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SMALE SPACES FROM SELF-SIMILAR GRAPH ACTIONS

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ABSTRACT. We show that, for contracting and regular self-similar graph actions, the shift maps on limit spaces are positively expansive local homeomorphisms. From this, we find that limit solenoids of contracting and regular self-similar graph actions are Smale spaces and that the unstable Ruelle algebras of the limit solenoids are strongly Morita equivalent to the Cuntz-Pimsner algebras by Exel and Pardo if self-similar graph actions satisfy the contracting, regular, pseudo free and G-transitive conditions.

1. Introduction. Exel and Pardo [**4**] generalized self-similar groups of Nekrashevych [**9, 10**] to self-similar graph actions. For a self-similar group (G, X), Nekrashevych constructed two dynamical systems $(\mathcal{J}_G, \sigma)$, called the limit dynamical system, and $(\mathcal{S}_G, \widehat{\sigma})$, called the limit solenoid, and two associated C^*-algebras $\mathcal{O}_G$ and $\mathcal{O}_\sigma$. Here, $\mathcal{O}_G$ is a universal Cuntz-Pimsner algebra with a correspondence over $C^*_r(G)$, and $\mathcal{O}_\sigma$ is a groupoid algebra of the Deaconu groupoid from $(\mathcal{J}_G, \sigma)$. Then, he showed that the limit solenoid of (G, X) is a Smale space and that the stable Ruelle algebra and the unstable Ruelle algebra, respectively, of the limit solenoid are strongly Morita equivalent to $\mathcal{O}_\sigma$ and $\mathcal{O}_G$, respectively. On the other hand, for a self-similar graph action (G, E), Exel and Pardo [**3**] defined a C^*-algebra $\mathcal{O}_{G,E}$ which is $*$-isomorphic to a Cuntz-Pimsner algebra for a correspondence over $C(E^0) \rtimes G$. They then showed that $\mathcal{O}_{G,E}$ includes $\mathcal{O}_G$ as a special case. Moreover, the limit dynamical system $(J_{(G,E)}, \sigma)$ and its Deaconu groupoid algebra $C^*(\Gamma_{(G,E)})$ are studied in [**18**] following Nekrashevych's development. Although the topological structure of $J_{(G,E)}$ is different from that of $\mathcal{J}_G$, it turned out that $(J_{(G,E)}, \sigma)$ and its groupoid algebra $C^*(\Gamma_{(G,E)})$ are natural generalizations of $(\mathcal{J}_G, \sigma)$ and $\mathcal{O}_\sigma$. Therefore, it is rational to expect that the limit solenoid of

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a self-similar graph action (G, E) under suitable conditions is a Smale space and that $\mathcal{O}_{G,E}$ and $C^*(\Gamma_{(G,E)})$ are related to Ruelle algebras of the limit solenoid.

In this paper, we show that the limit solenoid is a Smale space and that $\mathcal{O}_{G,E}$ is strongly Morita equivalent to the unstable Ruelle algebra if (G, E) is a contracting, regular and pseudo free self-similar graph action and E is G-transitive. The main techniques used here are positive expansiveness of the shift maps in the limit dynamical systems and groupoid equivalence in the sense of Muhly, Renault and Williams **[8]**. When

$$\sigma\colon J_{(G,E)} \longrightarrow J_{(G,E)}$$

is a surjective positively expansive map, the inverse limit system induced from $(J_{(G,E)}, \sigma)$, which is topologically conjugate to the limit solenoid, is a Smale space (see **[11, 16]**). For the unstable Ruelle algebra of the limit solenoid and $\mathcal{O}_{G,E}$, which is $*$-isomorphic to a groupoid algebra, we borrow ideas from **[12]** to reduce the groupoid for the unstable Ruelle algebra on a transversal that is determined by a fixed left-hand-sided infinite path. Then, it becomes much easier to compare the groupoid algebras using strong Morita equivalence.

2. Self-similar graph actions. We introduce the basic definitions and properties of self-similar graph actions to be used later. All material in this section is taken from **[4, 9, 10]**. The reader is referred to these for more details.

2.1. Directed graphs. Suppose that $E = (E^0, E^1, d, r)$ is a directed graph where E^0 is the set of vertices, E^1 is the set of edges and d and r are domain and range maps, respectively. A directed graph E is called *finite* if E^0 and E^1 are finite sets. A vertex is called a *sink* if it does not emit any edge and a *source* if it does not receive any edge.

Let E be a directed graph. A *finite path* of length $n \geq 1$ in E is a finite sequence

$$a = a_1 \cdots a_n$$

such that $a_i \in E^1$ and $r(a_i) = d(a_{i+1})$ for $i = 1, \ldots, n-1$. The *domain* of a is defined to be $d(a) = d(a_1)$ and the *range* of a is $r(a) = r(a_n)$. A vertex $v \in E^0$ is considered a path of length zero with $d(v) = r(v) = v$. For every integer $n \geq 0$, we denote by E^n the set of paths of length n in E and denote by E^* the set of finite paths in E, i.e.,

$$E^* = \bigcup_{n=0}^{\infty} E^n.$$

If a and b are paths in E such that $r(a) = d(b)$, then ab is the path obtained by concatenating a and b.

We consider the left-infinite path space and the right-infinite path space

$$E^{-\omega} = \{\cdots a_{-2}a_{-1} \colon a_i \in E^1 \text{ and } r(a_i) = d(a_{i+1})\}$$

and

$$E^{\omega} = \{a_0 a_1 \cdots \colon a_i \in E^1 \text{ and } r(a_i) = d(a_{i+1})\}.$$

We also use the two-sided-infinite path space

$$E^{\pm\omega} = \{\cdots a_{-2}a_{-1} \cdot a_0 a_1 \cdots \colon a_i \in E^1 \text{ and } r(a_i) = d(a_{i+1})\}.$$

The left-infinite path space $E^{-\omega}$ has the product topology of the discrete set E^1. For each $a \in E^*$, define the *cylinder set* $Z(a)$ as

$$Z(a) = \{\alpha \in E^{-\omega} \colon \alpha = \cdots a_{-n-1}a_{-n} \cdots a_{-1} \text{ such that } a_{-n} \cdots a_{-1} = a\}.$$

Then, the product topology on $E^{-\omega}$ has $\{Z(a) \colon a \in E^*\}$ as its basis. Similarly, the collection of appropriate cylinder sets are bases of topologies of E^{ω} and $E^{\pm\omega}$.

2.2. Self-similar graph actions. Let $E = (E^0, E^1, d, r)$ be a directed graph and G a group. An automorphism of E is a bijection

$$f \colon E^0 \cup E^1 \longrightarrow E^0 \cup E^1$$

such that, for $i = 0$ and 1, $f(E^i) \subset E^i$, $f \circ d = d \circ f$ and $f \circ r = r \circ f$ hold. We say that G acts on E if there is a group homomorphism from G to the group of graph automorphisms of E. We denote the (left) actions of G on E^0 and E^1 by

$$(g, v) \longmapsto g(v) \in E^0$$

and

$$(g, e) \longmapsto g(e) \in E^1$$

for $g \in G$, $v \in E^0$ and $e \in E^1$.

Definition 2.1 ([**4, 9, 10**])**.** Suppose that E is a finite directed graph with no sink nor source and that G is a group acting on E such that the induced action on E^* is *faithful*. We call the pair (G, E) a *self-similar graph action* if, for all $g \in G$ and $e \in E^1$, there exists a unique $h \in G$ such that

$$g(eb) = g(e)h(b)$$

for every $b \in E^*$ with $r(e) = d(b)$.

Remark 2.2 ([**2,** subsection 7.2])**.** The faithful condition of G-action implies the uniqueness of h in Definition 2.1. See [**3, 4**] for more general cases.

For all $g \in G$ and $a, b \in E^*$ such that $ab \in E^*$, by induction, there is a unique element $h \in G$ such that $g(ab) = g(a)h(b)$. We call the unique element h the *restriction* of g at a and denote it by $g|_a$. For $c = g(a)$ and $h = g|_a$, we formally write the above equality as

$$g \cdot a = c \cdot h.$$

We will need the following basic properties of restrictions [**4, 9, 10**]: for $g, h \in G$ and $a, b \in E^*$,

$$g|_{ab} = (g|_a)|_b, \qquad (gh)|_a = g|_{h(a)}h|_a, \qquad (g|_a)^{-1} = g^{-1}|_{g(a)}.$$

Standing assumption. In this paper, we assume that every graph is a connected finite directed graph with no sink nor source, and every group is a finitely generated countable group.

