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MAPS PRESERVING QUASI-ISOMETRIES ON HILBERT C^*-MODULES

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ABSTRACT. Let $\mathcal{K}(\mathcal{H})$ be the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$. Let E be a Hilbert $\mathcal{K}(\mathcal{H})$-module and $\mathcal{L}(E)$ the C^*-algebra of all adjointable maps on E. In this paper, we prove that, if $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ is a unital surjective bounded linear map, which preserves quasi-isometries in both directions, then there are unitary operators $U, V \in \mathcal{L}(E)$ such that

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{\mathrm{tr}}V$$

for all $T \in \mathcal{L}(E)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

1. Introduction. Let $\mathcal{A}$ be a C^*-algebra. A (right) inner-product $\mathcal{A}$-module is a linear space E, which is a right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in E$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, together with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ satisfying the following conditions:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$,
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$,
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$

for all $x, y, z \in E$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A Hilbert $\mathcal{A}$-module (Hilbert C^*-module) is an inner product $\mathcal{A}$-module E which is complete under the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. Let E be a Hilbert $\mathcal{A}$-module. A map $T : E \to E$ is called *adjointable* if there is a map $T^* : E \to E$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in E$. It is easy to see that T must be $\mathcal{A}$-linear (i.e., $T(xa) = T(x)a$ for all $x \in E$ and $a \in \mathcal{A}$) and bounded [**11,** page 8]. The set of all adjointable maps is denoted by $\mathcal{L}(E)$, which is a C^*-algebra. For every pair of vectors $x, y \in E$, we use $\theta_{x,y}$ to denote the rank 1 linear operator on E, defined by $\theta_{x,y}(z) = x\langle y, z \rangle$ for any

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$z \in E$. The closed linear subspace of $\mathcal{L}(E)$ spanned by $\{\theta_{x,y} : x, y \in E\}$ is denoted by $\mathcal{K}(E)$. In fact, $\mathcal{K}(E)$ is a closed ideal of $\mathcal{L}(E)$ and is called the algebra of "compact" operators.

Definition 1.1 ([**15**]). If $T \in \mathcal{L}(E)$ and ${T^*}^2 T^2 = T^* T$, then it is called a *quasi-isometry.*

Recall that, if $(E, \langle \cdot, \cdot \rangle)$ is a Hilbert $\mathcal{A}$-module and K is a nonzero positive invertible element of $\mathcal{L}(E)$, we define $\langle x, y \rangle_K = \langle Kx, y \rangle$ for each $x, y \in E$. Then, $\langle \cdot, \cdot \rangle_K$ becomes an inner product on E, and $E_K = (E, \langle \cdot, \cdot \rangle_K)$ becomes a Hilbert $\mathcal{A}$-module, see [**2**].

If T^* is the adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle$, then $T^\sharp = K^{-1} T^* K$ is the K-adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle_K$ since

$$\begin{aligned} \langle Tx, y \rangle_K = \langle KTx, y \rangle = \langle Tx, Ky \rangle = \langle x, T^* Ky \rangle = \langle x, KK^{-1} T^* Ky \rangle \\ = \langle Kx, K^{-1} T^* Ky \rangle = \langle x, K^{-1} T^* Ky \rangle_K \end{aligned}$$

for each $x, y \in E$. It is easy to see that $\sharp$ is an involution on $\mathcal{L}(E)$. We say that $S \in \mathcal{L}(E)$ is K-self-adjoint if $K^{-1} S^* K = S$, i.e., $S^\sharp = S$. The set of all adjointable linear operators on E with respect to the inner product $\langle \cdot, \cdot \rangle_K$ is the same as $\mathcal{L}(E, \langle \cdot, \cdot \rangle)$.

Definition 1.2. An operator $T \in \mathcal{L}(E)$ is called a K-*quasi-isometry* if ${T^\sharp}^2 T^2 = T^\sharp T$. It is called K-*unitary* if $UK^{-1} U^* K = K^{-1} U^* KU = I$.

Utilizing [**10,** page 47] and [**13,** page 158], the next lemma is apparent.

