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ON A PROBLEM OF BHARANEDHAR AND PONNUSAMY INVOLVING PLANAR HARMONIC MAPPINGS

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ABSTRACT. In this paper, we give a negative answer to a problem presented by Bharanedhar and Ponnusamy **[1]** concerning univalency of a class of harmonic mappings. More precisely, we show that for all values of the involved parameter, this class contains a non-univalent function. Moreover, several results on a new subclass of close-to-convex harmonic mappings, motivated by the work of Ponnusamy and Sairam Kaliraj **[16]**, are obtained.

1. Introduction. In this paper, we consider univalency criteria for complex-valued harmonic functions f in the open unit disk $\mathbb{D}$. It is well known that such functions can be written as $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f, respectively. Let $\mathcal{H}$ be the class of harmonic functions normalized by the conditions $f(0) = f_z(0) - 1 = 0$, which have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad z \in \mathbb{D}. \tag{1.1}$$

Since the Jacobian of f is given by $|h'|^2 - |g'|^2$, by Lewy's theorem (see **[10]**), it is locally univalent and sense-preserving if and only if $|g'| < |h'|$, or equivalently, the dilatation $\omega = g'/h'$ with $h'(z) \neq 0$ has the property $|\omega| < 1$ in $\mathbb{D}$. The subclass of $\mathcal{H}$ that is univalent and

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sense-preserving in $\mathbb{D}$ is denoted by $\mathcal{S}_{\mathcal{H}}$. Univalent harmonic functions are also called harmonic mappings.

The classical family $\mathcal{S}$ of analytic univalent and normalized functions in $\mathbb{D}$ is a subclass of $\mathcal{S}_{\mathcal{H}}$ with $g(z) \equiv 0$. The family of all functions $f \in \mathcal{S}_{\mathcal{H}}$ with the additional property that $f_{\overline{z}}(0) = 0$ is denoted by $\mathcal{S}_{\mathcal{H}}^0$. There exist reciprocal transformations between the classes $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{H}}^0$ (see [**5, 6**]). Observe that the family $\mathcal{S}_{\mathcal{H}}^0$ is compact and normal; however, the family $\mathcal{S}_{\mathcal{H}}$ is not compact. For recent results involving univalent harmonic mappings, the interested reader is referred to [**1, 2, 3, 4, 7, 8, 11, 12**], [**14**]–[**21**], and the references therein.

A domain Ω is said to be *close-to-convex* if $\mathbb{C}\backslash\Omega$ can be represented as a union of non intersecting half-lines. Following Kaplan's results [**9**], an analytic function F is called close-to-convex if there exists a univalent convex analytic function ϕ defined in $\mathbb{D}$ such that

$$\mathrm{Re}\left(\frac{F'(z)}{\phi'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Furthermore, a planar harmonic mapping $f : \mathbb{D} \to \mathbb{C}$ is close-to-convex if it is injective and $f(\mathbb{D})$ is a close-to-convex domain. We denote by $\mathcal{C}_{\mathcal{H}}^0$ the class of close-to-convex harmonic mappings.

This paper is organized as follows. In Section 2, we give a negative answer to a problem posed by Bharanedhar and Ponnusamy in [**1**]. In Section 3, we study a subclass of close-to-convex harmonic mappings, which is motivated by work of Ponnusamy and Sairam Kaliraj [**16**]. Coefficient estimates, a growth theorem, a covering theorem and an area theorem, for mappings of this class, are obtained.

2. A problem of Bharanedhar and Ponnusamy. Recently, Mocanu [**11**] proposed the following conjecture involving the univalency of planar harmonic mappings.

Conjecture 2.1. *Let*

$$\mathcal{M} = \left\{ f = h + \overline{g} \in \mathcal{H} : g' = zh' \ \textit{and} \ \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2},\ z \in \mathbb{D} \right\}.$$

Then, $\mathcal{M} \subset \mathcal{S}_{\mathcal{H}}^0$.

