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AN ANALYTIC NOVIKOV CONJECTURE FOR SEMIGROUPS

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ABSTRACT. In this article, we formulate a version of the analytic Novikov conjecture for semigroups rather than groups and show that the descent argument from coarse geometry generalizes effectively to this new situation.

1. Introduction. For the purposes of this article, a *semigroup* is a set P equipped with an associative binary operation $P \times P \to P$, such that we have a *unit element* $e \in P$ where $pe = ep = p$ for all $p \in P$, and the *left cancellation property* holds, that is to say, $pq = pr$ implies $q = r$ for all $p, q, r \in P$. Note that the left cancellation property tells us that the unit element e is unique.

In [**7**], both the reduced and maximal C^*-algebras associated to a semigroup are defined, issues associated to amenability examined and K-theory groups computed. The computations of K-theory groups lead to a natural question, namely, whether a version of the Baum-Connes conjecture [**1**] could be formulated for semigroups.

In this paper, we make a first step towards such a conjecture, formulating an *analytic assembly map*

$$\beta\colon K_n^P(EP) \longrightarrow K_n C_r^*(P),$$

where $C_r^*(P)$ is the reduced C^*-algebra of the semigroup P, and EP is the classifying space for free P-actions. We conjecture that this map is injective for torsion-free semigroups.

We also show that the descent argument from the coarse Baum-Connes conjecture, as explained, for example, in [**12**], or, more generally, in [**11**], still works in the semigroup case. Thus, the analytic Novikov conjecture holds for semigroups where the space EP is a finite P-CW-complex and has a compatible coarse structure where the

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coarse Baum-Connes conjecture is satisfied. We conclude the article by looking at some simple examples where the descent argument applies.

The descent argument works in the same way as it does for groups; however, to carry it out, we need to generalize parts of the general theory of equivariant homology for group actions to the semigroup case. These generalizations are fortunately mainly straightforward, and the details may be found in Sections 4 and 5.

The broad strategy of the proof is as follows.

• We define a notion of generalized homology theory for P-spaces, where P is a semigroup.

• We prove that a natural transformation of generalized homology theories for P-spaces is an isomorphism for finite P-CW-complexes if it is an isomorphism for homogeneous P-spaces.

• We show that the K-homology of homotopy fixed-point defines a generalized homology theory for P-spaces and that there is a natural transformation from P-equivariant K-homology to the K-homology of homotopy fixed point sets. We use the above argument to prove that this natural transformation is an isomorphism for finite P-CW-complexes.

• A commutative diagram relates the above transformation to the analytic assembly map and the coarse Baum-Connes conjecture, from which we deduce the descent result.

2. Semigroup actions. Let P be a semigroup. Let X be a set. A *left P-action* on X is a map $P \times X \to X$, written $(p, x) \mapsto px$, such that $p(qx) = (pq)x$ for all $p, q \in P$ and $x \in X$.

Similarly, a *right P-action* on X is a map $X \times P \to X$, written $(x, p) \mapsto xp$, such that $(xp)q = x(pq)$ for all $p, q \in P$ and $x \in X$.

For a set X equipped with a left P-action, and a subset $A \subseteq X$, we write

$$pA = \{pa \mid a \in A\}, \qquad PA = \bigcup_{p \in P} pA.$$

Given sets X and Y with left P-actions, a map $f \colon X \to Y$ is called *equivariant* if $f(px) = pf(x)$ for all $x \in X$ and $p \in P$. We similarly talk about equivariant maps between sets equipped with right P-actions.

A *P-space* is a topological space equipped with a continuous right P-action. Similarly as for groups acting on spaces, we distinguish certain types of P-spaces. Given P-spaces X and Y, we write $\mathrm{Map}_P(X,Y)$ to denote the set of all continuous equivariant maps from X to Y. It is a topological space, with the compact open topology.

Definition 2.1. Let X be a P-space. Then, we call X:

- *free* if, for all $x \in X$, there is an open neighborhood $U \ni x$ such that $Up \cap U = \emptyset$ for all $p \in P \backslash \{e\}$, where e is the unit element of P;
- *cocompact* if there is a compact subset $K \subseteq X$ such that $X = KP$.

A *fundamental domain* in a P-space X is a subset $D \subseteq X$ such that every element $x \in X$ can be written uniquely as $x = sp$, where $s \in D$ and $p \in P$.

We call an equivariant continuous map $f\colon X \to Y$ *proper* if, whenever $Z \subseteq Y$ is cocompact, the inverse image $f^{-1}[Z] \subseteq X$ is also cocompact.

Example 2.2. We call a subset $S \subseteq P$ a *generating set* if every element of P is a product of elements of S. The *Cayley graph* $\mathrm{Cay}(P;S)$ is an oriented labeled graph with set of vertices P. The element $s \in S$ is an oriented edge from p to q if $p = sq$. There is a right P-action on the space $\mathrm{Cay}(P;S)$ defined by right-multiplication on the vertices and extending to be linear on the edges. The P-space $\mathrm{Cay}(P;S)$ is cocompact if S is finite. The P-action on the vertices is free; it is also free on the edges if P contains no elements of order 2.

Example 2.3. The *infinite join*, see **[8]**, $P * P * P * \cdots$ of countably many copies of the semigroup P is a free and weakly contractible P-space.

Let X and Y be metric spaces. Recall (see **[13]**) that a (not necessarily continuous) map $f\colon X \to Y$ is called a *coarse map* if:

- for all $R > 0$, there exists an $S > 0$ such that, if $d(x,y) < R$, then $d(f(x), f(y)) < S$;
- let $B \subseteq Y$ be bounded. Then, $f^{-1}[B] \subseteq X$ is also bounded.

A *coarse P-space* is a proper metric space X equipped with a P-action such that, for each $p \in P$, the map $p\colon X \to X$ is both coarse and continuous.

Note that, for a generating set S, the Cayley graph $\mathrm{Cay}(P;S)$ is an example of a coarse P-space.

Definition 2.4. Let P be a semigroup. We call an equivalence relation $\sim$ on P a *right congruence* if, whenever $p \sim q$ and $p, q, r \in P$, we have $pr \sim qr$.