Lemma 1.3. *Suppose that $\mathcal{A}$ is a C^*-algebra. Then, the following conditions are equivalent*:

(i) *For all $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$.*
(ii) *For all ideals I and J of $\mathcal{A}$, $IJ = \{0\}$ implies $I = \{0\}$ or $J = \{0\}$.*
(iii) *For all closed ideals I and J of $\mathcal{A}$, $IJ = \{0\}$ implies $I = \{0\}$ or $J = \{0\}$.*

Recall that a C^*-algebra is said to be prime if it satisfies one of the conditions of Lemma 1.3. In particular, it shows that topological and algebraic primeness are equivalent in the setting of C^*-algebras.

The Gelfand-Naimark theorem states that an arbitrary C^*-algebra $\mathcal{A}$ has a representation $\psi : \mathcal{A} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We know that ψ is a $*$-homomorphism. If ψ is one-to-one, then it is called a *faithful representation.* A representation $\psi : \mathcal{A} \to B(\mathcal{H})$ of a C^*-algebra $\mathcal{A}$ is *irreducible* if the closed vector subspaces of $\mathcal{H}$, being ψ-invariant, are only $\{0\}$ and $\mathcal{H}$. A C^*-algebra $\mathcal{A}$ is primitive if its zero ideal is primitive, that is, if $\mathcal{A}$ has a faithful non-zero irreducible representation. If $\mathcal{H}$ is a nonzero Hilbert space, then the identity representation of $B(\mathcal{H})$ on $\mathcal{H}$ is irreducible by [**13,** page 158], and $B(\mathcal{H})$ is primitive.

Theorem 1.4. ([**13,** Theorem 5.4.5])**.** *Any primitive C^*-algebra is a prime C^*-algebra.*

A linear map φ from a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is called a $*$-Jordan *homomorphism* if $\varphi(a^2) = \varphi(a)^2$ and $\varphi(a^*) = \varphi(a)^*$ for every $a \in \mathcal{A}$. A well-known result of Herstein [**10,** Theorem 3.1] states that a $*$-Jordan homomorphism onto a prime C^*-algebra is either a $*$-homomorphism or a $*$-anti-homomorphism.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces and $\varphi : \mathcal{X} \to \mathcal{Y}$ is a map. We say that φ is a preserving map in both directions whenever

$$x \in \mathcal{X} \text{ has the property } p \Longleftrightarrow \varphi(x) \in \mathcal{Y} \text{ has the property } p.$$

Linear preserver problems appear in many areas of mathematics, especially in matrix theory and operator theory. These problems are often studied as the form of linear maps preserving some properties in Banach algebras or other linear spaces. Many mathematicians have investigated several linear preserver problems, see [**1, 3, 6, 7, 9, 12, 14, 16**].

Suppose that E is a Hilbert $\mathcal{A}$-module. We assume that $\mathcal{X} = \mathcal{Y} = \mathcal{L}(E)$ and characterize surjective continuous linear maps $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$, which preserve quasi-isometries in both directions.

2. Linear maps that preserve quasi-isometries. In this section, we intend to characterize unital surjective linear maps from $\mathcal{L}(E)$ onto itself which preserve quasi-isometries. We need the following well-known theorem.

Theorem 2.1 ([**12,** page 208])**.** *Suppose that $\mathcal{H}$ is a Hilbert space. If $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective linear isometry, then there are unitary* *operators U and V in $\mathcal{B}(\mathcal{H})$ such that φ is either of the form*

$$\varphi(T) = UTV,$$

or of the form

$$\varphi(T) = UT^{\mathrm{tr}}V$$

for each $T \in \mathcal{B}(H)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Suppose that $\mathcal{H}$ is a Hilbert space and $\mathcal{K}(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$. Bakić and Guljaš [**5**] discussed the concept of an orthonormal basis for Hilbert C^*-modules and proved that each Hilbert C^*-module E over the C^*-algebra $\mathcal{K}(\mathcal{H})$ possesses an orthonormal basis.