2.3. Universal C^*-algebra $\mathcal{O}_{G,E}$. For a self-similar graph action (G, E), $\mathcal{O}_{G,E}$ is the universal unital C^*-algebra generated by a set

$$\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\} \cup \{u_g : g \in G\}$$

subject to the following relations:

(1) $\{p_x : x \in E^0\}$ is a set of mutually orthogonal projections;
(2) $\{s_e : e \in E^1\}$ is a set of partial isometries;
(3) $s_e^* s_e = p_{d(e)}$ for every $e \in E^1$;
(4) $p_x = \sum_{e \in r^{-1}(x)} s_e s_e^*$ for every $x \in E^0$ where $r^{-1}(x)$ is finite and nonempty;

(5) the map $u\colon G \to \mathcal{O}_{G,E}$ defined by $g \mapsto u_g$ is a unitary representation of G;
(6) $u_g s_e = s_{g(e)} u_{g|_e}$ for every $g \in G$ and $e \in E^1$; and
(7) $u_g p_x = p_{g(x)} u_g$ for every $g \in G$ and $x \in E^0$.

See [**3, 4**] for more details regarding $\mathcal{O}_{G,E}$.

Remark 2.3 ([**4,** Proposition 8.1])**.** We can extend the action of G on E^* to E^ω: for every $g \in G$, $\xi = x_0 x_1 \cdots \in E^\omega$ and $n \geq 0$, $g(\xi) = \eta = y_0 y_1 \cdots \in E^\omega$ is defined as

$$g(x_0 \cdots x_n) = y_0 \cdots y_n.$$

We will need the following properties of self-similar graph actions.

Definition 2.4 ([**4, 9, 10**])**.** Suppose that (G, E) is a self-similar graph action.

(1) We say that (G, E) is *contracting* if there is a finite subset N of G satisfying the following: for every $g \in G$, there is an $n \geq 0$ such that $g|_a \in N$ for every $a \in E^*$ of length $|a| \geq n$. If the action is contracting, the smallest finite subset of G satisfying this condition is called the *nucleus* of the group and is denoted by $\mathcal{N}$.

(2) We say that (G, E) is *regular* if, for every $g \in G$ and every $w \in E^\omega$, either $g(w) \neq w$ or there is a neighborhood of w such that every point in the neighborhood is fixed by g.

(3) We say that (G, E) is *pseudo free* if $\mathrm{Fix}_g = \{a \in E^*\colon g(a) = a\}$ is a finite set for every $g \in G$.

(4) We say that E is *G-transitive* if, for any two vertices u and v of E, there is a finite sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that, for each $i \in \{1, \ldots, n\}$, either there is a $g_i \in G$ such that

$$g_i(u_{i-1}) = u_i,$$

or there is a path $a_i \in E^*$ such that

$$d(a_i) = u_{i-1} \quad \text{and} \quad r(a_i) = u_i.$$

Definition 2.5 ([**9, 10**])**.** Two paths $\cdots a_{-2} a_{-1}$ and $\cdots b_{-2} b_{-1}$ in $E^{-\omega}$ are said to be *asymptotically equivalent* if there are a finite set $I \subset G$

and a sequence $g_n \in I$ such that

$$g_n(a_{-n} \cdots a_{-1}) = b_{-n} \cdots b_{-1}$$

for every $n \in \mathbb{N}$.

For two-sided infinite space $E^{\pm\omega}$, we say that two paths $\cdots a_{-2}a_{-1} \cdot a_0a_1 \cdots$ and $\cdots b_{-2}b_{-1} \cdot b_0b_1 \cdots$ are *asymptotically equivalent* if there are a finite set $H \subset G$ and a sequence $h_n \in H$ such that

$$h_n(a_na_{n+1} \cdots) = b_nb_{n+1} \cdots$$

for every $n \in \mathbb{Z}$.

Remark 2.6. When (G, E) is a contracting self-similar graph action, we can use the nucleus $\mathcal{N}$ of G, instead of the arbitrary finite subset of G, to determine asymptotic equivalence. See [**10,** subsection 2.3] for details.

2.4. Limit dynamical systems. The quotient of $E^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of (G, E) and is denoted by $J_{(G,E)}$. Since the asymptotic equivalence relation is invariant under the shift map

$$\sigma \colon E^{-\omega} \longrightarrow E^{-\omega},$$

defined by

$$\cdots a_{-2}a_{-1} \longmapsto \cdots a_{-3}a_{-2},$$

the shift map σ induces a continuous surjection on $J_{(G,E)}$. By abuse of notation, we denote the induced map on $J_{(G,E)}$ by σ if there is no confusion. The dynamical system $(J_{(G,E)}, \sigma)$ is called the *limit dynamical system* of (G, E). See [**9, 10**] for details.

Theorem 2.7 ([**18,** Lemma 2.9, Proposition 3.1])**.** *If (G, E) is a self-similar graph action, then*:

(1) *$J_{(G,E)}$ is a compact metrizable space, and*

(2) *$\sigma \circ q = q \circ \sigma$ where $q \colon E^{-\omega} \to J_{(G,E)}$ is the quotient map.*

Theorem 2.8 ([**18,** Lemma 5.4])**.** *If (G, E) is a contracting and regular self-similar graph action, then $\sigma \colon J_{(G,E)} \to J_{(G,E)}$ is a surjective local homeomorphism.*

2.5. Limit solenoids. Suppose that (G, E) is a self-similar graph action with the two-sided infinite path space $E^{\pm\omega}$. We denote the quotient of $E^{\pm\omega}$ by the asymptotic equivalence relation by $S_{(G,E)}$. Then, the shift map σ on $E^{\pm\omega}$ induces a homeomorphism of $S_{(G,E)}$, which is also denoted σ. We call the dynamical system $(S_{(G,E)}, \sigma)$ the *limit solenoid* of the self-similar graph action (G, E).

The proofs of the following properties of limit solenoids are identical to those of [**10,** Proposition 2.4] and [**18,** Lemma 2.9].

Theorem 2.9. *If (G, E) is a self-similar graph action, then*:

(1) *$S_{(G,E)}$ is a compact metrizable space, and*

(2) *$\sigma \circ q = q \circ \sigma$ where $q\colon E^{\pm\omega} \to S_{(G,E)}$ is the quotient map.*

Suppose that $(J_{(G,E)}, \sigma)$ is the limit dynamical system of (G, E). We define the inverse limit of $(J_{(G,E)}, \sigma)$

$$\varprojlim(J_{(G,E)}, \sigma) = \{(x_0, x_1, x_2, \ldots)\colon x_i \in J_{(G,E)} \text{ and } \sigma(x_i) = x_{i-1}\}.$$

Then, $\varprojlim(J_{(G,E)}, \sigma)$ carries an induced homeomorphism, which we also denote as σ, given by

$$\sigma\colon (x_0, x_1, x_2, \ldots) \longmapsto (\sigma(x_0), x_0, x_1, x_2, \ldots).$$

Theorem 2.10 ([**9,** Proposition 5.7.3])**.** *The limit solenoid $(S_{(G,E)}, \sigma)$ is topologically conjugate to the inverse limit system $(\varprojlim(J_{(G,E)}, \sigma), \sigma)$.*

3. Quotient maps and shift maps.

For a self-similar graph action (G, E), we show that the quotient maps

$$q\colon E^{-\omega} \longrightarrow J_{(G,E)}$$

and

$$q\colon E^{\pm\omega} \longrightarrow S_{(G,E)}$$

are open maps and that the shift map

$$\sigma\colon J_{(G,E)} \longrightarrow J_{(G,E)}$$

is a positively expansive map if (G, E) is contracting and regular.

3.1. Quotient maps. Suppose that (G,E) is a self-similar graph action with the left-infinite path space $E^{-\omega}$. First, we consider a principal groupoid defined by the asymptotic equivalence relation on $E^{-\omega}$

$$H = \{(\xi,\eta) \in E^{-\omega} \times E^{-\omega} \colon \xi \text{ is asymptotically equivalent to } \eta\}.$$

Then, $E^{-\omega}$ is the unit space of H and its *coarse moduli space*

$$|H| = E^{-\omega}/H = \{[\xi] \colon \xi \in E^{-\omega}\}$$

is $J_{(G,E)}$. Here, $[\xi] = \{\eta \in E^{-\omega} \colon (\xi,\eta) \in H\}$, i.e., $\eta \in [\xi]$ if and only if $q(\eta) = q(\xi)$.

Remark 3.1. In order to give a locally compact Hausdorff topology on H, for each natural number n, we define a binary relation $\sim_n$ on $E^{-\omega}$ by $\xi \sim_n \eta$ if and only if

(1) there are $\xi', \eta' \in E^{-\omega}$ such that ξ' is asymptotically equivalent to η', and
(2) $\xi_{-n} \cdots \xi_{-1} = \xi'_{-n} \cdots \xi'_{-1}$ and $\eta_{-n} \cdots \eta_{-1} = \eta'_{-n} \cdots \eta'_{-1}$.