Observe that, if we have a right congruence $\sim$ we have a right P-action on the set of equivalence classes $P/\sim$ defined by writing $([p])q = [pq]$, where $[r]$ is the equivalence class containing an element $r \in P$. The quotient $P/\sim$ can be considered a P-space with the discrete topology.

Definition 2.5. A *homogeneous P-space* is a P-space X such that there is an equivariant homeomorphism $X \to P/\sim$ for some right congruence $\sim$.

We now define a class of P-spaces of particular importance to us, called *P-CW-complexes.* Firstly, write

$$D^{n+1} = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 \leq 1\}$$

and

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}.$$

Note that $S^n \subseteq D^{n+1}$. An *n-dimensional P-cell* is a P-space of the form $X \times D^n$, where X is a homogeneous P-space, and P acts trivially on D^n.

Given a P-space Y and P-cell $X \times D^n$ equipped with a continuous equivariant map $f \colon X \times S^{n-1} \to Y$, we can form a P-space

$$(X \times D^n) \cup_{X \times S^{n-1}} Y = \frac{(X \times D^n) \amalg Y}{\sim},$$

where $(x, s) \sim f(x, s)$ if $(x, s) \in X \times S^{n-1}$.

We call the P-space $(X \times D^n) \cup_{X \times S^{n-1}} Y$ the space obtained from Y by *attaching* the P-cell $X \times D^n$ by the map f.

Definition 2.6. A *finite P-CW-complex* is a P-space X together with a sequence of subspaces

$$X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X$$

such that:

- each inclusion $X^i \hookrightarrow X^{i+1}$ is equivariant;
- the space X^0 is a finite disjoint union of homogeneous P-spaces;
- the space X^k is equivariantly homeomorphic to the space obtained from X^{k-1} by attaching finitely many k-dimensional P-cells.

The above sequence $X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X$ is called a *CW-decomposition* of X. Assuming $X^n \neq X^{n-1}$, the number n is called the *dimension* of the cell decomposition.

Note that any finite P-CW-complex is cocompact. The following is fairly clear.

Proposition 2.7. *Let X be a finite P-CW-complex. Then, X is free if and only if it has a CW-composition in which, for all k, every k-dimensional P-cell takes the form $P \times D^n$.*

3. The coarse Baum-Connes conjecture. Let X be a proper metric space. Recall that a Hilbert space H is called an *X-module* if the C^*-algebra of bounded linear operators $\mathcal{L}(H)$ is equipped with a $*$-homomorphism $\rho\colon C_0(X) \to \mathcal{L}(H)$.

Let $\mathcal{K}(H)$ be the C^*-algebra of compact operators on H. Then, we call an X-module H *ample* if $\overline{\rho[C_0(X)]H} = H$ and $\rho(f) \in \mathcal{K}(H)$ implies $f = 0$.

Definition 3.1. Let H be an X-module, and let $T \in \mathcal{L}(H)$. Then:

- we call T *locally compact* if $\rho(f)T, T\rho(f) \in \mathcal{K}(H)$ for all $f \in C_0(X)$;
- we call T *pseudolocal* if $\rho(f)T - T\rho(f) \in \mathcal{K}(H)$ for all $f \in C_0(X)$;
- we define the *support* of T, $\mathrm{Supp}(T) \subseteq X \times X$, to be the set of pairs $(x, y) \in X \times X$ such that, for all open sets $U \ni x$ and $V \ni y$, we have $f \in C_0(U)$ and $g \in C_0(V)$ such that $\rho(f)T\rho(g) \neq 0$;

• we call T *controlled* if the support $\mathrm{Supp}(T)$ is contained in a neighborhood of the diagonal, $\Delta_R = \{(x,y) \in X \times X \mid d(x,y) < R\}$, for some $R > 0$.

Definition 3.2. Let H be an ample X-module. Then, we define $D^*(X)$ to be the smallest C^*-subalgebra of $\mathcal{L}(H)$ containing all pseudolocal and controlled operators. We define $C^*(X)$ to be the smallest C^*-subalgebra of $\mathcal{L}(H)$ containing all locally compact and controlled operators.

Now, $C^*(X)$ is a C^*-ideal in $D^*(X)$, so that we have a short exact sequence

$$0 \longrightarrow C^*(X) \longrightarrow D^*(X) \longrightarrow \frac{D^*(X)}{C^*(X)} \longrightarrow 0.$$

Further, as shown in [**6, 12**], the K-theory group of the quotient, $K_n(D^*(X)/C^*(X))$, is isomorphic to the K-homology group $K_{n-1}(X)$, and the K-theory group $K_n C^*(X)$ does not depend on a particular choice of X-module. Thus, looking at the boundary maps in the long exact sequence of K-theory groups [**14, 16**], we obtain a map

$$\alpha \colon K_*(X) \longrightarrow K_* C^*(X),$$

called the *coarse assembly map*.

The *coarse Baum-Connes conjecture* asserts that this map is an isomorphism whenever the space X has bounded geometry and is uniformly contractible; the reader is again referred to [**6, 12**] for details, including precisely what the terms bounded geometry and uniformly contractible mean.

The coarse Baum-Connes conjecture is known to be true for a vast number of spaces, perhaps most notably, bounded geometry and uniformly contractible spaces which can be uniformly embedded in Hilbert space, see [**17**], but, as shown in [**5**], is false in general.

4. Equivariant homology.

Definition 4.1. Let $f, g : X \to Y$ be equivariant maps between P-spaces. A *P-homotopy* between f and g is an equivariant continuous map $H \colon X \times [0,1] \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$.

We call H a *proper P-homotopy* if H is a proper equivariant continuous map.

Above, the space $X \times [0,1]$ is given the P-action defined by the formula $(x,t)p = (xp,t)$ where $p \in P$, $x \in X$ and $t \in [0,1]$.

If a P-homotopy exists between maps f and g, we call them *P-homotopic*, and write $f \simeq_P g$. The notion of being P-homotopic is an equivalence relation. The equivalence relation of being properly P-homotopic is similarly defined.