If $e\mathcal{K}(\mathcal{H})e = \mathbb{C}e$, where $e \in \mathcal{K}(\mathcal{H})$ is a projection, then it is called a *minimal projection*. Suppose that $e_0 \in \mathcal{K}(\mathcal{H})$ is a minimal projection and $E_{e_0} = \{xe_0 : x \in E\}$. From [**5,** Remark 4], E_{e_0} is an invariant subspace for all $\mathcal{K}(\mathcal{H})$-linear operators on E, and E_{e_0} is a Hilbert space with respect to the inner product $(xe_0, ye_0) = \mathrm{tr}(\langle xe_0, ye_0\rangle)$ for all $x, y \in E$, where tr denotes the usual trace. In addition, there exists an orthonormal basis $(\nu_\lambda)_{\lambda\in I}$ for E such that $\langle \nu_\lambda, \nu_\lambda\rangle = e_0$ for all $\lambda \in I$. By Fourier expansion, $\nu_\lambda = \nu_\lambda\langle \nu_\lambda, \nu_\lambda\rangle = \nu_\lambda e_0$ for all $\lambda \in I$; hence, $\nu_\lambda \in E_{e_0}$. In fact $(\nu_\lambda)_{\lambda\in I}$ is an orthonormal basis for the Hilbert space E_{e_0}. Therefore, E_{e_0} contains an orthonormal basis for E. This implies that E_{e_0} as a submodule of E contains a dense submodule of E generated by $(\nu_\lambda)_{\lambda\in I}$. Thus, E_{e_0} is dense in E.

Theorem 2.2 ([**5,** Theorem 5])**.** *Let E be a Hilbert $\mathcal{K}(\mathcal{H})$-module and e_0 a minimal projection in $\mathcal{K}(\mathcal{H})$. Then, the map $\psi : \mathcal{L}(E) \to \mathcal{B}(E_{e_0})$ defined by $\psi(T) = T|_{E_{e_0}}$ is a $*$-isomorphism of C^*- algebras.*

From [**5,** Theorem 6], T is a compact operator in $\mathcal{K}(E)$ if and only if $\psi(T) = T|_{E_{e_0}}$ is a compact operator on Hilbert space E_{e_0}. Therefore, $\varphi : \mathcal{K}(E) \to \mathcal{K}(E_{e_0})$, defined by $\psi(T) = T|_{E_{e_0}}$, is a $*$-isomorphism. The above statements are true for the map $\phi : \mathcal{L}(E) \to \mathcal{B}(E_{e_0})$ defined by $\phi(T) = T^*|_{E_{e_0}}$ as a $*$-anti-isomorphism.

Corollary 2.3. *Let $\mathcal{H}$ be a Hilbert space, E a Hilbert $\mathcal{K}(\mathcal{H})$-module and e_0 a minimal projection in $\mathcal{K}(\mathcal{H})$. Then, the C^*-algebra $\mathcal{L}(E)$ is prime.*

Proof. Since E_{e_0} is a Hilbert space, $\mathcal{B}(E_{e_0})$ is primitive. From Theorem 2.2, we deduce that $\mathcal{L}(E)$ is primitive, whence, from Theorem 1.4, we conclude that $\mathcal{L}(E)$ is prime. $\square$

Proposition 2.4. *Let E be a Hilbert $\mathcal{K}(\mathcal{H})$-module and $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ a $*$-isomorphism or a $*$-anti-isomorphism. Then, there are unitary operators $U, V \in \mathcal{L}(E)$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{\mathrm{tr}}V$$

for all $T \in \mathcal{L}(E)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

Proof. The proof is similar to that of [**4,** Main theorem]. From Theorem 2.2, the map $\psi : \mathcal{L}(E) \to \mathcal{B}(E_{e_0})$ defined by $\psi(T) = T|_{E_{e_0}}$, is a $*$-isomorphism or the map $\phi : \mathcal{L}(E) \to \mathcal{B}(E_{e_0})$, defined by $\phi(T) = T^*|_{E_{e_0}}$, a $*$-anti-isomorphism of C^*-algebras. Consider the linear operator $\Phi : \mathcal{B}(E_{e_0}) \to \mathcal{B}(E_{e_0})$ given by $\Phi = \psi\varphi\psi^{-1}$ or $\Phi = \psi\varphi\phi^{-1}$. Clearly, Φ is a $*$-isomorphism or a $*$-anti-isomorphism, respectively; thus, it is surjective and an isometry. From Theorem 2.1, there are unitary operators u, v on E_{e_0} such that Φ is either of the form $\Phi(S) = uSv$ or of the form $\Phi(S) = uS^{\mathrm{tr}}v$, where $S \in \mathcal{B}(E_{e_0})$, and S^{tr} is the transpose of S with respect to an arbitrary but fixed orthonormal basis of E_{e_0}. $\square$

Corollary 2.5. *Let E be a Hilbert $\mathcal{K}(\mathcal{H})$-module, and let $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ be a $\sharp$-isomorphism. Then, there are K-unitary operators $U, V \in \mathcal{L}(E)$ such that either*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{\mathrm{tr}}V$$

for all $T \in \mathcal{L}(E)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

In order to achieve our next result, we utilize the strategy of [**9**].