Then, it is easy to see that $\sim_n$ is an equivalence relation due to the asymptotical equivalence between ξ' and η'. Now, we let

$$H_n = \{(\xi,\eta) \in E^{-\omega} \times E^{-\omega} \colon \xi \sim_n \eta\}$$

with the subspace topology of $E^{-\omega} \times E^{-\omega}$.

Since the complement of H_n is open in $E^{-\omega} \times E^{-\omega}$, each H_n is a compact Hausdorff space satisfying $H \subset H_{n+1} \subset H_n$ with the inclusion map $i_n \colon H \to H_n$. We give H the initial topology induced from $(H_n, i_n)_{n\in\mathbb{N}}$. Then, H is a compact Hausdorff space by [**17,** Example 29.10, Theorem 29.11]. It is not difficult to verify that the initial topology on H is compatible with the groupoid structure. A left Haar system on H is described in [**14,** Example I.2.5(c)]. See [**14,** Section I.2] for more details.

We learned the following from an unpublished lecture note by Freed [**5**].

Proposition 3.2 ([**5,** Lemma 15.66])**.** *The quotient map* $q \colon E^{-\omega} \to J_{(G,E)}$ *is an open map.*

Proof. Let H be as above. We identify $E^{-\omega} = H^{(0)}$ and $J_{(G,E)} = |H|$. For every open set U in $E^{-\omega}$, $q(U)$ is open in $J_{(G,E)}$ if and only if $q^{-1} \circ q(U)$ is open in $E^{-\omega}$. On the other hand, the structure of the groupoid H implies $d \circ r^{-1}(U) = q^{-1} \circ q(U)$, where d and r are the domain and range maps, respectively, of H. Since H is a locally compact Hausdorff groupoid with a left Haar system, d and r are open maps by [**14,** Proposition I.2.4]. Hence, $d \circ r^{-1}(U)$ is an open subset of $E^{-\omega}$, and thus, is $q^{-1} \circ q(U)$ for every open set $U \subset E^{-\omega}$. Therefore, the quotient map q is an open map. □

When (G, E) is a contracting self-similar graph action such that E is an n-bouquet, every asymptotic equivalence class on $E^{-\omega}$ has no more than $|\mathcal{N}|$ elements by [**9,** Proposition 3.2.6]. We can obtain the same result for finite graphs.

Proposition 3.3. *Suppose that* (G, E) *is a contracting self-similar graph action. For each* $x \in J_{(G,E)}$, $|q^{-1}(x)| \leq |\mathcal{N}|$, *where* $|\cdot|$ *is the cardinality and* $\mathcal{N}$ *is the nucleus.*

Proof. We fix one element $\xi = \cdots x_{-n} \cdots x_{-1} \in q^{-1}(x)$ and consider an arbitrary element $\eta = \cdots y_{-n} \cdots y_{-1} \in E^{-\omega}$. Let

$$X = \{\{g_n\} \colon g_n \in \mathcal{N} \text{ for every } n \in \mathbb{N} \text{ and } g_{n-1} = g_n|_{x_{-n}} \text{ for every } n \geq 2\}.$$

Then, $\eta \in q^{-1}(x)$ if and only if there is at least one sequence $\{g_n\} \in X$ such that $g_n(x_{-n}) = y_{-n}$ for every $n \in \mathbb{N}$. Thus, we have an injective map $q^{-1}(x) \to X$ that sends each $\eta \in q^{-1}(x)$ to one of such sequences in X, which implies $|q^{-1}(x)| \leq |X|$.

In order to show $|X| \leq |\mathcal{N}|$, we consider X as a subset of $\prod \mathcal{N}$. For every $n \in \mathbb{N}$, let $X_n = \mathcal{N}$ and

$$p_n \colon \prod \mathcal{N} = \prod X_n \longrightarrow X_1 \times \cdots \times X_n$$

be the projection map given by

$$(g_1, \ldots, g_n, g_{n+1}, \ldots) \longmapsto (g_1, \ldots, g_n).$$

Due to

$$g_n|_{x_{-n}x_{-n+1}} = (g_n|_{x_{-n}})|_{x_{-n+1}} = g_{n-1}|_{x_{-n+1}} = g_{n-2},$$

we observe that, for $(g_1, \dots, g_n) \in p_n(X)$, g_n determines $g_{n-1}, \dots, g_1$. Therefore, $|p_n(X)| \leq |X_n| = |\mathcal{N}|$ for every $n \in \mathbb{N}$. Since a map $p_{n+1}(X) \to p_n(X)$ defined by $(g_1, \dots, g_n, g_{n+1}) \mapsto (g_1, \dots, g_n)$ is surjective, we have $|p_n(X)| \leq |p_{n+1}(X)|$. Thus, the sequence $\{|p_n(X)|\}$ is a bounded increasing sequence of natural numbers, and $\{|p_n(X)|\}$ is a convergent sequence by the monotone convergence theorem. Hence, there is a natural number N such that $|p_N(X)| = |p_{N+k}(X)|$ for every $k \geq 1$ since $\{|p_n(X)|\}$ is a convergent sequence of natural numbers. Then, for each $(g_1, \dots, g_N) \in p_N(X)$, there is a unique $(g_1, \dots, g_N, g_{N+1}) \in p_{N+1}(X)$ and, by induction, a unique $(g_1, \dots, g_N, \dots, g_{N+k}) \in p_{N+k}(X)$, for every $k \in \mathbb{N}$. Therefore, we can choose an element $(g_1, \dots, g_N, \dots, g_{N+k}, \dots) \in p_N^{-1}(g_1, \dots, g_N) \subset X$ for each $(g_1, \dots, g_N) \in p_N(X)$. We define

$$s_{N+k} \colon p_N(X) \longrightarrow p_{N+k}(X)$$

by

$$(g_1, \dots, g_N) \longmapsto (g_1, \dots, g_N, \dots, g_{N+k})$$

and

$$t \colon p_N(X) \longrightarrow X$$

by

$$(g_1, \dots, g_N) \longmapsto (g_1, \dots, g_N, \dots, g_{N+k}, \dots).$$

Then, it is clear that s_{N+k} is bijective and $p_{N+k} \circ t = s_{N+k}$ for every $k \in \mathbb{N}$.

Now, we show that $t \colon p_N(X) \to X$ is surjective. Then, we will have $|X| \leq |p_N(X)| \leq |\mathcal{N}|$. Assume that $t \colon p_N(X) \to X$ is not surjective. Then, $X \backslash t(p_N(X))$ is not an empty set so that there is an $h = (h_1, \dots, h_N, \dots) \in X \backslash t(p_N(X))$. When we compare h and each $(g_1, \dots, g_N, \dots) \in t(p_N(X))$, there is at least one index n such that $h_n \neq g_n$, i.e., $(h_1, \dots, h_n) \neq (g_1, \dots, g_n)$, so that, for every $k \in \mathbb{N}$,

$$(h_1, \dots, h_n, \dots, h_{n+k}) \neq (g_1, \dots, g_n, \dots, g_{n+k}).$$

Here, it is clear that $n > N$ due to the fact that $p_N(h) = (h_1, \dots, h_N) \in p_N(X)$. Since $t(p_N(X))$ has finitely many elements, there is a natural number K such that $(h_1, \dots, h_{N+K}) \neq (g_1, \dots, g_{N+K})$ for every $(g_1, \dots, g_N, \dots) \in t(p_N(X))$, i.e.,

$$(h_1, \dots, h_{N+K}) \notin p_{N+K} \circ t(p_N(X)).$$

However, $(h_1, \dots, h_{N+K}) = p_{N+K}(h) \in p_{N+K}(X)$ means that there exists at least one $(a_1, \dots, a_N, \dots, a_{N+K}) \in p_{N+K}(X)$ such that

$$\begin{aligned}(h_1, \dots, h_{N+K}) &= (a_1, \dots, a_N, \dots, a_{N+K}) \\ &= s_{N+K}(a_1, \dots, a_N) \\ &= p_{N+K} \circ t(a_1, \dots, a_N) \in p_{N+K} \circ t(p_N(X)),\end{aligned}$$

a contradiction. Hence, $t\colon p_N(X) \to X$ is a surjective map, which implies that $|X| \leq |p_N(X)| \leq |\mathcal{N}|$. Therefore, we have $|q^{-1}(x)| \leq |X| \leq |p_N(X)| \leq |\mathcal{N}|$ for every $x \in J_{(G,E)}$. □

By the same argument, we have similar results for the limit of the solenoid:

Proposition 3.4.

