation form an unbounded sequence resulting in three different asymptotic behaviors of λ_n depending upon α.

4. Trace formula. Now, we shall turn to deriving the trace the formula for the operator L.

In [**22**], the next theorem is proven.

Theorem 4.1. *Given operators A_0 and B so that $A_0^{-1} \in \sigma_1$, $D(A_0) \subset D(B)$ and supposing that a number $\delta \in [0,1)$ exists such that $BA_0^{-\delta}$ is bounded and $\omega \in [0,1)$, $\omega+\delta < 1$, $A_0^{-(1-\delta-\omega)}$ is the trace class operator. Then, there exists a subsequence $\{n_m\}_{m=1}^{\infty}$ of natural numbers so that*

$$\lim_{m\to\infty} \sum_{j=0}^{n_m} (\mu_j - \lambda_j - (B\varphi_j, \varphi_j)) = 0,$$

where $\{\mu_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ are eigenvalues of the operators A_0+B, and A_0 and $\{\varphi_j\}_{j=1}^{\infty}$ are eigenvectors of A_0.

Taking $A_0 = L_0$ and $A_0+B = L$, $B = Q$, the validity of the theorem for L is easily seen, if

$$\alpha > \frac{4}{3}. \tag{4.1}$$

(This estimation is found from condition $L_0^{-1} \in \sigma_1$, which means that eigenvalues of L_0^{-1} form a convergent series; thus, $4\alpha/(\alpha+4) > 1$, yielding (4.1).)

Now, we prove the next lemma, which plays an important role in obtaining the trace formula.

Lemma 4.2. *Under assumptions* (1)–(3) *from Section* 1, *the series follows*:

$$\sum_{k=1}^{\infty}\Bigg\{\sum_{m=0}^{\infty} H_{k,m}\Bigg[\int_0^1 q_k(t)\sin^2(\alpha_k t)\,dt + \int_0^1 2q_k(t)\sin(\alpha_k t)\,\mathrm{sh}(\alpha_k t)\frac{\cos\alpha_k}{\mathrm{ch}\,\alpha_k}\,dt + \int_0^1 q_k(t)\frac{\cos^2\alpha_k}{\mathrm{ch}^2\,\alpha_k}\,\mathrm{sh}^2(\alpha_k t)\,dt\Bigg]$$
$$+ \sum_{m=0}^{\infty} H'_{k,m}\Bigg[\int_0^1 q_k(t)\sin^2(\beta_k t)\,dt + \int_0^1 2q_k(t)\sin(\beta_k t)\,\mathrm{sh}(\beta_k t)\frac{\cos\beta_k}{\mathrm{ch}\,\beta_k}\,dt + \int_0^1 q_k(t)\frac{\cos^2\beta_k}{\mathrm{ch}^2\,\beta_k}\,\mathrm{sh}^2(\beta_k t)\,dt\Bigg]\Bigg\}.$$

(*Here $H'_{k,m}$ is the same as $H_{k,m}$ with β_k instead of α_k and α_0, and β_0 are the roots of form $y \pm iy$*) *converges absolutely.*

Proof. Consider the series with the terms

$$H_{k,m}\left[\int_0^1 q_k(t)\sin^2(\alpha_m t)\,dt\right] = -H_{k,m}\int_0^1 q_k(t)\frac{\cos(2\alpha_m t)}{2}\,dt. \tag{4.2}$$

Here, we use condition (2) on $q(t)$. Since

$$\left|\int_0^1 q_k(t)\cos 2\alpha_m(t)\,dt\right| \le \int_0^1 |q_k(t)|\,dt \le \|q_k(t)\|_{\sigma_1};$$

thus, $\sum_{k=1}^{\infty}\int_0^1 q_k(t)\,dt <$ const. On the other hand, from the asymptotics in (3.10), $H_{k,m} \sim C_1/\alpha_m^2$ and $H'_{k,m} \sim C_2/\beta_m^2$ Therefore, we obtain the double sum with terms (4.2) convergent by condition (3) and asymptotics (3.10) and (3.11). Consider the double series with the terms

$$2H_{k,m}\frac{\cos\alpha_m}{\operatorname{ch}\alpha_m}\int_0^1 q_k(t)\sin(\alpha_m t)\operatorname{sh}(\alpha_m t)\,dt. \tag{4.3}$$