A continuous equivariant map $f\colon X \to Y$ is called a *P-homotopy equivalence* if there is a continuous equivariant map $g\colon Y \to X$ such that $g \circ f \simeq_P \mathrm{id}_X$ and $f \circ g \simeq_P \mathrm{id}_Y$. We write $X \simeq_P Y$ when a P-homotopy equivalence $X \to Y$ exists.

Definition 4.2. A *locally finite P-homology theory*, h_*^P, graded over $\mathbb{Z}$, consists of a sequence of functors, h_n^P (where $n \in \mathbb{Z}$), from the category of P-spaces and proper equivariant maps to the category of abelian groups satisfying the following axioms.

• Let $f, g : X \to Y$ be equivariant continuous maps that are properly P-homotopic. Then, the maps $f_*, g_* : h_n^P(X) \to h_n^P(Y)$ induced by the functor h_n^P are equal for all n.

• Let $X = A \cup B$ be a P-space, where $A, B \subseteq X$ are open, and $PA \subseteq A$, $PB \subseteq B$. Consider the inclusions $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$ and $l : B \hookrightarrow X$. Let

$$\alpha = (i_*, -j_*) : h_n^P(A \cap B) \longrightarrow h_n^P(A) \oplus h_n^P(B)$$

and

$$\beta = k_* + l_* : h_n^P(A) \oplus h_n^P(B) \longrightarrow h_n^P(X).$$

Then, we have natural maps $\partial : h_n^P(X) \to h_{n-1}^P(A \cap B)$ fitting into a long exact sequence

$$\longrightarrow h_n^P(A \cap B) \xrightarrow{\alpha} h_n^P(A) \oplus h_n^P(B) \xrightarrow{\beta} h_n^P(X) \xrightarrow{\partial} h_{n-1}^P(A \cap B) \longrightarrow .$$

• $h_n(\emptyset) = \{0\}$ for all n.

We call the first of these axioms *homotopy invariance*. The long exact sequence in the second axiom is called the *Mayer-Vietoris sequence* associated to the decomposition $X = A \cup B$.

We can also talk about locally finite P-homology theories on subcategories of the category of P-spaces and proper equivariant maps, for instance, on the category of free P-spaces.

Lemma 4.3. *Let X and Y be P-spaces, let $f : X \times S^{n-1} \to Y$ be proper equivariant continuous map, and let $Z = (X \times D^n) \cup_{X \times S^{n-1}} Y$. Then, we have a natural long exact sequence*

$$\longrightarrow h_n^P(X \times S^{n-1}) \longrightarrow h_n^P(X) \oplus h_n^P(Y) \longrightarrow h_n^P(Z) \xrightarrow{\partial} h_{n-1}^P(X \times S^{n-1}) \longrightarrow .$$

Further, the map ∂ arises from a Mayer-Vietoris sequence associated to a decomposition of Z.

Proof. Let $\pi : (X \times D^n) \amalg Y \to Z$ be the quotient map. We can choose open sets $U \subseteq \pi[X \times D^n]$ and $V \supseteq \pi[Y]$ such that:

- $U \cup V = Z$;
- $PU \subseteq U$, $PV \subseteq V$;
- $U \simeq_P X \times D^n \simeq_P X$;
- $V \simeq_P Y$;
- $U \cap V \simeq_P X \times S^{n-1}$.

Applying the Mayer-Vietories sequence of the decomposition $X = U \cup V$ along with homotopy invariance, we get a long exact sequence

$$\longrightarrow h_n^P(X \times S^{n-1}) \longrightarrow h_n^P(X) \oplus h_n^P(Y) \longrightarrow h_n^P(Z) \xrightarrow{\partial} h_{n-1}^P(X \times S^{n-1}) \longrightarrow . \qquad \square$$

Definition 4.4. Let h_*^P and k_*^P be locally finite P-homology theories. A *natural transformation* $\tau : h_*^P \to k_*^P$ is a sequence of natural transformations $\tau : h_n^P \to k_n^P$ which preserve Mayer-Vietoris sequences.

Lemma 4.5. *Let X be a P-space, and let $\tau : h_*^P \to k_*^P$ be a natural transformation between P-homology theories such that the maps τ :*

$h_n^P(X) \to k_n^P(X)$ *are isomorphisms. Then, the maps* $\tau : h_n^P(X \times S^k) \to k_n^P(X \times S^k)$ *are all isomorphisms.*

Proof. Observe that

$$X \times S^0 = X_1 \amalg X_2,$$

where X_1 and X_2 are both equivariantly homeomorphic to X. Certainly, $X_1 \cap X_2 = \emptyset$; thus, $h_n^P(X_1 \cap X_2) = 0$ for all n, and the Mayer-Vietoris sequence provides that $h_n^P(X \times S^0) = h_n^P(X) \oplus h_n^P(X)$. Similarly, $k_n^P(X \times S^0) = k_n^P(X) \oplus k_n^P(X)$.

It immediately follows that the map $\tau : h_n^P(X \times S^0) \to k_n^P(X \times S^0)$ is an isomorphism.

Now suppose the map $\tau : h_n^P(X \times S^{k-1}) \to k_n^P(X \times S^{k-1})$ is an isomorphism for all n. We can write $S^k = A \cup B$, where $A \cong D^n$, $B \cong D^n$ and $A \cap B \simeq S^{k-1}$, so that $X \times A \simeq_P B \times A \simeq_P X$. Then, we have a commutative diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} h_n^P(X \times S^{k-1}) & \to & h_n^P(X) \oplus h_n(X) & \to & h_n^P(S^k \times X) & \to & h_{n-1}^P(S^{k-1} \times X) & \to & h_{n-1}^P(X) \oplus h_{n-1}^P(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k_n^P(X \times S^{k-1}) & \to & k_n^P(X) \oplus k_n(X) & \to & k_n^P(S^k \times X) & \to & k_{n-1}^P(S^{k-1} \times X) & \to & k_{n-1}^P(X) \oplus h_{n-1}^P(X). \end{array}$$

By the five lemma **[15]**, we see the map $\tau : h_n^P(X \times S^k) \to k_n^P(X \times S^k)$ is an isomorphism for all n. The desired result now follows by induction. □

Theorem 4.6. *Let* $\tau : h_*^P \to k_*^P$ *be a natural transformation of* P*-homology theories such that* $\tau : h_m^P(X) \to k_m^P(X)$ *is an isomorphism whenever* X *is a homogeneous* P*-space. Then,* $\tau : h_m^P(Z) \to k_m^P(Z)$ *is an isomorphism whenever* Z *is a finite* P*-*CW*-complex.*

Proof. Let Z be a finite P-CW-complex. Then, we have subsets

$$Z^0 \subseteq Z^1 \subseteq \cdots \subseteq Z^n = Z,$$

where Z^0 is a finite disjoint union of homogeneous P-spaces, and Z^k is equivariantly homeomorphic to the space obtained from Z^{k-1} by attaching finitely many k-dimensional P-cells. Certainly, the map $\tau : h_m^P(Z^0) \to k_m^P(Z^0)$ is an isomorphism for all m.