(1) *The quotient map* $q\colon E^{\pm\omega} \to S_{(G,E)}$ *is an open map.*

(2) *For each* $x \in S_{(G,E)}$, $|q^{-1}(x)| \leq |\mathcal{N}|$.

3.2. Shift maps. For a contracting and regular self-similar graph action (G, E), we show that the shift map $\sigma\colon J_{(G,E)} \to J_{(G,E)}$ is positively expansive.

Definition 3.5 (**[15]**)**.** Let (X, d) be a metric space. A continuous map $f\colon X \to X$ is called *positively expansive* if there exists a constant $\rho > 0$ such that, for any distinct points $x, y \in X$, there exists an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \rho$.

Suppose that X is a locally compact metrizable space with diagonal $\Delta = \{(x, x)\colon x \in X\}$ and that $f\colon X \to X$ is a continuous map.

Definition 3.6 (**[15]**)**.** An *expansivity neighborhood* for f is a closed neighborhood $N \subset X \times X$ of Δ such that, for any distinct $x, y \in X$, there is an $n \geq 0$ such that $(f^n(x), f^n(y)) \notin N$. We say that f is *weakly positively expansive* if it has an expansivity neighborhood.

Theorem 3.7 (**[15,** Theorem 4]**).** *Let* $f\colon X \to X$ *be a continuous map on a locally compact metrizable space* X*. Then,* f *is positively expansive if and only if it is weakly positively expansive with respect to some metric compatible with the topology of* X*.*

Now, we consider a self-similar graph action (G, E). For each natural number m, we define

$$U_m = \{(\cdots a_{-1}, \cdots b_{-1}) \in E^{-\omega} \times E^{-\omega} \colon g(a_{-m} \cdots a_{-1}) = b_{-m} \cdots b_{-1} \text{ for some } g \in \mathcal{N}\}$$

and

$$V_m = (q \times q)(U_m) \subset J_{(G,E)} \times J_{(G,E)}.$$

Lemma 3.8. *For every natural number m, V_m is a closed neighborhood of the diagonal Δ of $J_{(G,E)} \times J_{(G,E)}$.*

Proof. First, we show that U_m is a closed subset of $E^{-\omega} \times E^{-\omega}$. Let

$$\xi = \cdots x_{-m} \cdots x_{-1} \quad \text{and} \quad \eta = \cdots y_{-m} \cdots y_{-1}$$

be elements of $E^{-\omega}$ such that (ξ, η) is a boundary element of U_m. Then, for a neighborhood $W = Z(x_{-m} \cdots x_{-1}) \times Z(y_{-m} \cdots y_{-1})$ of (ξ, η) we have $W \cap U_m \neq \emptyset$. Choose an element $(\alpha, \beta) \in W \cap U_m$ such that

$$\alpha = \cdots a_{-m} \cdots a_{-1} \quad \text{and} \quad \beta = \cdots b_{-m} \cdots b_{-1}.$$

Since (α, β) is an element of U_m, there is a group element $g \in \mathcal{N}$ such that $g(a_{-m} \cdots a_{-1}) = b_{-m} \cdots b_{-1}$. On the other hand, $(\alpha, \beta) \in W$ means

$$\alpha \in Z(x_{-m} \cdots x_{-1}) \quad \text{and} \quad \beta \in Z(y_{-m} \cdots y_{-1}),$$

which imply

$$a_{-m} \cdots a_{-1} = x_{-m} \cdots x_{-1}$$

and

$$b_{-m} \cdots b_{-1} = y_{-m} \cdots y_{-1}.$$

Thus, we have

$$g(x_{-m} \cdots x_{-1}) = y_{-m} \cdots y_{-1},$$

and (ξ, η) is included in U_m; hence, U_m is a closed subset of $E^{-\omega} \times E^{-\omega}$. Then, $V_m = (q \times q)(U_m)$ is a closed subset of $J_{(G,E)} \times J_{(G,E)}$ since $E^{-\omega}$ and $J_{(G,E)}$ are compact spaces and the quotient map $q \colon E^{-\omega} \to J_{(G,E)}$ is continuous.

Moreover, U_m is an open set in $E^{-\omega} \times E^{-\omega}$. Let $(\alpha, \beta) \in U_m$ be given by $\alpha = \cdots a_{-m} \cdots a_{-1}$ and $\beta = \cdots b_{-m} \cdots b_{-1}$. Then, the existence of some $g \in \mathcal{N}$ such that

$$g(a_{-m} \cdots a_{-1}) = b_{-m} \cdots b_{-1}$$

implies

$$Z(a_{-m}\cdots a_{-1})\times Z(b_{-m}\cdots b_{-1})\subset U_m.$$

Thus, (α,β) is an interior point of U_m, and U_m is an open subset of $E^{-\omega}\times E^{-\omega}$. Hence, $V_m=(q\times q)(U_m)$ is open in $J_{(G,E)}\times J_{(G,E)}$ by Proposition 3.2.

In order to show $\Delta\subset V_m$, consider any $(z,z)\in\Delta$ and $\zeta=\cdots z_{-m}\cdots z_{-1}\in q^{-1}(z)$. Then, it is trivial that $(\zeta,\zeta)\in U_m$ and $(q\times q)(\zeta,\zeta)=(z,z)\in V_m$. Therefore, V_m is a closed neighborhood of the diagonal Δ. □

In order to show that V_m is an expansivity neighborhood for the shift map, we need to extend [**10,** Lemma 6.3] a little further.

Lemma 3.9. *If (G,E) is a contracting and regular self-similar graph action, then there is a natural number k_0 such that, for every $k\geq k_0$, any $w\in E^k$ and any two elements $g,h\in\mathcal{N}$, either $g(w)\neq h(w)$ or $g(w)=h(w)$ and $g|_w=h|_w$ hold.*

Proof. It is proven in [**10,** Lemma 6.3] that there is a natural number k_0 such that, for any $w\in E^{k_0}$ and any two elements $g,h\in\mathcal{N}$, either $g(w)\neq h(w)$ or $g(w)=h(w)$ and $g|_w=h|_w$ hold. For every $k>k_0$, let $w_0\in E^{k_0}$, $w_1\in E^{k-k_0}$ and $w_2\in E^*$ be arbitrary words with the conditions $r(w_0)=d(w_1)$ and $r(w_1)=d(w_2)$ so that $w_0w_1\in E^k$ and $w_0w_1w_2\in E^*$. We must show that, for any $g,h\in\mathcal{N}$, $g(w_0w_1)=h(w_0w_1)$ implies $g|_{w_0w_1}=h|_{w_0w_1}$.

If $g(w_0w_1)=h(w_0w_1)$, then $w_0\in E^{k_0}$ implies

$$g(w_0w_1)=g(w_0)g|_{w_0}(w_1)=h(w_0)h|_{w_0}(w_1)=h(w_0w_1)$$

such that $g(w_0)=h(w_0)$ and $g|_{w_0}=h|_{w_0}$ hold. Thus, for any $w_2\in E^*$ such that $w_0w_1w_2$ is allowed, we obtain

$$\begin{aligned}g(w_0w_1w_2)&=g(w_0)g|_{w_0}(w_1w_2)=g(w_0w_1)g|_{w_0w_1}(w_2)\\&=h(w_0)h|_{w_0}(w_1w_2)=h(w_0w_1)h|_{w_0w_1}(w_2)=h(w_0w_1w_2).\end{aligned}$$

Therefore, we have $g|_{w_0w_1}=h|_{w_0w_1}$, and this completes the proof. □

Lemma 3.10. *Suppose that $m\geq k_0+1$ is any natural number, where k_0 is given in Lemma 3.9, and that U_m and V_m are as in Lemma 3.8. Then, V_m is an expansivity neighborhood for $\sigma\colon J_{(G,E)}\to J_{(G,E)}$.*

Proof. We prove the following. If $(x, y) \in J_{(G,E)} \times J_{(G,E)}$ satisfies $(\sigma^n x, \sigma^n y) \in V_m$ for every $n \geq 0$, then $x = y$.