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert space, E a Hilbert $\mathcal{K}(\mathcal{H})$-module, and let $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ be a unital surjective bounded linear map. If φ preserves quasi-isometries in both directions, then there are unitary operators $U, V \in \mathcal{L}(E)$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{\mathrm{tr}}V,$$

where $T^{\rm tr}$ is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

Proof. Choose a self-adjoint operator S in $\mathcal{L}(E)$. Then, $\exp(itS)^* = \exp(-itS)$ for each $t \in \mathbb{R}$. Clearly, $\exp(itS)^{*2}\exp(itS)^2 = \exp(itS)^* \exp(itS)$. Therefore,

$$\varphi(\exp(itS))^{*2}\varphi(\exp(itS))^2 = \varphi(\exp(itS))^*\varphi(\exp(itS)).$$

Thus,

$$\begin{aligned}\varphi\bigg(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\bigg)^{*2}\varphi\bigg(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\bigg)^2 \\ = \varphi\bigg(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\bigg)^{*}\varphi\bigg(I + itS + \frac{(it)^2}{2!}S^2 + \cdots\bigg).\end{aligned}$$

Hence,

$$\begin{aligned}&\bigg(I - it\varphi(S)^* - \frac{t^2}{2}\varphi(S^2)^* + \cdots\bigg)^2\bigg(I + it\varphi(S) - \frac{t^2}{2}\varphi(S^2) + \cdots\bigg)^2 \\ &= \bigg(I - it\varphi(S)^* - \frac{t^2}{2}\varphi(S^2)^* + \cdots\bigg)\bigg(I + it\varphi(S) - \frac{t^2}{2}\varphi(S^2) + \cdots\bigg);\end{aligned}$$

thus,

$$\begin{aligned}I + 2it(\varphi(S) - \varphi(S)^*) + t^2(4\varphi(S)^*\varphi(S) - \varphi(S^2) - \varphi(S)^2 \\ - \varphi(S^2)^* - \varphi(S)^{*2}) + \cdots \\ = I + it(\varphi(S) - \varphi(S)^*) + t^2\bigg(\varphi(S)^*\varphi(S) - \frac{1}{2}\varphi(S^2) - \frac{1}{2}\varphi(S^2)^*\bigg) + \cdots .\end{aligned}$$

It follows that

$$\varphi(S) = \varphi(S)^* \tag{2.1}$$

and

$$\begin{aligned}4\varphi(S)^*\varphi(S) - \varphi(S^2) - \varphi(S)^2 - \varphi(S^2)^* - \varphi(S)^{*2} \\ = \varphi(S)^*\varphi(S) - \frac{1}{2}\varphi(S^2) - \frac{1}{2}\varphi(S^2)^*\end{aligned} \tag{2.2}$$

for every self-adjoint operator $S \in \mathcal{L}(E)$. By (2.1), (2.2) and similar to **[9,** Proof of Theorem 3.3.], for every $T \in \mathcal{L}(E)$, we have:

(i) $\varphi(T^*) = \varphi(T)^*$;
(ii) $\varphi(T^2) = \varphi(T)^2$.

Therefore, φ is a $*$-Jordan homomorphism. It is well known that every $*$-Jordan homomorphism onto a prime C^*-algebra is a $*$-homomorphism or a $*$-anti-homomorphism. Since $\mathcal{L}(E)$ is a prime C^*-algebra, φ is a $*$-homomorphism or a $*$-anti-homomorphism.