Let Y be a P-space such that the map $\tau : h_m^P(Y) \to h_m^P(Y)$ is an isomorphism for all m. Suppose that we have an attaching map

$f : X \times S^{n-1} \to Y$ for a homogeneous P-space X. Let $Y' = (X \times D^n) \cup_{X \times S^{n-1}} Y$. Then, it follows by Lemma 4.3, Lemma 4.5 and the five lemma [**15**] that the map $\tau : h^P_m(Y') \to h^P_m(Y')$ is an isomorphism for all m.

Hence, this proves the desired result by induction. □

The following is proved similarly.

Theorem 4.7. *Let $\tau : h^P_* \to k^P_*$ be a natural transformation of P-homology theories such that $\tau : h^P_m(P) \to k^P_m(P)$ is an isomorphism for all m. Then, $\tau : h^P_m(Z) \to k^P_m(Z)$ is an isomorphism whenever Z is a free finite P-CW-complex.*

5. Homotopy fixed point sets. Let E be a P-space. Given an equivariant continuous map $f : X \to Y$, we have an induced map $f_* : \mathrm{Map}_P(E, X) \to \mathrm{Map}_P(E, Y)$ defined by the formula $f_*(g)(\lambda) = f(g(\lambda))$ where $g \in \mathrm{Map}_P(E, X)$ and $\lambda \in E$.

Proposition 5.1. *Let E be a finite free P-CW-complex. Let X be (non-equivariantly) weakly contractible. Then, the space $\mathrm{Map}_P(E, X)$ is also weakly contractible.*

Proof. Let E be a zero-dimensional free P-CW-complex. Then, E is a disjoint union of finitely many, say k, copies of P. Hence,

$$\underset{P}{\mathrm{Map}}(E, X) = \underset{P}{\mathrm{Map}}(P, X)^k \cong X^k,$$

which is weakly contractible since so is X.

More generally, if E is a free P-CW-complex, it is obtained by attaching finitely many cells to a zero-dimensional P-CW-complex. Thus, to prove the result, it suffices to show by induction that, if Y is a free P-space where $\mathrm{Map}_P(Y, X)$ is weakly contractible, and E is a space obtained by attaching a free P-cell to Y, then $\mathrm{Map}_P(E, X)$ is weakly contractible.

Let $f : P \times S^{n-1} \to Y$ be a continuous equivariant map, and let

$$E = (P \times D^n) \cup_{P \times S^{n-1}} Y.$$

It is a standard example from algebraic topology, for example, **[15]**, that the inclusion $S^{n-1} \to D^n$ is a cofibration. Hence, the inclusion $P \times S^{n-1} \to P \times D^n$ has the homotopy extension property for P-homotopies, and thus, the push-out $Y \to E$ also has this property. Now, the cofibration $Y \to E$ has cofibre

$$E/Y \cong \frac{P \times D^n}{P \times S^{n-1}} \cong P \times S^n.$$

Hence, we have a *fibration* $\mathrm{Map}_P(E, X) \to \mathrm{Map}_P(Y, X)$, with fibre $\mathrm{Map}_P(P \times S^n, X)$. Pick $f_0 \in \mathrm{Map}_P(Y, X)$. Then, the fibre is the inverse image of f_0, that is, the set of P-equivariant maps $f : E \to X$ which, restrict to f_0 on Y, which, by definition of E, is

$$\mathop{\mathrm{Map}}_P \left(\frac{E}{Y}, X \right) \cong \mathop{\mathrm{Map}}_P (P \times S^n, X).$$

Now, by hypothesis, the space $\mathrm{Map}_P(Y, X)$ is weakly contractible. We know that the space X is weakly contractible, and $\mathrm{Map}_P(P \times S^n, X) = \mathrm{Map}(S^n, X)$; thus, the homotopy groups of $\mathrm{Map}(S^n, X)$ are all zero, as those of the space X are all zero. Therefore, by the long exact sequence of homotopy groups associated to a fibration, the homotopy groups of the space $\mathrm{Map}_P(E, X)$ are all zero, that is to say, $\mathrm{Map}_P(E, X)$ is weakly contractible, and we are finished. □

The next corollary immediately follows from the above by looking at mapping fibres.

Corollary 5.2. *Let E be a finite free P-CW-complex. Let X and Y be P-spaces, and let $f : X \to Y$ be an equivariant map that is* (*non-equivariantly*) *a weak equivalence. Then, the induced map $f_* : \mathrm{Map}_P(E, X) \to \mathrm{Map}_P(E, Y)$ is a weak equivalence.*

Definition 5.3. We define a *classifying space for free P-actions EP* to be a free P-space that is weakly contractible.

Proposition 5.4. *A classifying space EPalways exists. If we can choose EP to be a finite free P-CW-complex, then it is unique up P-homotopy equivalence.*

Proof. By Example 2.3, a free and weakly contractible P-space EP exists. Suppose that we can choose EP to be a finite free P-CW-complex. Let X be another free and weakly contractible P-CW-complex. Then, by Proposition 5.1, the spaces $\mathrm{Map}_P(EP, X)$ and $\mathrm{Map}_P(X, EP)$ are weakly contractible. In particular, they are non-empty; thus, we have continuous equivariant maps $f : EP \to X$ and $g : X \to EP$.