Let $\xi = \cdots x_{-m} \cdots x_{-1} \in q^{-1}(x)$ and $\eta = \cdots y_{-m} \cdots y_{-1} \in q^{-1}(y)$. Since the shift maps on $E^{-\omega}$ and $J_{(G,E)}$ and the quotient map are commutative to each other, $(\sigma^n x, \sigma^n y) \in V_m$ means $(\sigma^n \xi, \sigma^n \eta) \in U_m$. Thus, for every $n \geq 0$, there is a group element $g_n \in \mathcal{N}$ such that $g_n(x_{-m-n} \cdots x_{-1-n}) = y_{-m-n} \cdots y_{-1-n}$. In order to obtain $x = y$, we show

$$g_n(x_{-m-n} \cdots x_{-1-n} \cdots x_{-1}) = y_{-m-n} \cdots y_{-1-n} \cdots y_{-1},$$

which implies an asymptotic equivalence between ξ and η such that

$$x = q(\xi) = q(\eta) = y.$$

For $n = 0$, 1, we have

$$\begin{aligned} g_0(x_{-m} \cdots x_{-1}) &= g_0(x_{-m} \cdots x_{-2} x_{-1}) \\ &= g_0(x_{-m} \cdots x_{-2}) g_0|_{x_{-m} \cdots x_{-2}}(x_{-1}) \\ &= y_{-m} \cdots y_{-2} y_{-1} \end{aligned}$$

and

$$\begin{aligned} g_1(x_{-m-1} \cdots x_{-2}) &= g_1(x_{-m-1} x_{-m} \cdots x_{-2}) \\ &= g_1(x_{-m-1}) g_1|_{x_{-m-1}}(x_{-m} \cdots x_{-2}) \\ &= y_{-m-1} y_{-m} \cdots y_{-2}. \end{aligned}$$

Since we choose $m - 1 \geq k_0$, by Lemma 3.9,

$$g_0(x_{-m} \cdots x_{-2}) = y_{-m} \cdots y_{-2} = g_1|_{x_{-m-1}}(x_{-m} \cdots x_{-2})$$

implies

$$g_0|_{x_{-m} \cdots x_{-2}} = (g_1|_{x_{-m-1}})|_{x_{-m} \cdots x_{-2}} = g_1|_{x_{-m-1} x_{-m} \cdots x_{-2}}$$

and

$$\begin{aligned} g_1(x_{-m-1} \cdots x_{-2} x_{-1}) &= g_1(x_{-m-1} \cdots x_{-2}) g_1|_{x_{-m-1} x_{-m} \cdots x_{-2}}(x_{-1}) \\ &= g_1(x_{-m-1} \cdots x_{-2}) g_0|_{x_{-m} \cdots x_{-2}}(x_{-1}) \\ &= y_{-m-1} y_{-m} \cdots y_{-2} y_{-1}. \end{aligned}$$

Then, by induction, we have $g_n(x_{-m-n} \cdots x_{-1-n} \cdots x_{-1}) = y_{-m-n} \cdots y_{-1-n} \cdots y_{-1}$ for every $n \geq 0$. Therefore, ξ is asymptotically equivalent to η, and V_m is an expansivity neighborhood for $\sigma \colon J_{(G,E)} \to J_{(G,E)}$. □

Now, we have the following from Theorem 2.8, Theorem 3.7 and Lemma 3.10.

Theorem 3.11. *If (G,E) is a contracting and regular self-similar graph action, then $\sigma\colon J_{(G,E)} \to J_{(G,E)}$ is a positively expansive surjective local homeomorphism.*

Since a local homeomorphism is an open map, [**13,** Theorem 2] implies the following.

Corollary 3.12. *If (G,E) is a contracting and regular self-similar graph action, then $\sigma\colon J_{(G,E)} \to J_{(G,E)}$ is expanding.*

Remark 3.13. The metric mentioned in Theorem 3.7 is given as follows. Let U_m and V_m be as above. Then,

$$g(x_{-m-1}x_{-m}\cdots x_{-1}) = g(x_{-m-1})g|_{x_{-m-1}}(x_{-m}\cdots x_{-1})$$

implies $U_{m+1} \subset U_m$ and $V_{m+1} \subset V_m$, and it is easy to see that $\{V_m\}$ satisfies the conditions of [**6,** page 185, Lemma 12]. For $x, y \in J_{(G,E)}$, we define

$$\tau(x,y) = \sup\{m \in \mathbb{N} \cup \{0\} \colon (x,y) \in V_m\} \quad \text{and} \quad \delta(x,y) = 2^{-\tau(x,y)}.$$

Let $\overline{d}(x,y)$ be the infimum of $\sum \delta(a_{i-1},a_i)$ over all finite sequences $a_0, a_1, \ldots, a_n$ in $J_{(G,E)}$ such that $a_0 = x$ and $a_n = y$. Then, $\overline{d}$ is the metric on $J_{(G,E)}$ induced from $\{V_m\}$ by the aforementioned citation.

Proposition 3.14. *For every $x, y \in J_{(G,E)}$, $\delta(x,y) = \overline{d}(x,y)$.*

Proof. First, we remark that $\tau(x,y) < \infty$ if and only if $x \neq y$ in $J_{(G,E)}$. Trivially, $\delta(x,x) = \overline{d}(x,x) = 0$ and $\delta(x,y) \geq \overline{d}(x,y)$ by the definition. In order to show $\delta(x,y) \leq \overline{d}(x,y)$, let $a_0, a_1, \ldots, a_n$ be any finite sequence in $J_{(G,E)}$ such that $a_0 = x$ and $a_n = y$. We observe that, if there is at least one $i \in \{1, \ldots, n\}$ such that $\tau(a_{i-1},a_i) \leq \tau(x,y)$, then

$$\delta(x,y) = 2^{-\tau(x,y)} \leq 2^{-\tau(a_{i-1},a_i)} \leq \sum_{j=0}^{n} 2^{-\tau(a_{j-1},a_j)} = \sum_{j=0}^{n} \delta(a_{j-1},a_j).$$

Thus, we assume that $\tau(a_{i-1}, a_i) \gneq \tau(x, y)$ for every $i = 1, \ldots, n$, and obtain a contradiction. For each a_i, choose $\alpha_i = \cdots a_{i,-2}a_{i,-1} \in q^{-1}(a_i)$. Then, we have

$$(\alpha_{i-1}, \alpha_i) \in U_{\tau(a_{i-1},a_i)} \subset U_{\tau(x,y)+1}$$

since $U_{m+1} \subset U_m$ for every m and $\tau(a_{i-1}, a_i) \geq \tau(x, y) + 1 \gneq \tau(x, y)$. Thus, there is a group element $g_i \in \mathcal{N}$ for every $i = 1, \ldots, n$ such that

$$g_i(a_{i-1,-\tau(x,y)-1} \cdots a_{i-1,-1}) = a_{i,-\tau(x,y)-1} \cdots a_{i,-1}$$

and

$$g_n \cdots g_1(a_{0,-\tau(x,y)-1} \cdots a_{0,-1}) = a_{n,-\tau(x,y)-1} \cdots a_{n,-1}.$$

Then, we have $(\alpha_0, \alpha_n) \in U_{\tau(x,y)+1}$ and $(q(\alpha_0), q(\alpha_n)) = (x, y) \in V_{\tau(x,y)+1}$ such that $\tau(x, y) \geq \tau(x, y)+1$, a contradiction. Hence, there is at least one $i \in \{1, \ldots, n\}$ such that $\tau(a_{i-1}, a_i) \leq \tau(x, y)$, which implies that $\delta(x, y) \leq \sum_{j=0}^{n} \delta(a_{j-1}, a_j)$. Since $a_0, \ldots, a_n$ is any finite sequence satisfying $a_0 = x$ and $a_n = y$, we conclude that $\delta(x, y) \leq \overline{d}(x, y)$. Therefore, $\delta(x, y) = \overline{d}(x, y)$ for all $x, y \in J_{(G,E)}$. □

4. Smale spaces. We omit the definitions and fundamental properties of Smale spaces and their corresponding C^*-algebras. The interested reader may consult [**11, 12**] for details.

For a contracting and regular self-similar graph action (G, E), where E is an n-bouquet, Nekrashevych showed [**10,** Proposition 6.10] that its limit solenoid is a Smale space. We extend his result to finite graphs.

Theorem 4.1. *If (G, E) is a contracting and regular self-similar graph action, then its limit solenoid $(S_{(G,E)}, \sigma)$ is a Smale space.*

Proof. When (G, E) satisfies contracting and regular conditions,

$$\sigma\colon J_{(G,E)} \longrightarrow J_{(G,E)}$$

is a positively expansive surjective local homeomorphism by Theorem 3.11. Then, Theorem 2.10 and [**16,** Lemma 4.18] imply the conclusion. □

4.1. Unstable Ruelle algebras. We show that, under contracting, regular, pseudo free and G-transitive conditions, the unstable Ruelle algebra of $(S_{(G,E)}, \sigma)$ is strongly Morita equivalent to the groupoid algebra $\mathcal{O}_{G,E}$ of Exel and Pardo.

Instead of the formal definition of unstable equivalence given in [**11**], we use [**10,** Proposition 6.8].