By applying the close-to-convexity criterion for analytic functions due to Kaplan [**9**], Bshouty and Lyzzaik [**3**] solved the above conjecture by establishing the following, stronger result:

Theorem A. $\mathcal{M} \subset \mathcal{C}_{\mathcal{H}}^0$.

Later, Ponnusamy and Sairam Kaliraj [**16,** Theorem 4.1] generalized Theorem A under the assumption that the analytic dilatation ω satisfies the condition

$$\mathrm{Re}\left(\frac{\lambda z\omega'(z)}{1-\lambda\omega(z)}\right) > -\frac{1}{2}$$

for all λ such that $|\lambda| = 1$. In particular, for $\omega(z) = \lambda k z^n$,

$$\left(|\lambda| = 1;\ 0 < k \leq \frac{1}{2n-1};\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}\right),$$

they gave the following result.

Theorem B. *Suppose that h and g are analytic in $\mathbb{D}$ such that*

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2},$$

and

$$g'(z) = \lambda k z^n h'(z)$$

$$\left(n \in \mathbb{N};\ |\lambda| = 1;\ 0 < k \leq \frac{1}{2n-1}\right).$$

Then, $f = h + \overline{g}$ is univalent and close-to-convex in $\mathbb{D}$.

Motivated by Theorem B, we introduce the following natural class of close-to-convex harmonic mappings, which will be studied in Section 3. Note that, for $n = 1$, we have the class $\mathcal{M}(\alpha, \zeta)$, which was studied in [**18**].

Definition 2.2. A harmonic mapping $f = h + \overline{g} \in \mathcal{H}$ is said to be in the class $\mathcal{M}(\alpha, \zeta, n)$ if h and g satisfy the conditions

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha, \quad -\frac{1}{2} \leq \alpha < 1, \tag{2.1}$$

and

$$g'(z) = \zeta z^n h'(z) \tag{2.2}$$

$$\left(\zeta \in \mathbb{C} \text{ with } |\zeta| \le \frac{1}{2n-1};\ n \in \mathbb{N}\right).$$

In 1995, Ponnusamy and Rajasekaran [**13**] derived the following starlikeness criterion for analytic functions.

Theorem C. *Suppose that F is a normalized analytic function in $\mathbb{D}$. If F satisfies the condition*

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) < \beta, \quad 1 < \beta \le \frac{3}{2},$$

then F is univalent and starlike in $\mathbb{D}$, i.e., $F(\mathbb{D})$ is a domain, starlike with respect to the origin.

Essentially motivated by Theorems A and C, Bharanedhar and Ponnusamy [**1,** page 763, Problem 1] posed the following problem, presented here in a slightly modified form.

Problem 2.3. *For $\beta \in (1, 3/2)$, define*

$$\mathcal{P}(\beta) = \left\{ f = h + \overline{g} \in \mathcal{H} :\ g' = zh' \ \textit{and}\ \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) < \beta,\ z \in \mathbb{D} \right\}.$$

Determine $\inf\{\beta \in (1, 3/2) :\ \mathcal{P}(\beta) \subset \mathcal{S}_{\mathcal{H}}^0\}$.

We recall the following result of Bshouty and Lyzzaik [**3**]:

Theorem D. *Suppose that $0 \le \lambda < 1/2$. Let $f = h + \overline{g}$ be the harmonic polynomial mapping with*

$$h(z) = z - \lambda z^2 \quad \text{and} \quad g(z) = \frac{z^2}{2} - \frac{2\lambda z^3}{3}.$$

If $0 \le \lambda \le 3/10$, then f is univalent in $\mathbb{D}$. However, for $3/10 < \lambda < 1/2$, f is not univalent in $\mathbb{D}$.

Remark 2.4. In view of Theorem D, we see that β can be restricted to the value on the interval $(1, 11/8]$ since

$$\sup_{z\in\mathbb{D}}\left\{\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)\right\}=\frac{11}{8}$$

for

$$h(z)=z-\frac{3}{10}z^2.$$

Now, we are ready to give a counterexample which shows that, for all $\beta\in(1,11/8]$, the class $\mathcal{P}(\beta)$ of Problem 2.3 contains a non-univalent function.