Similarly, the spaces $\mathrm{Map}_P(EP, EP)$ and $\mathrm{Map}_P(X, X)$ are weakly contractible; thus, the sets $\pi_0 \mathrm{Map}_P(EP, EP)$ and $\pi_0 \mathrm{Map}_P(X, X)$ are trivial. Since $g \circ f, \mathrm{id}_{EP} \in \mathrm{Map}_P(EP, EP)$, they must be P-homotopic. Similarly, $f \circ g$ and $\mathrm{id}_P \in \mathrm{Map}_P(X, X)$ are P-homotopic. In other words, the composites $g \circ f$ and $f \circ g$ are both P-homotopic to identity maps, and we are done. □

Note that, if G is a group with torsion, then the classifying space BG is never a finite CW-complex, and thus, the universal cover EG is never a finite G-CW-complex. The same is true in the semigroup world since, if a semigroup has torsion, it contains a finite subgroup, which, of course, is a subgroup with torsion.

Definition 5.5. Let X be a P-space. We define the *homotopy fixed point set* of X to be the space $X^{hP} = \mathrm{Map}_P(EP, X)$.

By the above, if EP is a finite free P-CW-complex, then, up to homotopy, the space X^{hP} does not depend on the version of the space EP chosen. Furthermore, by Proposition 5.1 and Corollary 5.2, if X is weakly contractible, then so is X^{hP}, and, if $f : X \to Y$ is a weak equivalence, then so is $f_* : X^{hP} \to Y^{hP}$.

Now, let X be a cocompact coarse P-space. Then, the C^*-algebra $C_0(X)$ is equipped with a left P-action defined by writing $(pf)(x) = f(xp)$ for all $f \in C_0(X)$, $p \in P$ and $x \in X$.

Let H be a Hilbert space equipped with a left P-action by isometries. Let $v_p : H \to H$ be the isometry associated to the element $p \in P$. Then, the C^*-algebra $\mathcal{L}(H)$ comes equipped with a left P-action by homomorphisms defined by writing

$$pT = v_p T v_p^*, \quad p \in P,\ T \in \mathcal{L}(H).$$

We call H an *equivariant X-module* if it comes equipped an equivariant $*$-homomorphism $\rho : C_0(X) \to \mathcal{L}(H)$.

Now, observe in the case of an ample equivariant X-module that the semigroup P acts on the C^*-algebras $C^*(X)$ and $D^*(X)$ on the left by $*$-homomorphisms. We can, therefore, form homotopy fixed point sets $C^*(X)^{hP}$ and $D^*(X)^{hP}$. If EP is a finite P-CW-complex, then the space EP is cocompact, and these sets are C^*-algebras, with addition, multiplication and involution defined pointwise, and the norm defined by taking the supremum

$$\|f\| = \sup\{\|f(x)\| \mid x \in EP\}$$

for $g \in C^*(X)^{hP}$ or $g \in D^*(X)^{hP}$. Further, $C^*(X)^{hP}$ is a C^*-ideal in $D^*(X)^{hP}$; thus, we can form the quotient $D^*(X)^{hP}/C^*(X)^{hP}$.

We write

$$K_n^{hP}(X) = K_{n+1}\left(\frac{D^*(X)^{hP}}{C^*(X)^{hP}}\right).$$

By the open mapping theorem, the quotient map $D^*(X) \to D^*(X)/C^*(X)$ has a continuous section (though not one that is a $*$-homomorphism). This implies that we have a natural isomorphism

$$K_n^{hP}(X) \cong K_{n+1}\left(\frac{D^*(X)}{C^*(X)}\right)^{hP}.$$

Proposition 5.6. *The sequence of functors K_*^{hP} is a locally finite P-homology theory.*

Proof. Let $U(X)$ be the stable unitary group of the C^*-algebra $D^*(X)/C^*(X)$. Then, the groups $K_{n-1}(X)$ and $K_{n-1}^{hP}(X)$ are the homotopy groups of $U(X)$ and $U(X)^{hP}$, respectively.

By proper homotopy-invariance of K-homology, the inclusions $i_0, i_1 : X \to X \times [0,1]$ defined by the formulae $i_0(x) = (x,0)$ and $i_1(x) = (x,1)$, respectively, induce weak equivalences $U(X) \to U(X \times [0,1])$. From Corollary 5.2, these maps both induce weak equivalences $U(X)^{hG} \to U(X \times [0,1])^{hG}$, and thus, isomorphisms $K_n^{hP}(X) \to K_n^{hP}(X \times [0,1])$. Proper P-homotopy-invariance of the functors K_n^{hP} now follows.

Let $X = A \cup B$ be a P-space, where $A, B \subseteq X$ are open, and $PA \subseteq A$, $PB \subseteq B$. Then, by looking at Mayer-Vietoris sequences in

K-homology, we have a weak fibration sequence

$$U(A \cap B) \longrightarrow U(A) \vee U(B) \longrightarrow U(X)$$

and so, by Corollary 5.2, a weak fibration sequence

$$U(A \cap B)^{hP} \longrightarrow U(A)^{hP} \vee U(B)^{hP} \longrightarrow U(X)^{hP}.$$

The existence of Mayer-Vietoris sequences for the sequence of functors K_*^{hP} now also follows. □

6. Semigroup C^*-algebras and assembly. Let P be a semigroup. Let $l^2(P)$ be the Hilbert space with an orthonormal basis indexed by P, that is to say, we have an orthonormal basis $\{e_p \mid p \in P\}$.

Given $p \in P$, we have an isometry

$$v_p : l^2(P) \longrightarrow l^2(P)$$

defined by the formula $v_p(e_q) = e_{pq}$. Note that, for this to be an isometry, we need the left-cancelation property. The semigroup P acts on the space $l^2(P)$ by the formula $pw = v_p(w)$ where $p \in P$ and $w \in L^2(P)$.

The next definition comes from [**7**].

Definition 6.1. The *reduced semigroup C^*-algebra* $C_r^*(P)$ is the smallest C^*-subalgebra of the space of bounded linear operators $\mathcal{L}(l^2(P))$ that contains the set of isometries $\{v_p \mid p \in P\}$.

Note that reduced group C^*-algebras are an obvious special case.