Definition 4.2 ([**10,** Proposition 6.8]). Suppose that $(S_{(G,E)}, \sigma)$ is the limit solenoid of a contracting and regular self-similar graph action (G, E). For $x, y \in S_{(G,E)}$, let

$$\xi = \cdots x_{-1} \cdot x_0 x_1 \cdots \in q^{-1}(x)$$

and

$$\eta = \cdots y_{-1} \cdot y_0 y_1 \cdots \in q^{-1}(y).$$

We say that x is *unstably equivalent* to y if and only if there exist $n \in \mathbb{Z}$ and $g \in \mathcal{N}$ such that

$$g(x_n x_{n+1} \cdots) = y_n y_{n+1} \cdots .$$

The unstable groupoid and its induced groupoid of $(S_{(G,E)}, \sigma)$ are given by

$$R^u = \{(x, y) \in S_{(G,E)} \times S_{(G,E)} : x \text{ is unstably equivalent to } y\}$$

and

$$R^u \rtimes \mathbb{Z} = \{(x, l-k, y) \in S_{(G,E)} \times \mathbb{Z} \times S_{(G,E)} : l, k \in \mathbb{N}, (\sigma^l(x), \sigma^k(y)) \in R^u\}.$$

It is a well-known fact that R^u and $R^u \rtimes \mathbb{Z}$ are locally compact Hausdorff groupoids. The groupoid C^*-algebra $C^*(R^u \rtimes \mathbb{Z})$ is called the *unstable Ruelle algebra* of the Smale space $(S_{(G,E)}, \sigma)$. See [**11, 12**] for details.

4.2. Strong Morita equivalence between $C^*(R^u \rtimes \mathbb{Z})$ and $\mathcal{O}_{G,E}$.

For a self-similar graph action (G, E), the following groupoid is constructed in [**4,** Theorem 8.19]:

$$\mathcal{G}_{G,E} = \left\{ \begin{array}{c} (\alpha; [\{g_i\}], l-k; \beta) : \alpha, \beta \in E^\omega,\ g_i \in G, l, k \in \mathbb{N}, \\ \text{there exists an } n \geq l \text{ such that} \\ g_i \cdot \alpha_i = \beta_{i-l+k} \cdot g_{i+1} \text{ for all } i \geq n. \end{array} \right\}$$

Here, $[\{g_i\}]$ is the equivalence class of $\{g_i\}$ under $\sim$ such that, for sequences of group elements, $\{g_i\} \sim \{h_i\}$ if and only if there is an

$m \geq 0$ such that $g_i = h_i$ for every $i \geq m$. A suitable topology of $\mathcal{G}_{G,E}$ is described in [**4,** Proposition 9.5].

Theorem 4.3 ([**4,** Theorem 9.6])**.** *If (G, E) is pseudo free, then the Cuntz-Pimsner algebra $\mathcal{O}_{G,E}$ is $*$-isomorphic to the groupoid algebra $C^*(\mathcal{G}_{G,E})$.*

Now, we show that there is a groupoid equivalence between $R^u \rtimes \mathbb{Z}$ and $\mathcal{G}_{G,E}$ in the sense of Muhly, Renault and Williams [**8**]. We begin by mentioning a well-known groupoid equivalence result reviewed in [**7,** Section 5]: Let Γ be a locally compact Hausdorff groupoid and X a locally compact Hausdorff space. If there is a continuous open surjection $\psi\colon X \to \Gamma^{(0)}$, we set

$$\Gamma^{\psi} = \{(\xi, \gamma, \eta)\colon \xi,\ \eta \in X,\ \gamma \in \Gamma,\ \psi(\xi) = d(\gamma),\ \psi(\eta) = r(\gamma)\}$$

with the subspace topology of $X \times \Gamma \times X$.

Lemma 4.4 ([**7,** Lemma 5.1])**.** *The groupoid Γ is equivalent to Γ^{ψ}.*

Suppose that (G, E) is a contracting and regular self-similar graph action with the two-sided infinite path space $E^{\pm\omega}$ and the induced unstable groupoid $R^u \rtimes \mathbb{Z}$. Then, $E^{\pm\omega}$ is a compact Hausdorff space, $R^u \rtimes \mathbb{Z}$ is a locally compact Hausdorff groupoid whose unit space is $S_{(G,E)}$ and $q\colon E^{\pm\omega} \to S_{(G,E)}$ is a continuous open surjection by Proposition 3.4. Thus, the following is true by Lemma 4.4.

Proposition 4.5. *The groupoid $R^u \rtimes \mathbb{Z}$ is equivalent to*

$$(R^u \rtimes \mathbb{Z})^q = \left\{ \begin{matrix} (\xi, (x, l-k, y), \eta)\colon \xi,\ \eta \in E^{\pm\omega},\ q(\xi) = x,\ q(\eta) = y, \\ l, k \in \mathbb{N},\ (\sigma^l x, \sigma^k y) \in R^u. \end{matrix} \right\}$$

In order to compare $(R^u \rtimes \mathbb{Z})^q$ with $\mathcal{G}_{G,E}$, whose unit space is E^{ω}, we need to reduce the unit space of $(R^u \rtimes \mathbb{Z})^q$. For this purpose, we use a *transversal* in [**8,** Example 2.7]. Fix a left-infinite word $z = \cdots z_{-2} z_{-1} \in E^{-\omega}$, and consider

$$T = \{z \cdot w \in E^{\pm\omega} \colon w \in E^{\omega}\}.$$

Then, T is trivially a closed subspace of $E^{\pm\omega}$. Since $(R^u \rtimes \mathbb{Z})^q$ has the subspace topology of $E^{\pm\omega} \times (R^u \rtimes \mathbb{Z}) \times E^{\pm\omega}$, so does

$$(R^u \rtimes \mathbb{Z})^q_T = \{\gamma \in (R^u \rtimes \mathbb{Z})^q \colon d(\gamma) \in T\}.$$

Then, $d|_{(R^u \rtimes \mathbb{Z})^q{}_T}$ and $r|_{(R^u \rtimes \mathbb{Z})^q{}_T}$, respectively, are open maps since they are projection maps to the first and the third coordinate spaces, respectively, of $(R^u \rtimes \mathbb{Z})^q{}_T$. Now, we show that T meets every orbit in the unit space of $(R^u \rtimes \mathbb{Z})^q$.

Lemma 4.6 ([**4,** Proposition 13.2])**.** *If (G, E) is a self-similar graph action such that E is G-transitive, then, for any vertices u and v of E there are $a \in E^*$, $p \in E^0$ and $g \in G$ such that a is a path from u to p and $g(p) = v$.*

Lemma 4.7. *If (G, E) is a contracting and regular self-similar graph action such that E is G-transitive, then, for every $\xi = \cdots x_{-1} \cdot x_0 x_1 \cdots \in E^{\pm\omega}$, there is an $\eta = \cdots z_{-1} \cdot w \in T$ such that*

$$(\xi, (q(\xi), l - k, q(\eta)), \eta) \in (R^u \rtimes \mathbb{Z})^q$$

for some nonnegative integers l, k.

Proof. For two vertices $r(z_{-1})$ and $d(x_0)$, by Lemma 4.6, there are a vertex p, a path a from $r(z_{-1})$ to p and a $g \in G$ such that $g(p) = d(x_0)$. Then,

$$g^{-1}(x_0 x_1 \cdots) = y_0 y_1 \cdots \in E^\omega$$

satisfies $d(g^{-1}(x_0 x_1 \cdots)) = d(g^{-1}(x_0)) = g^{-1}(d(x_0)) = p$. Thus, we have

$$\eta = \cdots z_{-2} z_{-1} \cdot a \cdot g^{-1}(x_0 x_1 \cdots) = \cdots z_{-2} z_{-1} \cdot a \cdot y_0 y_1 \cdots \in T.$$

Now, we verify that $\sigma^n(\xi) = \cdots x_{-n-2} x_{-n-1} \cdot x_{-n} \cdots x_{-1} x_0 \cdots$ is unstably equivalent to η, where n is the length of a. For $g^{-1} \in G$, the contracting condition implies that there is a natural number m such that $g^{-1}|_b \in \mathcal{N}$ for every $b \in E^m$. Then, we obtain from Remark 2.3 that

$$g^{-1}(x_0 x_1 \cdots) = y_0 y_1 \cdots = g^{-1}(x_0 \cdots x_{m-1}) g^{-1}|_{x_0 \cdots x_{m-1}}(x_m \cdots)$$

such that $g^{-1}|_{x_0 \cdots x_{m-1}} \in \mathcal{N}$ and $g^{-1}|_{x_0 \cdots x_{m-1}}(x_m x_{m+1} \cdots) = y_m y_{m+1} \cdots$. Therefore, $\eta = \cdots z_{-1} \cdot a \cdot g^{-1}(x_0 \cdots) \in T$ satisfies $(\xi, (q(\xi), n - 0, q(\eta)), \eta) \in (R^u \rtimes \mathbb{Z})^q$. $\square$

Thus, T meets every orbit in the unit space of $(R^u \rtimes \mathbb{Z})^q$, and T is a transversal to $(R^u \rtimes \mathbb{Z})^q$. Then, we have the following from **[8,** Example 2.7].