Consider the harmonic function given by $f_\gamma=h+\overline{g}\in\mathcal{H}$, where

$$h(z)=\frac{1}{\gamma}[1-(1-z)^\gamma],\quad 1<\gamma\le\frac{7}{4},$$

and

$$g(z)=\frac{1}{\gamma(1+\gamma)}[1-(1+\gamma z)(1-z)^\gamma],\quad 1<\gamma\le\frac{7}{4}.$$

Clearly, we have $g'=zh'$. It follows that

$$1+\frac{zh''(z)}{h'(z)}=\frac{1-\gamma z}{1-z},$$

and therefore,

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)<\frac{1+\gamma}{2},\quad 1<\frac{1+\gamma}{2}\le\frac{11}{8},$$

that is,

$$f_\gamma=h+\overline{g}\in\mathcal{P}((1+\gamma)/2)\subset\mathcal{P}(\beta).$$

In what follows, we shall prove that the function f_γ is not univalent in $\mathbb{D}$. It is easy to verify that both the analytic and co-analytic parts of f_γ have real coefficients, and thus, $f_\gamma(z)=\overline{f_\gamma(\overline{z})}$ for all $z\in\mathbb{D}$. In particular,

$$\operatorname{Re}(f_\gamma(re^{i\theta}))=\operatorname{Re}(f_\gamma(re^{-i\theta}))$$

for some $r\in(0,1)$ and $\theta\in(-\pi,0)\cup(0,\pi)$. It suffices to show that there exist $r_0\in(0,1)$ and $\theta_0\in(-\pi,0)\cup(0,\pi)$ such that

$$\operatorname{Im}(f_\gamma(r_0e^{i\theta_0}))=\operatorname{Im}(f_\gamma(r_0e^{-i\theta_0}))=0.$$

In view of the relation

$$\begin{aligned}\operatorname{Im}(f_\gamma(z)) = \operatorname{Im}(h(z) - g(z)) &= \operatorname{Im}\left(\frac{1-(1-z)^{\gamma+1}}{\gamma+1}\right)\\ &= -\operatorname{Im}\left(\frac{e^{(\gamma+1)\log(1-z)}}{\gamma+1}\right),\end{aligned}$$

we see that

$$\begin{aligned}\operatorname{Im}(f_\gamma(re^{i\theta})) &= -\operatorname{Im}\left(\frac{e^{(\gamma+1)\log(1-re^{i\theta})}}{\gamma+1}\right)\\ &= -\frac{e^{(\gamma+1)\log|1-re^{i\theta}|}}{\gamma+1}\sin[(\gamma+1)\arg(1-re^{i\theta})],\end{aligned}$$

and

$$\begin{aligned}-\operatorname{Im}(f_\gamma(re^{-i\theta})) &= \frac{e^{(\gamma+1)\log|1-re^{-i\theta}|}}{\gamma+1}\sin[(\gamma+1)\arg(1-re^{-i\theta})]\\ &= \operatorname{Im}(f_\gamma(re^{i\theta})).\end{aligned}$$

By noting that

$$\arg(1-re^{i\theta}) \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right),$$

we deduce that, for each $1 < \gamma \leq 7/4$, there exist $r_0 \in (0,1)$ and $\theta_0 \in (-\pi, 0) \cup (0, \pi)$ such that

$$\sin[(\gamma+1)\arg(1-r_0e^{i\theta_0})] = 0.$$

It follows that

$$\operatorname{Im}(f_\gamma(r_0e^{i\theta_0})) = \operatorname{Im}(f_\gamma(r_0e^{-i\theta_0})) = 0.$$

Therefore, there exist two distinct points $z_1 = r_0e^{i\theta_0}$ and $z_2 = r_0e^{-i\theta_0}$ in $\mathbb{D}$ such that $f_\gamma(z_1) = f_\gamma(z_2)$, which shows that the function $f_\gamma(z)$ is not univalent in $\mathbb{D}$. Thus, we conclude that the conditions given in Problem 2.3 are not satisfied for any $\beta \in (1, 11/8]$.