For large m values, this is equivalent to

$$\frac{2\cos\alpha_m}{\operatorname{ch}(\alpha_m)\alpha_m^2}\int_0^1 q_k(t)\sin(\alpha_m t)\operatorname{sh}(\alpha_m t)\,dt. \tag{4.4}$$

Since

$$\left|\frac{\operatorname{sh}(\alpha_m t)}{\operatorname{ch}\alpha_m}\right| < 1 \quad \text{for } t \in [0,1],$$

from the condition $q(t) \in \sigma_1$ and the asymptotics of α_m follows convergence of the series with the terms in (4.3). Then, for large m values,

$$H_{k,m}\int_0^1 q_k(t)\frac{\cos^2\alpha_m}{\operatorname{ch}^2\alpha_m}\operatorname{sh}^2(\alpha_m t)\,dt \sim \frac{1}{4\alpha_m^2}\int_0^1 q_k(t)\frac{\cos^2\alpha_m}{\operatorname{ch}^2\alpha_m}\operatorname{sh}^2(\alpha_m t)\,dt. \tag{4.5}$$

Relation (4.5), asymptotics of α_m and condition (1.3) yield convergence of the series with the terms in (4.5).

Using the asymptotics (3.11), convergence of the series could also be justified with terms β_m instead of α_m. □

From Theorem 1.1, Theorem 4.1 and Lemma 4.2 we obtain:

$$\lim_{m\to\infty}\sum_{k=1}^{n_m}(\mu_k - \lambda_k)$$

$$= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} H_{k,m}\left[\int_0^1 q_k(t)\cos(2\alpha_m t)\,dt - \int_0^1 q_k(t)\frac{2\cos\alpha_m}{\operatorname{ch}\alpha_m}\sin(\alpha_m t)\,dt - \int_0^1 \frac{\cos^2\alpha_m}{\operatorname{ch}^2\alpha_m} q_k(t)\operatorname{sh}^2(\alpha_m t)\,dt\right]$$

$$+ \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} H'_{k,m}\left[\int_0^1 q_k(t)\cos(2\beta_m t)\,dt - \int_0^1 q_k(t)\frac{2\cos\beta_m}{\operatorname{ch}\beta_m}\sin(\beta_m t)\,dt - \int_0^1 \frac{\cos^2\beta_m}{\operatorname{ch}^2\beta_m} q_k(t)\operatorname{sh}^2(\beta_m t)\,dt\right].$$

For evaluating the sum of the first series, we select the function of the complex variable $G(z)$ to be:

$$G(z) = \frac{\cos(2zt)}{((\operatorname{tg} z/z) - (\operatorname{th} z/z) + 2z^2 + 2(\gamma_k/z^2))z\cos^2 z}.$$

It is easy to see that it has poles at the roots of equation (3.7). Thus,

$$\operatorname*{res}_{z=\alpha_m} G(z) = \frac{\cos(2\alpha_m t)}{((\operatorname{tg} z/z) - (\operatorname{th} z/z) + 2z^2 + 2(\gamma_k/z^2))'|_{z=\alpha_m}\alpha_m\cos^2\alpha_m}$$
$$= -H_{k,m}\cos(2\alpha_m t),$$

$$\operatorname*{res}_{z=\beta_m} G(z) = \frac{\cos(2\beta_m t)}{((\operatorname{tg} z/z) - (\operatorname{th} z/z) + 2z^2 + 2(\gamma_k/z^2))'|_{z=\beta_m}\beta_m\cos^2\beta_m}$$
$$= -H'_{k,m}\cos(2\beta_m t),$$