More generally, let H be a Hilbert space equipped with a right P-action by isometries. Let $v_p : H \to H$ be the isometry associated to the element $p \in P$. Then, $C_r^*(P)$ is isomorphic to the smallest C^*-subalgebra of $\mathcal{L}(H)$ containing the set of isometries $\{v_p \mid p \in P\}$.

As noted in the previous section, the C^*-algebra $\mathcal{L}(H)$ has a left P-action defined by writing $pT = v_p T v_p^*$, $p \in P$, $T \in \mathcal{L}(H)$.

Let X be a free cocompact P-space. Then, we can write

$$X = U_1 \cup \cdots \cup U_n,$$

where each U_i is an open P-space that is equivariantly homeomorphic to $P \times W_i$ for some space W_i with trivial P-action.

Now, let H be an equivariant X-module, meaning H is a Hilbert space with a P-action and an equivariant $*$-homomorphism $\rho : C_0(X) \to \mathcal{L}(H)$. Then, on each set E_i, H restricts to an equivariant U_i-module through composing the $*$-homomorphism ρ with the inclusion $C_0(U_i) \hookrightarrow C_0(X)$.

Definition 6.2. We call an equivariant X-module H *P-adequate* if X is a finite union of open P-spaces U, where:

• each U is an open P-space that is equivariantly homeomorphic to $P \times W$ for some space W with trivial P-action.

• The restriction of H is U is equivariantly isomorphic to a U-module of the form $l^2(P) \otimes H'_W$, where H'_W is an ample W-module. Here, the $*$-homomorphism $C_0(U) \to \mathcal{L}(l^2(P) \otimes H'_W)$ is the composite

$$C_0(U) \longrightarrow C_0(P \times W) \longrightarrow \mathcal{L}(l^2(P) \otimes H'_W),$$

defined in the obvious way.

Definition 6.3. Let H be a P-adequate X-module. We define $D^*_P(X)$ to be the smallest C^*-subalgebra of $\mathcal{L}(H)$ containing all pseudolocal, controlled operators that are fixed under the action of P. We define $C^*_P(X)$ to be the smallest C^*-subalgebra of $\mathcal{L}(H)$ containing all locally compact, controlled operators that are fixed under the action of P.

Theorem 6.4. *Let P be a semigroup. Let X be a cocompact free coarse P-space. Then, the C^*-algebras $C^*_r(P)$ and $C^*_P(X)$ are Morita equivalent.*

Proof. Let H be a P-adequate equivariant X-module. Decompose X into a finite union of n open P-sets U as in Definition 6.2. Let

$$T : H \longrightarrow H$$

be P-fixed, locally compact and controlled. Then, by definition of locally compact, and the P-action, when restricted to U, T can be considered to be a $P \times P$ matrix of compact operators. Overall, T is an $n \times n$ matric (T_{ij}) where each T_{ij} is a $P \times P$ matrix of compact operators. Since T is translation-invariant under P, so is each T_{ij}.

Since T is also controlled, we see that $T \in C_r^*(\Gamma) \otimes \mathcal{K}$. More generally, taking limits, if $T \in C_P^*(X)$, then $T \in C_r^*(\Gamma) \otimes \mathcal{K}$.

Conversely, any operator of the form $v_P \otimes k$, where $p \in P$ and k is compact, is both controlled and locally compact. Any element $M_n(C_r^*(\Gamma) \otimes \mathcal{K})$ is a limit of sums of such operators, and thus,

$$M_n(C_r^*(\Gamma) \otimes \mathcal{K}) \subseteq C_P^*(X).$$

The desired result now follows. $\square$

The above argument was essentially made for groups in [**12,** Lemma 5.14]. Although the cited argument was problematic, it works here since we have restricted ourselves to free actions.

Now, let X be any P-space. Then, we have a short exact sequence

$$0 \longrightarrow C_P^*(X) \longrightarrow D_P^*(X) \longrightarrow \frac{D_P^*(X)}{C_P^*(X)} \longrightarrow 0.$$

By the above, when X is cocompact, we can identify the K-theory groups $K_*(C_P^*(X))$ and $K_* C_r^*(P)$. Thus, looking at the boundary maps in the long exact sequence of K-theory groups (see, for example, [**14, 16**]), we obtain a map

$$\beta : K_{*+1}\left(\frac{D_P^*(X)}{C_P^*(X)}\right) \longrightarrow K_* C_r^*(P)$$

called the *analytic assembly map*. This assembly map is a generalization of the corresponding map for groups, see, for example, [**12**].

Definition 6.5. Let X be a coarse P-space. Then, we define the P-equivariant K-homology groups of X by writing

$$K_n^P(X) = K_{n+1}\left(\frac{D_P^*(X)}{C_P^*(X)}\right).$$

Definition 6.6. We say that a torsion-free semigroup P satisfies the *analytic Novikov conjecture* if we have a cocompact classifying space EP such that the map

$$\beta : K_n^P(EP) \longrightarrow K_n C_r^*(P)$$

is injective.

We restrict our attention to torsion-free semigroups, since, in the case of groups, the map β is not, in general, injective for groups with torsion, although it *is* conjectured to be rationally injective. However, all of our arguments here are for torsion-free semigroups.

Furthermore, the equivariant K-homology groups $K_n^P(EP)$ are not necessarily easy to compute. The issue is that, unlike the group case, even if X is a free P-space, we do not have the formula $K_n^P(X) \cong K_n(X/P)$, which holds in the classical case. We go into further detail on this issue in the final section when we look at examples.

7. Descent. The *descent argument*, outlined in this section, tells us that the coarse Baum-Connes conjecture, along with certain mild extra conditions, implies the analytic Novikov conjecture.