The image domain of f_γ for $\gamma = 5/4$ is given in Figures 1 and 2 to illustrate our counterexample.

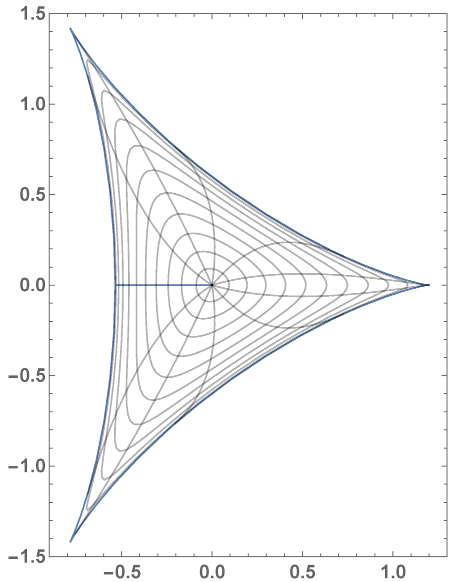

FIGURE 1. The image of the mapping $f_{5/4}$.

3. The subclass $\mathcal{M}(\alpha, \zeta, n)$ of close-to-convex harmonic mappings. Recall the following lemma, due to Suffridge [**17**], which will be required in the proof of Theorem 3.2.

Lemma 3.1. *If $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ satisfies condition* (2.1), *then*

$$|a_k| \le \frac{1}{k!} \prod_{j=2}^{k} (j - 2\alpha), \quad k \in \mathbb{N} \setminus \{1\}, \tag{3.1}$$

with the extremal function given by

$$h(z) = \int_0^z \frac{dt}{(1 - \delta t)^{2-2\alpha}}, \quad |\delta| = 1;\ z \in \mathbb{D}.$$

We now derive the coefficient estimates for the class $\mathcal{M}(\alpha, \zeta, n)$.

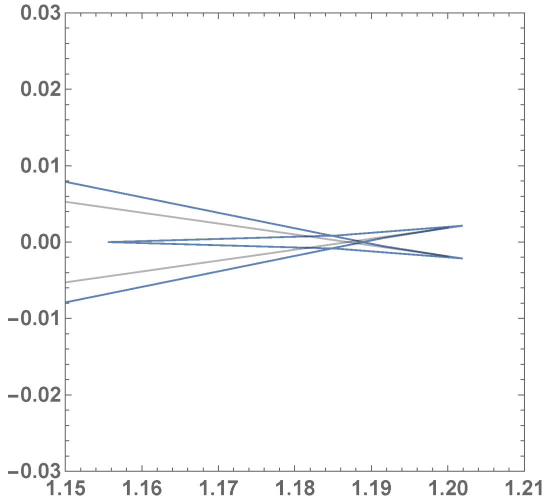

FIGURE 2. An enlarged view of the right cusp of the image of $f_{5/4}$.

Theorem 3.2. *Let* $f = h + \overline{g} \in \mathcal{M}(\alpha, \zeta, n)$ *be of the form* (1.1). *Then, the coefficients* a_k, $k \in \mathbb{N} \setminus \{1\}$, *of* h *satisfy* (3.1). *Moreover, the coefficients* b_k, $k = n+1, n+2, \ldots;\ n \in \mathbb{N}$, *of* g *satisfy*:

$$|b_{n+1}| \leq \frac{|\zeta|}{n+1}$$

and

$$|b_{k+n}| \leq \frac{|\zeta|}{(k+n)(k-1)!} \prod_{j=2}^{k} (j - 2\alpha), \quad k \in \mathbb{N} \setminus \{1\}.$$

