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STOCHASTIC PREDATOR-PREY MODEL WITH LESLIE-GOWER AND HOLLING-TYPE II SCHEMES WITH REGIME SWITCHING

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ABSTRACT. A predator-prey model with Leslie-Gower and Holling-type II schemes with regime switching will be considered, that is, both white and color noises are taken into account. We firstly show that there exists a globally unique solution to the stochastic predator-prey model by use of the comparison theorem. Then, asymptotic properties of the system will be examined and the conditions under which the system is stochastically persistent will be given. Moreover, lastly, we analyze the optimal harvesting policy of the stochastic prey-predator model with Markovian switching.

1. Preliminaries. Interest in mathematical models for populations with interaction between species has been on the rise. Many models in theoretical ecology take the Lotka-Volterra model of an interacting species as a starting point. Recently, many results have been obtained regarding Lotka-Volterra models in random environment. Mao, et al., [**7**], considered a stochastic Lotka-Volterra model for a system with n interacting components, and [**2**] continued the study of [**7**] to investigate the dynamics properties of a stochastic Lotka-Volterra model. In [**4**], the stochastic Lotka-Volterra competitive system

$$dx_i(t) = x_i(t)\left[\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right)dt + \sigma_i(t)\,dB_i(t)\right], \quad i = 1, 2, \dots n,$$

was discussed.

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Recently, more and more attention has been focused on stochastic population models with Markovian switching. Stochastic population dynamics under regime switching was considered in [**5, 6, 9**]. Zhu and Yin [**10**] examined competitive Lotka-Volterra model in random environments, and [**11**] further gave certain long-run-average limits of the solution for the competitive Lotka-Volterra model.

Ji, et al. [**3**], developed a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation. This provided the stochastic system, which takes the following form:

$$\begin{cases} dx(t) = x(t)\big(a - bx(t) - \frac{cy(t)}{m+x(t)}\big)\,dt + \sigma_1 x(t)\,dB_1(t), \\ dy(t) = y(t)\big(r - \frac{fy(t)}{m+x(t)}\big)\,dt + \sigma_2 y(t)\,dB_2(t), \end{cases} \tag{1.1}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions, b, c and sf are positive constants and σ_i, $i = 1, 2$, represent noise intensities. In [**3**], it is shown that there is a unique positive solution to the system with positive initial value $x_0 > 0$, $y_0 > 0$. The long time behavior of the system is mainly investigated therein. The following result is proven in [**3**].

Under the conditions H:

$$a > \frac{\sigma_1^2}{2}, \qquad r > \frac{\sigma_2^2}{2}, \qquad \frac{a - (\sigma_1^2)/2}{c} > \frac{r - (\sigma_2^2)/2}{f},$$

the solution to (1.1) has the property

$$\lim_{t\to\infty} \frac{\int_0^t y(s)/(m(\alpha(s)) + x(s))\,ds}{t} = \frac{r - \sigma_2^2/2}{f},$$

which yields that the Leslie-Gower term in system (1.1) is globally stable in the time average. Moreover, referring to [**1**], Ji, et al. [**3**], presented the following definition of the persistence in mean.

Definition 1.1. The system is said to be *persistent in mean* if it satisfies

$$\liminf_{t\to\infty} \frac{\int_0^t x(s)\,ds}{t} > 0,$$

and

$$\liminf_{t\to\infty} \frac{\int_0^t y(s)\,ds}{t} > 0.$$

Under the assumption H, **[3]** shows that model (1.1) satisfies

$$\lim_{t\to\infty}\frac{\int_0^t x(s)\,ds}{t}=\frac{a-\sigma_1^2/2}{b}-\frac{c(r-\sigma_2^2/2)}{bf}>0, \tag{1.2}$$

and

$$\liminf_{t\to\infty}\frac{\int_0^t y(s)\,ds}{t}\geq\frac{m(r-\sigma_2^2/2)}{f}r-\frac{\sigma_2^2}{2}>0. \tag{1.3}$$

Relations (1.2) and (1.3) tell us that the system (1.1) is persistent in mean.