Other poles of that function are $\pi/2 + \pi k$, which are zeros of $\cos z$ and zero. Residues at the points $\pi/2 + \pi k$ are

$$\operatorname*{res}_{z=\pi/2+\pi k} G(z) = \cos((2k+1)t\pi).$$

Take as the contour of integration the rectangular contour C_N, with vertices at $A_N \pm iB_m$, where $A_N = \pi N$, $B_m = \pi m$. Let it bypass the imaginary roots of (3.7), with β_ms along the semicircle from the right and $-\beta_m$s and zero (also roots of equation (1.1) since the left hand side function is odd) from the left. Consider

$$\lim_{r\to 0}\int\limits_{\substack{0\le\varphi\le\pi/2\\ z=re^{i\varphi}}}\int\limits_0^1 \frac{\cos 2zt\ q_k(t)\,dt}{\sin z\cos z-\ \operatorname{th} z\cos^2 z+2z^3\cos^2 z+2(\gamma_k/z)\cos^2 z}\,dz$$

$$=\lim_{r\to 0}\int\limits_{\substack{0\le\varphi\le\pi/2\\ z=re^{i\varphi}}}\int\limits_0^1 \frac{2[1-(2zt)^2/2!+(2zt)^4/4!\cdots]q_k(t)\,dt}{\sin 2z-\operatorname{th} z(1+\cos 2z)+2z^3(1+\cos 2z)+2(\gamma_k/z)\cos^2 z}dz$$

$$=\lim_{r\to 0}\int\limits_{\substack{0\le\varphi\le\pi/2\\ z=re^{i\varphi}}}\int\limits_0^1 \frac{2[1-(2zt)^2/2!+(2zt)^4/4!\cdots]\,dt\,dz}{K(z)}$$

$$=\int\limits_{\substack{0\le\varphi\le\pi/2\\ z=re^{i\varphi}}}\int\limits_0^1 \frac{-(2zt)^2/2!q_k(t)\,dt}{4\gamma_k};$$

here,

$$\begin{aligned}K(z)&=2z-(2z)^3/3!+\cdots-[z-1/3z^3+(2/15)z^5+\cdots)\\&\quad\cdot[2-(2z)^2/2!+\cdots]+2z^3(2-(2z)^2/2!+\cdots]\\&\quad+2\gamma_k/z\cos^2 z(2-(2z)^2/2!+\cdots),\end{aligned}$$

which vanishes under the condition $\int_0^1 t^2 q_k(t)\,dt<\infty$. By using the asymptotics

$$G(z)\sim\frac{2\cos 2zt}{z^3\cos 2z},$$

when $|z|\to\infty$, it can be shown that the integral along the selected contour vanishes.

For evaluating the sums for the second and third terms of the series, we select the functions

$$F(z)=\frac{\sin tz\operatorname{sh} tz}{\operatorname{ch} z((\operatorname{tg} z/z)-(\ \operatorname{th} z/z)+2z^2+2(\gamma_k/z^2))z\cos z}$$

and

$$g(z) = \frac{\operatorname{sh}^2 zt}{\operatorname{ch}^2 z((\operatorname{tg} z/z) - (\operatorname{th} z/z) + 2z^2 + 2(\gamma_k/z^2))z}.$$

By using asymptotics for large $|z|$ values, it can be shown that the integrals along extended contours vanish. Therefore,

$$\lim_{N\to\infty} \int_0^1 \int_{C_N} G(z)\, dz q_k(t)\, dt = \lim_{N\to\infty} \int_0^1 \sum_{n=1}^{N} \cos(2n+1)\pi t q_k(t)\, dt$$
$$= \sum_{k=1}^{\infty} \frac{q_k(\pi) - q_k(0)}{4} = \frac{\operatorname{tr} q(\pi) - \operatorname{tr} q(0)}{4}.$$

Thus, the next theorem is proven.

Theorem 4.3. *Under conditions* (1)–(3), *the trace of operator* L *is*

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{\operatorname{tr} q(\pi) - \operatorname{tr} q(0)}{4}.$$

REFERENCES

1. B.A. Aliev, *Asymptotic bahavior of eigenvalues of one boundary value problem for the elliptic differential operator equation of second order*, Ukrainian J. Math. **58** (2006), 1146–1153.