The bounds are sharp for the extremal function given by

$$f(z) = \int_0^z \frac{dt}{(1-\delta t)^{2-2\alpha}} + \overline{\int_0^z \frac{\zeta t^n}{(1-\delta t)^{2-2\alpha}}\, dt}, \quad |\delta| = 1;\ z \in \mathbb{D}.$$

Proof. By equating the coefficients of z^{k+n-1} on both sides of (2.2), we see that

$$(k+n) b_{k+n} = \zeta k a_k, \quad k, n \in \mathbb{N};\ a_1 = 1. \tag{3.2}$$

In view of Lemma 3.1 and (3.2), we obtain the desired result of Theorem 3.2. □

Theorem 3.3. *Let* $f \in \mathcal{M}(\alpha,\zeta,n)$ *with* $0 \le \alpha < 1$ *and* $0 \le \zeta < 1/(2n-1)$, $n \in \mathbb{N}$. *Then:*

$$\Phi(r;\alpha,\zeta,n) \le |f(z)| \le \Psi(r;\alpha,\zeta,n), \quad r=|z|<1, \tag{3.3}$$

where

$$\Phi(r;\alpha,\zeta,n) = \begin{cases} \log(1+r) - \frac{\zeta\, r^{n+1}\, {}_2F_1(1,n+1;n+2;-r)}{n+1} & \alpha=1/2,\\ \frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} - \frac{\zeta\, r^{n+1}\, {}_2F_1(n+1,2-2\alpha;n+2;-r)}{n+1} & \alpha\neq 1/2,\end{cases}$$

and

$$\Psi(r;\alpha,\zeta,n) = \begin{cases} -\log(1-r) + \frac{\zeta\, r^{n+1}\, {}_2F_1(1,n+1;n+2;r)}{n+1} & \alpha=1/2,\\ \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1} + \frac{\zeta\, r^{n+1}\, {}_2F_1(n+1,2-2\alpha;n+2;r)}{n+1} & \alpha\neq 1/2.\end{cases}$$

All of these bounds are sharp. The extremal function is $f_{\alpha,\zeta,n} = h_\alpha + \overline{g_{\alpha,\zeta,n}}$, *or its rotations, where*

$$f_{\alpha,\zeta,n}(z) = \begin{cases} -\log(1-z) + \overline{\frac{\zeta\, z^{n+1}\, {}_2F_1(1,n+1;n+2;z)}{n+1}} & \alpha=1/2,\\ \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} + \overline{\frac{\zeta\, z^{n+1}\, {}_2F_1(n+1,2-2\alpha;n+2;z)}{n+1}} & \alpha\neq 1/2.\end{cases} \tag{3.4}$$

Proof. Assume that $f = h + \overline{g} \in \mathcal{M}(\alpha,\zeta,n)$. Also, let Γ be the line segment joining 0 and z. Then

$$\begin{aligned} |f(z)| &= \left|\int_\Gamma \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \overline{\xi}} d\overline{\xi}\right| \le \int_\Gamma (|h'(\xi)| + |g'(\xi)|)\,|d\xi| \\ &= \int_\Gamma (1+|\zeta||\xi|^n)|h'(\xi)||d\xi|. \end{aligned} \tag{3.5}$$

Moreover, let $\widetilde{\Gamma}$ be the preimage under f of the line segment joining 0 and $f(z)$. Then, we obtain

$$\begin{aligned} |f(z)| &= \int_{\widetilde{\Gamma}} \left|\frac{\partial f}{\partial \xi}\, d\xi + \frac{\partial f}{\partial \overline{\xi}}\, d\overline{\xi}\right| \ge \int_{\widetilde{\Gamma}} (|h'(\xi)| - |g'(\xi)|)\,|d\xi| \\ &= \int_{\widetilde{\Gamma}} (1-|\zeta||\xi|^n)|h'(\xi)||d\xi|. \end{aligned} \tag{3.6}$$

By observing that h is a convex analytic function of order α, $0 \leq \alpha < 1$, it follows that

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leq |h'(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}}, \quad |z| = r < 1. \tag{3.7}$$

By virtue of (3.5)–(3.7), we see that

$$\begin{aligned}\Phi(r;\alpha,\zeta,n) := \int_0^r \frac{(1-|\zeta|\rho^n)\,d\rho}{(1+\rho)^{2(1-\alpha)}} \\ \leq |f(z)| \leq \int_0^r \frac{(1+|\zeta|\rho^n)\,d\rho}{(1-\rho)^{2(1-\alpha)}} \\ =: \Psi(r;\alpha,\zeta,n),\end{aligned}$$

which yields the desired inequalities (3.3).