Proposition 4.8. *If (G, E) is a contracting and regular self-similar graph action such that E is G-transitive, then $(R^u \rtimes \mathbb{Z})^q$ is equivalent to $(R^u \rtimes \mathbb{Z})^q{}_T^T$.*

We show that $(R^u \rtimes \mathbb{Z})^q{}_T^T$ is equivalent to $\mathcal{G}_{G,E}$. First, recall that

$$T = \{z \cdot w \in E^{\pm\omega} : z \in E^{-\omega} \text{ is fixed}, w \in E^{\omega}\},$$
$$(R^u \rtimes \mathbb{Z})^q{}_T^T = \{(\xi, (q(\xi), l-k, q(\eta)), \eta) \in (R^u \rtimes \mathbb{Z})^q : \xi, \eta \in T\}$$

and

$$\mathcal{G}_{G,E} = \left\{ \begin{array}{c} (\alpha; [\{g_i\}], l-k; \beta) : \alpha,\ \beta \in E^{\omega},\ g_i \in G,\ l, k \in \mathbb{N}, \\ \text{there exists an } n \geq l \text{ such that} \\ g_i \cdot \alpha_i = \beta_{i-l+k} \cdot g_{i+1} \text{ for all } i \geq n. \end{array} \right\}$$

We simplify $(R^u \rtimes \mathbb{Z})^q{}_T^T$ and $\mathcal{G}_{G,E}$. On $(R^u \rtimes \mathbb{Z})^q{}_T^T$, consider

$$\xi = \cdots z_{-2} z_{-1} \cdot x_0 x_1 \cdots \quad \text{and} \quad \eta = \cdots z_{-2} z_{-1} \cdot y_0 y_1 \cdots .$$

Then, $(q(\xi), l-k, q(\eta)) \in R^u \rtimes \mathbb{Z}$ means that $\sigma^l(q(\xi)) = q(\sigma^l(\xi))$ is unstably equivalent to $\sigma^k(q(\eta)) = q(\sigma^k(\eta))$, which is equivalent to the existence of a natural number $m \geq l$ and some $g_m \in \mathcal{N}$ exist such that

$$g_m(x_m x_{m+1} \cdots) = y_{m-l+k} y_{m-l+k+1} \cdots .$$

Remark 4.9. Let m and g_m be as above.

(1) For every $j > m$, let $g_j = g_m|_{x_m \cdots x_{j-1}}$. Then, we have $g_j \cdot x_j = y_{j-l+k} \cdot g_{j+1}$ by Remark 2.3.

(2) A natural number m and a nucleus element g_m are not unique. However, if n is another natural number with $n \geq m$ and h_n is another nucleus element such that

$$h_n(x_n x_{n+1} \cdots) = y_{n-l+k} y_{n-l+k+1} \cdots ,$$

then, for every $j \geq \max\{m,n\} + k_0 = n + k_0$, where k_0 is the number given in Lemma 3.9, we have

$$\begin{aligned} g_j = g_m|_{x_m \cdots x_{j-1}} &= (g_m|_{x_m \cdots x_{n-1}})|_{x_n \cdots x_{j-1}} = g_n|_{x_n \cdots x_{j-1}} \\ &= h_n|_{x_n \cdots x_{j-1}} = h_j \end{aligned}$$

by Lemma 3.9, in other words, $[\{g_j\}] = [\{h_j\}]$ where $[\{g_j\}]$ was defined at the beginning of this subsection.

Lemma 4.10. *Suppose that (G, E) is a contracting and regular self-similar graph action such that E is G-transitive and that T is the above transversal to $(R^u \rtimes \mathbb{Z})^q$. Then, for every $x \in S_{(G,E)}$, $q^{-1}(x) \cap T$ has at most one element.*

Proof. For $x \in S_{(G,E)}$ such that $q^{-1}(x) \cap T \neq \emptyset$, we denote $\alpha, \beta \in q^{-1}(x) \cap T$ as

$$\alpha = \cdots z_{-2} z_{-1} \cdot a_0 a_1 \cdots \quad \text{and} \quad \beta = \cdots z_{-2} z_{-1} \cdot b_0 b_1 \cdots$$

and show $\alpha = \beta$.

First, we note that, by Lemma 3.9, there is a natural number k such that, for every $w \in E^k$ and $g \in \mathcal{N}$, either $g(w) \neq w$ or $g(w) = w$ and $g|_w = 1$. Since α and β are elements of $q^{-1}(x)$, α and β are asymptotically equivalent, and there is a $g_{-k} \in \mathcal{N}$ such that

$$g_{-k}(z_{-k} \cdots z_{-1} \cdot a_0 a_1 \cdots) = z_{-k} \cdots z_{-1} \cdot b_0 b_1 \cdots .$$

By the definition of the G-action on E^ω (see Remark 2.3), the above equality means that, for every $l \geq 0$,

$$\begin{aligned} g_{-k}(z_{-k} \cdots z_{-1} \cdot a_0 \cdots a_l) &= g_{-k}(z_{-k} \cdots z_{-1}) \cdot g_{-k}|_{z_{-k} \cdots z_{-1}}(a_0 \cdots a_l) \\ &= z_{-k} \cdots z_{-1} \cdot b_0 \cdots b_l. \end{aligned}$$

Then, $g_{-k}(z_{-k} \cdots z_{-1}) = z_{-k} \cdots z_{-1}$ implies $g_{-k}|_{a_{-k} \cdots a_{-1}} = 1$ by Lemma 3.9. Hence, we have $a_0 \cdots a_l = b_0 \cdots b_l$ for every $l \geq 0$, which induces $\alpha = \beta$. Therefore, $q^{-1}(x) \cap T$ has at most one element for every $x \in S_{(G,E)}$. ☐

Now, we no longer need $q(\xi)$ and $q(\eta)$ since $q|_T$ is a homeomorphism by Proposition 3.4 and Lemma 4.10. Thus, when (G, E) is a contracting and regular self-similar graph action, $(R^u \rtimes \mathbb{Z})^q{}^T_T$ is isomorphic to a groupoid

$$\mathcal{A} = \left\{ \begin{matrix} (\xi; l-k; \eta) \colon \xi, \eta \in T,\ l, k \in \mathbb{N},\ \text{there exist an } m \geq l \text{ and a} \\ g_m \in \mathcal{N} \text{ such that } g_m(x_m x_{m+1} \cdots) = y_{m-l+k} y_{m-l+k+1} \cdots \end{matrix} \right\}$$

by a map $(\xi, (q(\xi), l-k, q(\eta)), \eta) \mapsto (\xi; l-k; \eta)$. Then, $\mathcal{A}$ with the induced topology from $(R^u \rtimes \mathbb{Z})^{\,q}{}_T^T$ is topologically isomorphic to $(R^u \rtimes \mathbb{Z})^{\,q}{}_T^T$.

Remark 4.11. We can explain the induced topology on $\mathcal{A}$ as follows. Let

$$\mathcal{A}_n = \{(\xi; 0; \eta) \colon \xi, \eta \in T,\ \text{there exists a } g \in \mathcal{N} \text{ such that } g(x_n x_{n+1} \cdots) = y_n y_{n+1} \cdots\}$$

with the subspace topology of $T \times T$, and

$$\mathcal{A}_\infty = \bigcup_{n=0}^{\infty} \mathcal{A}_n$$

with the inductive limit topology. Then, the map $\mathcal{A}_\infty \times \mathbb{Z} \to \mathcal{A}$ sending $((\xi; 0; \eta), n)$ to $(\xi; n; \sigma^n(\eta))$ is a bijection, and the product topology of $\mathcal{A}_\infty \times \mathbb{Z}$ is transferred to $\mathcal{A}$. Since R^u has an inductive limit topology (see **[11, 12]** for details), it is routine to verify that this topology is the same as the induced topology.

On the other hand, with

$$\mathcal{G}_{G,E} = \left\{ \begin{matrix} (\alpha; [\{g_i\}], l-k; \beta) \colon \alpha, \beta \in E^\omega, g_i \in G,\ l,\ k \in \mathbb{N}, \\ \text{there exists an } n \in \mathbb{N} \text{ such that} \\ g_i \cdot \alpha_i = \beta_{i-l+k} \cdot g_{i+1} \text{ for all } i \geq n \end{matrix} \right\},$$

it is not difficult to observe that $g_i \cdot \alpha_i = \beta_{i-l+k} \cdot g_{i+1}$ for every $i \geq n$ is the same as $g_n(\alpha_n \alpha_{n+1} \cdots) = \beta_{n-l+k} \beta_{n-l+k+1} \cdots$ by Remark 2.3. In addition, we can say a little more about $[\{g_i\}]$.