Now, we show that φ is injective. Let $S \in \mathcal{L}(E)$ be self-adjoint and $\varphi(S) = 0$. Then, $\varphi(S + I) = I$ and $\varphi(S - I) = -I$ since $\varphi(I) = I$. Clearly, I and $-I$ are quasi-isometries and, since φ preserves quasi-isometries in both directions, $S+I$ and $S-I$ are quasi-isometries; thus, $2S^4 + 10S^2 = 0$ and $2S^3 + S = 0$. Therefore, $S = 0$. Let $T \in \mathcal{L}(E)$ be an arbitrary element and $\varphi(T) = 0$. There exist self-adjoint operators $S_1, S_2 \in \mathcal{L}(E)$ such that $T = S_1 + iS_2$ and

$$\varphi(S_1) + i\varphi(S_2) = \varphi(T) = 0 = \varphi(T)^* = \varphi(S_1) - i\varphi(S_2).$$

Thus, $\varphi(S_1) = 0$ and $\varphi(S_2) = 0$; hence, $S_1 = S_2 = 0$. Therefore, $T = 0$, which implies that φ is injective. Since φ is injective, it is a $*$-automorphism or a $*$-anti-automorphism. By Proposition 2.4, the proof is complete. $\square$

Now, we want to give a form for an operator that preserves K-quasi-isometries in both directions.

Corollary 2.7. *Let $\mathcal{H}$ be a Hilbert space, E a $\mathcal{K}(\mathcal{H})$-module and*

$$\varphi : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$$

a unital surjective bounded linear map. If φ preserves K-quasi-isometries in both directions, then there are K-unitary operators $U, V \in \mathcal{L}(E)$ such that either

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{\mathrm{tr}}V$$

for all $T \in \mathcal{L}(E)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

In Corollary 2.7, if $V = U^{-1}$, then $\varphi(T) = UTU^{-1}$ or $\varphi(T) = UT^{\mathrm{tr}}U^{-1}$.

$$\begin{aligned} UT^\sharp U^{-1} &= (UTU^{-1})^\sharp \; (\varphi(T^\sharp) = \varphi(T)^\sharp) \\ &= (U^\sharp)^{-1} T^\sharp U^\sharp. \end{aligned}$$

Straightforward computation shows that

$$U^*KUK^{-1}T^* = T^*U^*KUK^{-1}$$

for each $T^* \in \mathcal{L}(E)$. Hence, $U^*KUK^{-1} \in \mathscr{Z}(\mathcal{L}(E))$. We know that $\mathscr{Z}(\mathcal{L}(E)) = \{\lambda I : \lambda \in \mathbb{C}\}$. Therefore, there is a $\lambda \in \mathbb{C}$ such that $U^*KUK^{-1} = \lambda I$. It follows that

$$U^\sharp U = \lambda I.$$

Moreover, U is invertible, so $UU^\sharp = \lambda I$. On the other hand, the operator $UU^\sharp$ is K-self-adjoint. Then, $\lambda = \pm 1$.

Corollary 2.8. *Let $\mathcal{H}$ be a Hilbert space, E a Hilbert $\mathcal{K}(\mathcal{H})$-module and $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ a unital surjective bounded linear map. If φ preserves K-quasi-isometries in both directions, then there exist $\lambda = \pm 1$ and a K-unitary operator $U \in \mathcal{L}(E)$ satisfying $UU^\sharp = U^\sharp U = \lambda I$ such that*

$$\varphi(T) = \lambda UTU^{-1} \quad \text{or} \quad \varphi(T) = \lambda UT^{\mathrm{tr}}U^{-1},$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of E.

Now, we give an example that shows the process of structure of φ.

Example 2.9. Suppose that $\mathcal{H} = \mathbb{C}^2$. Then,

$$\mathcal{K}(\mathcal{H}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};\ a_{ij} \in \mathbb{C},\ i,j \in \{1,2\} \right\}.$$

Consider $E = \mathcal{K}(\mathcal{H})$ as a Hilbert $\mathcal{K}(\mathcal{H})$-module with inner product $\langle A, B\rangle = A^*B$, where A^* is the conjugate transpose of A. Set

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{K}(\mathcal{H})$$

and

$$(\nu_\lambda)_{\lambda \in \{0,1\}} = \left\{ \nu_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \nu_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

We have the following assertions:

(i) e_0 is a minimal projection, since

$$e_0\mathcal{K}(\mathcal{H})e_0 = \left\{ e_0 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_0;\ a_{ij} \in \mathbb{C},\ i,j \in \{1,2\} \right\}$$
$$= \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix};\ a_{11} \in \mathbb{C} \right\} = \{e_0 a_{11};\ a_{11} \in \mathbb{C}\}.$$

Also, $e_0^* = e_0$ and $e_0^2 = e_0$.