In this paper, both white and color noises will be taken into account in the stochastic predator-prey model with modified Leslie-Gower and Holling-type II schemes with Markovian switching, which reads

(1.4)
$$\begin{cases} dx(t)=x(t)\big(a(\alpha(t))-bx(t)-\frac{cy(t)}{m(\alpha(t))+x(t)}\big)\,dt+x(t)\sigma_1(\alpha(t))\,dB_1(t),\\ dy(t)=y(t)\big(r(\alpha(t))-\frac{fy(t)}{m(\alpha(t))+x(t)}\big)\,dt+y(t)\sigma_2(\alpha(t))\,dB_2(t),\end{cases}$$

where $B_i(t)$, $i=1,2$, are independent standard Brownian motions, σ_i, $i=1,2$, represent noise intensities, and b, c and f are positive constants. Note that the system (1.4) has a globally unique solution. We continue our work with the long-time behavior of the system. We further show that (1.4) is stochastically persistent in mean. Finally, the optimal harvesting policy of the prey $x(t)$ will be taken into account later in this paper. The corresponding harvesting system has the stochastic form

(1.5)
$$\begin{cases} dx(t)=x(t)\big(a(\alpha(t))-E-bx(t)-\frac{cy(t)}{m(\alpha(t))+x(t)}\big)\,dt+x(t)\sigma_1(\alpha(t))\,dB_1(t),\\ dy(t)=y(t)\big(r(\alpha(t))-\frac{fy(t)}{m(\alpha(t))+x(t)}\big)\,dt+y(t)\sigma_2(\alpha(t))\,dB_2(t),\end{cases}$$

where $B_i(t)$, $i=1,2$, are independent standard Brownian motions, σ_i, $i=1,2$, represent noise intensities, b, c and f are positive constants, and E denotes the harvesting efforts, which are deterministic. The optimal harvesting policy of system (1.5) will be given. Therefore, by taking both color and white noise into account, the new stochastic population system (1.4) and (1.5) has some desired properties: a global positive solution, stochastic persistence in mean and the optimal harvesting policy of prey $x(t)$.

Since $x(t)$ and $y(t)$ are population sizes or densities of the prey and the predator at time t, respectively, we are only interested in the positive solutions herein.

For convenience and simplicity in the following discussion, we define

$$\check{m} = \max_{\alpha \in S}(m(\alpha)) \quad \text{and} \quad \widehat{m} = \min_{\alpha \in S}(m(\alpha)).$$

The organization of the paper is as follows. We recall the fundamental theory regarding the stochastic differential equation with Markovian switching in Section 2. Section 3 is devoted to the stochastic prey-predator system with Leslie-Gower and Holling-type II schemes with Markovian switching. Finally, the paper concludes by considering the optimal harvesting policy for the prey $x(t)$ in Section 4.

2. Stochastic differential equation with Markovian switching. Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\alpha(t)$, $t \geq 0$, be a right-continuous Markov chain in the probability space tasking values in a finite state space $S = \{1, 2, \ldots, N\}$, with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{\alpha(t+\Delta t) = j \mid \alpha(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & i = j, \end{cases}$$

where $\Delta > 0$. Here, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}$. We assume that the Markov chain $\alpha(t)$ is independent of the Brownian motion, and almost every sample path of $\alpha(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

In addition, we assume, as a standing hypothesis, that the Markov chain is irreducible. The algebraic interpretation of irreducibility is $\operatorname{rank}(\Gamma) = N-1$. Under this condition, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_1, \ldots, \pi_N) \in R^{1\times N}$ which can be determined by solving the following linear equation

$$\pi\Gamma = 0,$$

subject to

$$\sum_{j=1}^{N} \pi_j = 1 \quad \text{and} \quad \pi_j > 0 \quad \text{for all } j \in S.$$