Lemma 7.1. *We have a natural transformation* $\theta_* : K_n^P(X) \to K_n^{hP}(X)$ *that is an isomorphism whenever* X *is a finite free* P*-CW-complex.*

Proof. Let $U(X)$ and $U_P(X)$ be the stable unitary groups of the C^*-algebras $D^*(X)/C^*(X)$ and $D^*_P(X)/C^*_P(X)$, respectively. Since the space EP is weakly contractible, we have a natural weak equivalence

$$\begin{aligned} j : U_P(X) &\simeq \mathrm{Map}(EP, U_P(X)) \\ &= \mathrm{Map}(EP, \underset{P}{\mathrm{Map}}(P, U_P(X))) \\ &= \underset{P}{\mathrm{Map}}(EP, \mathrm{Map}(P, U_P(X))) \\ &= \mathrm{Map}(P, U_P(X))^{hG}. \end{aligned}$$

Let

$$i : D^*_P(X)/C^*_P(X) \longrightarrow D^*(X)/C^*(X)$$

be defined by the inclusions

$$C^*_P(X) \hookrightarrow C^*(X) \quad \text{and} \quad D^*_P(X) \hookrightarrow D^*(X).$$

Then, we have a natural map

$$k : \mathrm{Map}(P, U_P(X)) \longrightarrow U(X),$$

defined by writing $k(f) = i_* f(e)$, where e is the identity element of the semigroup P.

Taking homotopy fixed point sets, we obtain a natural map

$$k' : \mathrm{Map}(P, U_P(X))^{hP} \longrightarrow U(X)^{hP}.$$

Composing with the map j, we have a natural map

$$\theta = k' \circ j : U_P(X) \longrightarrow U(X)^{hP},$$

and thus, a natural induced map $\theta_* : K_n^P(X) \to K_n^{hP}(X)$.

Let $c : P \to +$ be the constant map onto the one-point space. Then, the composition

$$c_* \circ i_* = i_* \circ c_* : U_P(P) \longrightarrow U(+)$$

is certainly a homotopy-equivalence, and the map

$$k : \mathrm{Map}(P, U_P(+)) \longrightarrow U(P)$$

is a homeomorphism, and thus, a weak equivalence. By Corollary 5.2, the map k' is also a weak equivalence. Thus, the map θ is a weak equivalence in this case, making the induced map $\theta_* : K_n^P(P) \to K_n^{hP}(P)$ an isomorphism.

By Theorem 4.7, the map $\theta_* : K_n^P(X) \to K_n^{hP}(X)$ is, therefore, an isomorphism whenever X is a finite free P-CW-complex. □

Now, let X be a cocompact coarse P-space. We can define a map

$$\eta_* : K_n(D_P^*(X)) \longrightarrow K_n(D^*(X)^{hP})$$

in much the same way as the map θ_* in the above lemma, and thus, whenever X is a cocompact coarse P-space, we have a commutative diagram

$$\begin{array}{ccccc} K_n(D_P^*(X)) & \xrightarrow{v_P} & K_n^P(X) & \xrightarrow{\beta} & K_n C_r^*(P) \\ \downarrow & & \downarrow & & \\ K_n(D^*(X)^{hP}) & \xrightarrow{v_{hP}} & K_n^{hP}(X), & & \end{array}$$

where β is the analytic assembly map.

Theorem 7.2. *Let X be a free coarse P-space that is a free finite P-CW-complex as a topological space. Suppose that the coarse Baum-*

Connes conjecture holds for X. Then, the analytic assembly map β is split-injective for P.

Proof. The coarse Baum-Connes conjecture for X implies that $K_n D^*(X) = 0$ for all n. Hence, by Proposition 5.1, $K_n(D^*(X)^{hP}) = 0$ for all n.

Now, by Lemma 7.1, the map $\theta_* : K_n^P(X) \to K_n^{hP}(X)$ in the above commutative diagram is an isomorphism. It follows that the map v_p is zero; thus, the map β is split-injective as required. □

Corollary 7.3. *Let P be a semigroup with a classifying space EP that is a coarse P-space and a finite P-CW-complex. Suppose that the coarse Baum-Connes conjecture holds for the space EP. Then, the analytic Novikov conjecture holds for the semigroup P.*

8. Ore semigroups.

Definition 8.1. A semigroup P is a *left Ore semigroup* if:

- for all $p, q, r \in P$, if $pq = pr$, or $qp = rp$, then $q = r$;
- for all $p, q \in P$, we have $pP \cap qP \neq \emptyset$.

It is shown in **[2]** that a semigroup P can be embedded into a group G such that $G = P^{-1}P = \{q^{-1}p \mid p, q \in P\}$ if and only if P is a left Ore semigroup.

One of the main results of **[4]** is a computation of the K-theory of the reduced C^*-algebra of an Ore semigroup. It is more precise than Corollary 7.3 but covers what appears to be a smaller class of semigroups. In order to state it, we need some more terminology.

Definition 8.2. Let P be a semigroup. A *right ideal* of P is a subset $X \subseteq P$ such that, for all $x \in X$ and $p \in P$, we have $xp \in X$.

We write

$$pX = \{px \mid x \in X\}, \qquad p^{-1}X = \{q \in P \mid pq \in X\}.$$

Let $\mathcal{J}$ be the smallest family of right ideals of P such that:

- $\emptyset, P \in \mathcal{J}$.
- If $X \in \mathcal{J}$ and $p \in P$, then pX, $p^{-1}X \in \mathcal{J}$.
- If $X, Y \in \mathcal{J}$, then $X \cap Y \in \mathcal{J}$.

The following comes from [**7**].

Definition 8.3. We call elements of $\mathcal{J}$ the *constructible right ideals* of P. We say that the constructible right ideals of P are *independent* if, for all right ideals $X_1, \ldots, X_n \in \mathcal{J}$, if the union

$$X = \bigcup_{j=1}^{n} X_j$$

is a right ideal, then $X = X_j$ for some j.

We refer to [**7**] for examples and further analysis. Now, let P be an Ore semigroup, and let $G = P^{-1}P$. For a right ideal X, form the group

$$G_X = \{g \in G \mid gX = X\}.$$

Let E_X be the orthogonal projection from the Hilbert space $l^2(P)$ to the Hilbert space $l^2(E)$. Let $D_r^\infty(P)$ be the smallest C^*-subalgebra of the space of bounded linear operators $\mathcal{L}(l^2(P))$ that contains the set of projections $\{E_{p^{-1}X} \mid p \in P, X \in \mathcal{J}\backslash\{\emptyset\}\}$.

For an index set I, we write $c_0(I) = \oplus_{i\in I}\mathbb{C}$. Let $\chi \subseteq \mathcal{J}\backslash\{\emptyset\}$ be a set containing precisely one representative for each set in the collection $G\backslash(P^{-1}(\mathcal{J}\backslash\{\emptyset\})$.