Now, we shall prove the sharpness of the result. We only need show that $f_{\alpha,\zeta,n}$, defined by (3.4), belongs to the class $\mathcal{M}(\alpha,\zeta,n)$ for each $\alpha \in [0,1)$. Suppose that

$$h_\alpha(z) = \begin{cases} -\log(1-z) & \alpha = 1/2, \\ \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \alpha \neq 1/2. \end{cases}$$

Then, we find that $h_\alpha(z)$ satisfies inequality (2.1) and the relation $g'_{\alpha,\zeta,n}(z) = \zeta z^n h'_\alpha(z)$ for each $\alpha \in [0,1)$. Moreover, for $0 \leq \alpha < 1$, $0 < \zeta < 1/(2n-1)$ with $n \in \mathbb{N}$, $0 < r < 1$, it is easy to see that

$$f_{\alpha,\zeta,n}(-r) = -\Phi(r;\alpha,\zeta,n) \quad \text{and} \quad f_{\alpha,\zeta,n}(r) = \Psi(r;\alpha,\zeta,n),$$

and therefore,

$$\left|f_{\alpha,\zeta,n}(-r)\right| = \Phi(r;\alpha,\zeta,n) \quad \text{and} \quad \left|f_{\alpha,\zeta,n}(r)\right| = \Psi(r;\alpha,\zeta,n).$$

This shows that the bounds are sharp. □

Next, we consider a covering theorem for functions in the class $\mathcal{M}(\alpha,\zeta,n)$.

Theorem 3.4. *Let $f \in \mathcal{M}(\alpha,\zeta,n)$ with $0 \leq \alpha < 1$ and $0 \leq \zeta < 1/(2n-1)$, $n \in \mathbb{N}$. Then, the range $f(\mathbb{D})$ contains the disk*

$$|\omega| < r(\alpha,\zeta,n) = \begin{cases} \log 2 - \frac{\zeta\, {}_2F_1(1,n+1;n+2;-1)}{n+1} & \alpha = 1/2, \\ \frac{2^{2\alpha-1}-1}{2\alpha-1} - \frac{\zeta\, {}_2F_1(n+1,2-2\alpha;n+2;-1)}{n+1} & \alpha \neq 1/2. \end{cases}$$

The bounds are sharp for the function $f_{\alpha,\zeta,n} = h_\alpha + \overline{g_{\alpha,\zeta,n}}$, *given by* (3.4) *or its rotations.*

Proof. By putting $r \to 1^-$ in the lower bound for $|f(z)|$ in Theorem 3.3, we obtain the desired result. The sharpness is similar to that of Theorem 3.2; thus, we omit the details. □

Now, we consider the area theorem of the mappings belonging to the class $\mathcal{M}(\alpha,\zeta,n)$. We denote $\mathcal{A}(f(\mathbb{D}_r))$ by the area of $f(\mathbb{D}_r)$, where $\mathbb{D}_r := r\mathbb{D}$ for $0 < r < 1$.