Lemma 4.12. *Suppose that (G, E) is a contracting and regular self-similar graph action and $(\alpha; [\{g_i\}], l-k; \beta) \in \mathcal{G}_{G,E}$. Then:*

(1) *g_i is an element of the nucleus for every large i, and*

(2) *the equivalence class $[\{g_i\}]$ is uniquely determined by $\alpha, \beta, l-k$.*

Proof.

(1) Let $n \in \mathbb{N}$ and $g_n \in G$ be such that

$$g_n(\alpha_n\alpha_{n+1}\cdots) = \beta_{n-l+k}\beta_{n-l+k+1}\cdots.$$

Then, the contracting condition implies that there is a natural number t such that $g_n|_{\alpha_n\cdots\alpha_{n+t-1}} = g_{n+t} \in \mathcal{N}$. Thus, we have $g_i \in \mathcal{N}$ for every $i \geq n+t$.

(2) We show that, if $(\alpha; [\{g_i\}], l-k; \beta)$ and $(\alpha; [\{h_i\}], l-k; \beta)$ are elements of $\mathcal{G}_{G,E}$, then $[\{g_i\}] = [\{h_i\}]$, i.e., there is an m such that $g_i = h_i$ for every $i \geq m$. Suppose that n_1 and n_2 are natural numbers such that $g_i \cdot \alpha_i = \beta_{i-l+k} \cdot g_{i+1}$ for every $i \geq n_1$ and $h_i \cdot \alpha_i = \beta_{i-l+k} \cdot h_{i+1}$ for every $i \geq n_2$. Let $n = \max\{n_1, n_2\}$. Without loss of generality, we may say that g_i and h_i are elements of $\mathcal{N}$ for every $i \geq n$ by (1). Let k_0 be the natural number given in Lemma 3.9. Then

$$g_n(\alpha_n\cdots\alpha_{n+k_0-1}) = \beta_{n-l+k}\cdots\beta_{n-l+k+k_0-1} = h_n(\alpha_n\cdots\alpha_{n+k_0-1})$$

implies $g_i = g_n|_{\alpha_n\cdots\alpha_{n+i-1}} = h_n|_{\alpha_n\cdots\alpha_{n+i-1}} = h_i$ for every $i \geq n+k_0$. Hence, we have $[\{g_i\}] = [\{h_i\}]$. □

Thus, we may delete $[\{g_i\}]$ from $(\alpha; [\{g_i\}], l-k; \beta)$, and $\mathcal{G}_{G,E}$ is topologically isomorphic to a groupoid

$$\mathcal{B} = \left\{ \begin{matrix} (\alpha; l-k; \beta)\colon \alpha, \beta \in E^\omega,\ l, k \in \mathbb{N},\ \text{there exist an } n \in \mathbb{N} \text{ and} \\ g_n \in \mathcal{N} \text{ such that } g_n(\alpha_n\alpha_{n+1}\cdots) = \beta_{n-l+k}\beta_{n-l+k+1}\cdots \end{matrix} \right\}$$

which has the induced topology from $\mathcal{G}_{G,E}$.

Similarly to the case of $\mathcal{A}$ in Remark 4.11, the induced topology on $\mathcal{B}$ can be explained from the product topology on

$$\mathcal{B}_n = \{(\alpha; 0; \beta)\colon \alpha, \beta \in E^\omega,\ \text{there exists a } g_n \in \mathcal{N} \text{ such that}$$
$$g_n(\alpha_n\alpha_{n+1}\cdots) = \beta_n\beta_{n+1}\cdots\},$$

the inductive limit topology on $\mathcal{B}_\infty$ and the induced topology from the product topology on $\mathcal{B}_\infty \times \mathbb{Z}$.

Now, we compare $(R^u \rtimes \mathbb{Z})^T_T{}^q$ and $\mathcal{G}_{G,E}$ via $\mathcal{A}$ and $\mathcal{B}$. Then, it is clear that there is a strong relation between $(R^u \rtimes \mathbb{Z})^T_T{}^q \simeq \mathcal{A}$ and $\mathcal{G}_{G,E} \simeq \mathcal{B}$. The only differences are the unit spaces $T = \{z \cdot w\colon z \in E^{-\omega}$ is fixed, $w \in E^\omega\}$ for $\mathcal{A}$ and E^ω for $\mathcal{B}$.

Let $\pi\colon T \to E^\omega$ be defined by $z \cdot w \mapsto w$ and $\overline{\pi}\colon (R^u \rtimes \mathbb{Z})q^T_T \to \mathcal{G}_{G,E}$ by

$$(\xi; l-k; \eta) \longmapsto (\pi(\xi); l-k; \pi(\eta)).$$

In order to simplify the notation, we denote the image of $(R^u \rtimes \mathbb{Z})q^T_T$ by $\overline{\pi}$ as I. It is easy to show that $\overline{\pi}$ is a continuous groupoid monomorphism so that I is a subgroupoid of $\mathcal{G}_{G,E}$. However, $\overline{\pi}$ is not an epimorphism since the unit space of I is

$$I^{(0)} = \{w \in E^\omega : d(w) = r(z_{-1})\} = \bigcup_{\substack{e \in E^1 \\ d(e)=r(z_{-1})}} Z(e),$$

which is a proper subset of E^ω if the graph E has more than one vertex.

Fortunately, $I^{(0)}$ is a transversal to $\mathcal{G}_{G,E}$. It is trivial to see that $I^{(0)}$ is a clopen subspace of E^ω since E is a finite graph. In the proof of Lemma 4.7, we showed that, for every $w \in E^\omega$, there are a finite path a and $g \in G$ such that $a \cdot g^{-1}(w) \in I^{(0)}$ and $(w; |a|-0; a \cdot g^{-1}(w)) \in \mathcal{G}_{G,E}$ where $|a|$ is the length of a; thus, $I^{(0)}$ meets every orbit in the unit space of $\mathcal{G}_{G,E}$. In order to show that $d|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}$ and $r|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}$ are open maps, we remark that $\mathcal{G}_{G,E}{}_{I^{(0)}} = d^{-1}(I^{(0)})$ is an open subspace of $\mathcal{G}_{G,E}$ since $I^{(0)}$ is an open subspace of E^ω. Then, every open subset U of $\mathcal{G}_{G,E}{}_{I^{(0)}}$ is an open set in $\mathcal{G}_{G,E}$ such that $d|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}(U) = d(U)$ and $r|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}(U) = r(U)$ are open sets by [**14,** Proposition I.2.4]. Hence, $d|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}$ and $r|_{\mathcal{G}_{G,E}{}_{I^{(0)}}}$ are open maps, and $I^{(0)}$ is a transversal to $\mathcal{G}_{G,E}$. Therefore, $\mathcal{G}_{G,E}$ is equivalent to $\mathcal{G}_{G,E}{}^{I^{(0)}}_{I^{(0)}}$ by [**8,** Example 2.7]. Moreover, it is clear that $\mathcal{G}_{G,E}{}^{I^{(0)}}_{I^{(0)}} = I$ and that I is isomorphic to $(R^u \rtimes \mathbb{Z})q^T_T$ by $\overline{\pi}$. Thus, we have the following.

Proposition 4.13. *If (G,E) is a contracting and regular self-similar graph action such that E is G-transitive, then $\mathcal{G}_{G,E}$ is equivalent to $(R^u \rtimes \mathbb{Z})q^T_T$.*

Combining Propositions 4.5, 4.8 and 4.13, we have a groupoid equivalence between $R^u \rtimes \mathbb{Z}$ and $\mathcal{G}_{G,E}$:

Theorem 4.14. *If (G,E) is a contracting and regular self-similar graph action such that E is G-transitive, then $R^u \rtimes \mathbb{Z}$ is equivalent to $\mathcal{G}_{G,E}$ in the sense of* [**8**].

Adding up Theorem 4.3, we summarize the above argument.

Theorem 4.15. *If (G, E) is a contracting, regular and pseudo free self-similar graph action such that E is G-transitive, then the unstable Ruelle algebra of $(S_{(G,E)}, \sigma)$ is strongly Morita equivalent to the Cuntz-Pimsner algebra $\mathcal{O}_{G,E}$ of* [**3**].

Remark 4.16. In [**2**], the authors stated that the stable Ruelle algebras of limit solenoids from self-similar graph actions are studied in [**1**].

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