Theorem 7.3 of [**4**] states the following.

Theorem 8.4. *Let P be a left Ore semigroup whose constructible right ideals are independent. Suppose that the group $G = P^{-1}P$ satisfies the Baum-Connes conjecture with coefficients in the G-C^*-algebras $c_0(P^{-1}(J\backslash\{\emptyset\}))$ and $D_r^\infty(P)$. Then, the groups $K_n(C_r^*(P))$ and $\oplus_{X\in\chi}K_n(C_r^*(G_X))$ are isomorphic.*

See [**1**] for more details on the Baum-Connes conjecture.

9. Examples. We conclude this article by looking at some simple examples where Corollary 7.3 applies. We can use Yu's theorem from **[17]** that the coarse Baum-Connes conjecture holds for any bounded geometry coarse space which can be uniformly embedded in Hilbert space. Of course, Yu's theorem is a heavy piece of machinery for this example; the author's proof of the coarse Baum-Connes conjecture for coarse CW-complexes in **[9, 10]** also suffices.

The explicit examples all satisfy the conditions of Theorem 8.4, so the result applies to them. In particular, by split-injectivity, we know that the K-theory group $K_*^P(EP)$ embeds as a subgroup of the K-theory group $K_*(C_r^* P)$, which the techniques from **[4]** can be used to compute.

9.1. The semigroup $\mathbb{N}$. The semigroup $\mathbb{N}$ acts freely on $\mathbb{R}^+$ by writing $(n,x) \mapsto n+x$, where $n \in \mathbb{N}$, and $x \in \mathbb{R}^+$. With the coarse structure defined by the metric, the space $\mathbb{R}^+$ is certainly uniformly embeddable in Hilbert space; thus, the coarse Baum-Connes conjecture holds.

Now, $\mathbb{R}^+$ is a finite free $\mathbb{N}$-CW-complex, with a single 0-cell, $\mathbb{N}$, and 1-cell $\mathbb{N} \times [0,1]$, with attaching map $f : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ defined by the formula $f(n,k) = n+k$.

Then, $\mathbb{R}_+$ is weakly contractible, so we can take $E\mathbb{N} = \mathbb{R}_+$, and, by Corollary 7.3, the analytic Novikov conjecture holds for $\mathbb{N}$.

Similarly, let $\mathbb{N}^\times$ be the group of non-zero natural numbers with group operation defined by multiplication. Then, $E\mathbb{N} = [1,\infty)$, with free $\mathbb{N}$-action defined by writing $(n,x) \mapsto nx$. As above, the analytic Novikov conjecture holds for $\mathbb{N}^\times$.

In actuality, the first example shows us why we do not have $K_*^P(EP) \cong K_*(EP/P)$ in general. In order to see this, observe, first of all, that the C^*-algebra $C_r^*(\mathbb{N})$ is simply the Toeplitz algebra. It is well known from basic K-theory, see **[14, 16]**, that we, therefore, have $K_1 C_r^*(\mathbb{N}) = 0$. It follows from the analytic Novikov conjecture for $\mathbb{N}$ that $K_1^{\mathbb{N}}(E\mathbb{N}) = 0$.

On the other hand, $\mathbb{R}^+/\mathbb{N} = S^1$, and $K_1(S^1) = \mathbb{Z}$; thus, $K_*^{\mathbb{N}}(\mathbb{R}^+) \ncong K_*(\mathbb{R}^+/\mathbb{N})$.

9.2. Free semigroups. The *free semigroup on n generators V_n* is the set of words in an alphabet with n letters, say $e_1, \dots, e_n$. Let

$S = \{e_1, \ldots, e_n\}$. Then, V_n certainly acts freely on the Cayley graph $\mathrm{Cay}(V_n; S)$, which is weakly contractible. Therefore, we can take $EV_n = \mathrm{Cay}(V_n; S)$.

The space EV_n is a finite V_n-CW-complex, with a single 0-cell, the set of vertices, and 1-cells $P \times \{e_1\} \times [0,1], \ldots, P \times \{e_n\} \times [0,1]$. The attaching map $f : P \times \{e_i\} \times \{0,1\} \to P$ is defined by writing $f(p, e_i, 0) = p$ and $f(p, e_i, 1) = pe_i$.

The space EV_n is certainly uniformly embeddable in an infinite-dimensional Hilbert space; thus, by Corollary 7.3, the analytic Novikov conjecture holds for the free semigroup V_n.

9.3. Products. Let P and Q be semigroups such that EP and EQ are finite free P- and Q-CW-complexes, respectively, and EP and EQ have compatible coarse structures where P and Q act, respectively, by coarse continuous maps. Suppose that EP and EQ are uniformly embeddable in Hilbert spaces H_P and H_Q, respectively. Then, we can take $E(P \times Q) = EP \times EQ$. The space $EP \times EQ$ is a free finite $P \times Q$-CW-complex, which is a coarse $P \times Q$-space uniformly embeddable in $H_P \times H_Q$. Thus, the analytic Novikov conjecture holds for $P \times Q$. In particular, by the above, the analytic Novikov conjecture holds for the semigroup $\mathbb{N}^m \times (\mathbb{N}^\times)^n$ for all m and n.

9.4. The $ax + b$ semigroup over $\mathbb{N}$. The *$ax + b$ semigroup over* $\mathbb{N}$ is defined in **[3]** as the set

$$P_{\mathbb{N}} = \left\{ \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \mid n \in \mathbb{N}^\times,\ k \in \mathbb{N} \right\},$$

with group operation defined by matrix multiplication. The group $P_{\mathbb{N}}$ acts freely and cocompactly on the space $[0,\infty) \times [1,\infty)$ by the formula

$$\begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ky \\ ny \end{pmatrix}.$$

As above, the space $[0,\infty) \times [1,\infty)$ has the structure of a finite $P_{\mathbb{N}}$-CW-complex. As a coarse space, it is uniformly embeddable in Hilbert space. Therefore, the analytic Novikov conjecture holds for $P_{\mathbb{N}}$.

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