Theorem 3.5. *Let* $f \in \mathcal{M}(\alpha,\zeta,n)$ *with* $0 \le \alpha < 1$. *Then, for* $0 < r < 1$, $\mathcal{A}(f(\mathbb{D}_r))$ *satisfies the inequalities*

$$2\pi \int_0^r \frac{\rho(1-|\zeta|^2\rho^{2n})}{(1+\rho)^{4(1-\alpha)}}\,d\rho \le \mathcal{A}(f(\mathbb{D}_r)) \le 2\pi \int_0^r \frac{\rho(1-|\zeta|^2\rho^{2n})}{(1-\rho)^{4(1-\alpha)}}\,d\rho. \tag{3.8}$$

Proof. Let $f = h + \overline{g} \in \mathcal{M}(\alpha,\zeta,n)$. Then, for $0 < r < 1$, we see that

$$\begin{aligned} \mathcal{A}(f(\mathbb{D}_r)) &= \iint_{\mathbb{D}_r} (|h'(z)|^2 - |g'(z)|^2)\,dx\,dy \\ &= \iint_{\mathbb{D}_r} (1 - |\zeta|^2|z|^{2n})|h'(z)|^2\,dx\,dy. \end{aligned} \tag{3.9}$$

By observing that h is a convex analytic function of order α, $0 \le \alpha < 1$, in view of (3.7) and (3.9), we obtain the desired inequalities (3.8) of Theorem 3.5. □

Remark 3.6. By setting $n = 1$ in Theorems 3.2, 3.3, 3.4 and 3.5, respectively, we get the corresponding results obtained in **[18]**.

Finally, we discuss the radius of close-to-convexity of a certain class of harmonic mappings related to the class $\mathcal{M}(\alpha,\zeta,n)$. The next lemma, due to Clunie and Sheil-Small **[5]**, will be required in the proof of Theorem 3.8.

Lemma 3.7. *If* h *and* g *are analytic in* $\mathbb{D}$ *with* $|h'(0)| > |g'(0)|$, *and* $h+\lambda g$ *is close-to-convex for each* λ $(|\lambda| = 1)$, *then* $f = h+\overline{g}$ *is harmonic close-to-convex in* $\mathbb{D}$.

Theorem 3.8. *Suppose that* $f = h + \overline{g}$ *satisfies inequality* (2.1) *with* $-1/2 < \alpha < 0$. *If* $g'(z) = z^n h'(z)$ *with* $n \in \mathbb{N} \setminus \{1\}$, *then* f *is close-to-convex in the disk*

$$|z| < \sqrt[n]{\frac{1+2\alpha}{1+2n+2\alpha}}, \quad n \in \mathbb{N} \setminus \{1\}.$$

Proof. Suppose that $F_\lambda(z) = h(z) - \lambda g(z)$ with $|\lambda| = 1$. It follows that

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zF_\lambda''(z)}{F_\lambda'(z)}\right) &= \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) + n\mathrm{Re}\left(\frac{\lambda z^n}{\lambda z^n - 1}\right) \\ &= \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) + \frac{n}{2}\left(1 - \frac{1 - |\lambda z^n|^2}{(1-\lambda z^n)(1-\overline{\lambda z^n})}\right). \end{aligned}$$

For $z = re^{i\theta}$ $(0 < r < 1)$, we see that

$$\begin{aligned} \frac{n}{2}\left(1 - \frac{1 - |\lambda z^n|^2}{(1-\lambda z^n)(1-\overline{\lambda z^n})}\right) &= \frac{n}{2}\left(1 - \frac{1 - r^{2n}}{1 + r^{2n} - 2\mathrm{Re}(\lambda z^n)}\right) \\ &\geq -\frac{nr^n}{1-r^n}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zF_\lambda''(z)}{F_\lambda'(z)}\right) d\theta &> \int_{\theta_1}^{\theta_2} \left(\alpha - \frac{nr^n}{1-r^n}\right) d\theta \\ &= \left(\alpha - \frac{nr^n}{1-r^n}\right)(\theta_2 - \theta_1) \\ &> -\pi, \quad \theta_1 < \theta_2 < \theta_1 + 2\pi \end{aligned}$$

for

$$|z| = r < \sqrt[n]{\frac{1+2\alpha}{1+2n+2\alpha}} =: r(\alpha, n).$$

From Lemma 3.7 and Kaplan's close-to-convexity criterion for analytic functions (see **[9]**), we deduce that f is close-to-convex in the disk $|z| < r(\alpha, n)$. □

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