2. N.M. Aslanova, *Trace formula of one boundary value for operator Sturm-Liouville equation*, Siberian J. Math **49** (2008), 1207–1215.

3. ______, *Investigation of spectrum and trace formula of operator Bessel equation*, Siberian Math J. **51** (2010), 721–737.

4. ______, *Study of the asymptotic eigenvalue distribution and trace formula of a second order operator-differential equation*, Bound. Value Prob. **2011** (2011).

5. N.M. Aslanova and M. Bayramoglu, *On generalized regularized trace formula of fourth order differential operator with operator coefficient*, Ukrainian J. Math. **66** (2014), 128–134.

6. M. Bayramoglu and N.M. Aslanova, *Eigenvalue distribution and trace formula of operator Sturm-Liouville equation*, Ukrainian J. Math. **62** (2010), 867–877.

7. ______, *Formula for second regularized trace of a problem with spectral parameter dependent boundary condition*, Hacettepe J. Math. Stat. **40** (2011), 635–647.

8. P.A. Binding, P.J. Browne and B.A. Watson, *Sturm-Liuville problems with boundary conditions rationally dependent on the eigen parameter*, J. Comp. Appl. Math. **148** (2002), 147–161.

9. S. Çalışkan, A. Bayramov, Z. Oer and S. Uslu, *Eigenvalues and eigenfunctions of discontinuous two-point boundary value problems with an eigenparameter in the boundary condition*, Rocky Mountain J. Math **43** (2013), 1871–1892.

10. A.G. Costyuchenko and B.M. Levitan, *On asymptotics of eigenvalue distribution of operator Sturm-Liouville equation*, Funct. Anal. Appl. **1** (1967), 86–96.

11. V.V. Dubrovskii, *Abstract trace formulas of elliptic smooth differential operators on compact manifolds*, Diff. Eqs. **27** (1991), 2164–2166.

12. C.G. Fulton, *Two point boundary value problems with eigenparameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh **77** (1977), 293–308.

13. I.M. Gelfand and B.M. Levitan, *On one identity on for eigenvalues of second order x differential operator*, Dokl. Akad. Nauk **88** (1953), 593–596.

14. V.I. Gorbachuk, *On asymptotics of eigenvalue distribution of boundary value problems for differential equations in class of vector-functions*, Ukrainian J. Math. **15** (1975), 657–664.

15. V.I. Gorbachuk and M.L. Gorbachuk *On some class of boundary value problems for Sturm-Liouville equation with operator equation*, Ukrainian J. Math. **24** (1972), 291–305.

16. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin 1984.

17. I.L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and their applications*, Mir, Moscow, 1971.

18. F.G. Maksudov, M. Bayramoglu and A.A. Adigezalov, *On regularized trace of Stutrm-Liouville operator on finite segment with unbounded operator coefficient*, Dokl. Akad. Nauk **277** (1984), 795–799.

19. M.A. Naymark, *Linear differential operators*, NAUKA, Moscow, 1969.

20. E.M. Russakovskii, *Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions*, Funct. Anal. Appl. **9** (1975), 358–359.

21. M.A. Rybak, *On asymptotics of eigenvalue distribution of some boundary value problems for Sturm-Liouville operator equation*, Ukrainian J. Math. **32** (1980), 248–252.

22. V.A. Sadovnichii and V.E. Podolskii, *Traces of operators with relatively compact perturbation*, Mat. Sbor. **193** (2002), 129–152.

Azerbaijan University of Architecture and Construction, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Differential equation Dept., 9, F. Agayev str., AZ1141, Baku, Azerbaijan
Email address: **nigar.aslanova@yahoo.com**

Azerbaijan University of Architecture and Construction, Institute of Mathematics and Mechanics of NAS of Azerbaijan, Differential equation Dept., 9, F. Agayev str., AZ1141, Baku, Azerbaijan
Email address: **mamed.bayramoglu@yahoo.com**

Azerbaijan State University of Economics (UNEC), 6, Istiglaliyyat str., AZ 1001, Baku, Azerbaijan
Email address: **xaligaslanov@yandex.ru**