(ii) For each

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in E,$$

we have

$$\sum_{\lambda=1}^{2} \nu_\lambda \langle \nu_\lambda, A\rangle = \nu_1\nu_1^* A + \nu_2\nu_2^* A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix} = A,$$

and, since

$$\langle \nu_\lambda, \nu_\mu\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } \lambda \neq \mu$$

and $\langle \nu_\lambda, \nu_\lambda\rangle = e_0$ for each $\lambda \in \{0,1\}$, by [**5,** Theorem 1], the orthonormal system $(\nu_\lambda)_{\lambda\in\{0,1\}}$ is an orthonormal basis for E.

(iii)

$$E_{e_0} = \{xe_0;\ x \in E\} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ a_{ij} \in \mathbb{C} \right\}$$
$$= \left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix};\ a_{11}, a_{21} \in \mathbb{C} \right\}.$$

For each

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \text{ in } E_{e_0},$$

we define $||A||^2 = |a_{11}|^2 + |a_{21}|^2$, so that E_{e_0} with the inner product $(A,B) = \mathrm{tr}(\langle A, B\rangle)$ is a Hilbert space, where $A, B \in E_{e_0}$. Indeed,

$$\begin{aligned}(A, A) &= \operatorname{tr}(\langle A, A\rangle) = \operatorname{tr}(A^*A) = \operatorname{tr}\left(\begin{bmatrix}\overline{a_{11}} & \overline{a_{21}}\\ 0 & 0\end{bmatrix}\begin{bmatrix}a_{11} & 0\\ a_{21} & 0\end{bmatrix}\right)\\ &= \operatorname{tr}\left(\begin{bmatrix}\overline{a_{11}}a_{11} + \overline{a_{21}}a_{21} & 0\\ 0 & 0\end{bmatrix}\right) = |a_{11}|^2 + |a_{21}|^2 = ||A||^2.\end{aligned}$$

(iv) Clearly, $\langle \nu_\lambda, \nu_\lambda\rangle = e_0$, $\nu_\lambda = \nu_\lambda e_0$ for each $\lambda \in \{0,1\}$. Therefore, by [**5,** Remark 4], $(\nu_\lambda)_{\lambda\in\{0,1\}} \in E_{e_0}$ is an orthonormal basis for E_{e_0}, too. Thus, E_{e_0} is a dense submodule for E.

Suppose that $\varphi : \mathcal{L}(E) \to \mathcal{L}(E)$ is an arbitrary unital surjective bounded linear map that preserves quasi-isometries in both directions. From Theorem 2.1, for each $T \in \mathcal{B}(E_{e_0})$, we have $\varphi(T) = uT^*v$ or $\varphi(T) = uT^{*\,\mathrm{tr}}v$, for some unitary operators $u, v \in \mathcal{B}(E_{e_0})$.

Suppose that

$$A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$$

are the corresponding matrices of u, v, respectively, where A, B are unitary matrices. Define $U(x) = \sum_{\lambda\in I} u(\nu_\lambda)\langle \nu_\lambda, x\rangle$, $V(x) = \sum_{\lambda\in I} v(\nu_\lambda) \times \langle \nu_\lambda, x\rangle$ for each $x \in E$ and $I = \{0,1\}$. Thus, $U(x) = \sum_{\lambda\in I} u(\nu_\lambda)\langle \nu_\lambda, x\rangle = \sum_{\lambda\in I} A\nu_\lambda\nu_\lambda^* x$ for each $x \in E$. Hence,

$$\begin{aligned}U &= \sum_{\lambda\in I} A\nu_\lambda\nu_\lambda^* = A\nu_1\nu_1^* + A\nu_2\nu_2^*\\ &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}a_{11} & 0\\ a_{21} & 0\end{bmatrix} + \begin{bmatrix}0 & a_{12}\\ 0 & a_{22}\end{bmatrix} = A.\end{aligned}$$

Similarly, we have $V = B$. Indeed, U, V are the extent of u, v on E, respectively. Therefore, by (iv), for each $T \in \mathcal{L}(E)$, we have $\varphi(T) = UTV$ or $UT^{\mathrm{tr}}V$.

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