Consider a stochastic differential equation with Markovian switching

$$dx(t) = f(x(t), t, \alpha(t))\, dt + g(x(t), t, \alpha(t))\, dB(t)$$

on $t \geq 0$, with initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \times R_+ \times S \longrightarrow R^n \quad \text{and} \quad g : R^n \times R_+ \times S \longrightarrow R^{n\times m}.$$

For the existence and uniqueness of the solution, we assume that the coefficients of the above equation satisfy the local Lipschitz condition and the linear growth condition, that is, for each $k = 1, 2, \ldots$, there is an $h_k > 0$ such that

$$|f(x, t, \alpha) - f(y, t, \alpha)| \bigvee |g(x, t, \alpha) - g(y, t, \alpha)| \leq h_k |x - y|$$

for all $t \geq 0$, $\alpha \in S$ and those $x, y \in R^n$ with $|x| \bigvee |y| \leq k$, and there is an $h > 0$ such that

$$|f(x, t, \alpha)| \bigvee |g(x, t, \alpha)| \leq h(1 + |x|)$$

for all $(x, t, \alpha) \in R^n \times R_+ \times S$.

Let $C^{2,1}(R^n \times R_+ \times S, R_+)$ denote the family of all nonnegative functions $V(x, t, \alpha)$ on $R^n \times R_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times R_+ \times S, R_+)$, define an operator LV from $R^n \times R_+ \times S$ to R by

$$\begin{aligned} LV(x, t, \alpha) &= V_t(x, t, \alpha) + V_x(x, t, \alpha) f(x, t, \alpha) \\ &\quad + \frac{1}{2}\text{trace}[g^T(x, t, \alpha) V_{xx}(x, t, \alpha) g(x, t, \alpha)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j). \end{aligned}$$

In particular, if V is independent of α, that is, $V(x, t, \alpha) = V(x, t)$, then

$$LV(x, t, \alpha) = V_t(x, t) + V_x(x, t,) f(x, t) + \frac{1}{2}\text{trace}[g^T(x, t) V_{xx}(x, t) g(x, t)].$$

3. Asymptotic properties of the prey-predator model with Leslie-Gower and Holling-type II schemes with regime switching. Here, we analyze stochastic prey-predator model with Leslie-Gower and Holling-type II schemes with regime switching. The stochastic system reads

$$\begin{cases} dx(t) = x(t)\big(a(\alpha(t)) - bx(t) - \frac{cy(t)}{m(\alpha(t))+x(t)}\big)\, dt + x(t)\sigma_1(\alpha(t))\, dB_1(t), \\ dy(t) = y(t)\big(r(\alpha(t)) - \frac{fy(t)}{m(\alpha(t))+x(t)}\big)\, dt + y(t)\sigma_2(\alpha(t))\, dB_2(t), \end{cases} \tag{3.1}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions, σ_i, $i = 1, 2$, represent noise intensities and b, c, f are positive constants. If the stochastic differential equation has a unique global (i.e., no explosion in a finite time) solution for any initial value, the coefficients of the equation are required to obey the linear growth condition and local Lipschitz condition (cf., **[8]**). We obtain that there is a unique local solution $X(t) = (x(t), y(t))$ on $t \in [0, \tau_e)$ with the initial value $(x_0, y_0) > 0$, $\alpha \in S$, where τ_e is the explosion time.

In order to proceed with our study, we assume the following assumptions (A_1) and (A_2).

(A_1) $0 < \min\{r(\alpha) - (1/2)\sigma_2^2(\alpha),\ \alpha \in S\} \leq \max\{r(\alpha) - (1/2)\sigma_2^2(\alpha), \alpha \in S\} = A$, and

(A_2) $\min\{a(\alpha) - (1/2)\sigma_1^2(\alpha), \alpha \in S\} = \Delta > 0$, $\Delta - c(P + \epsilon) > 0$,

where ϵ is a sufficiently small positive constant, and $P =: (1/f)\sum_{\alpha=1}^{N} \pi_\alpha(r(\alpha) - (\sigma_2^2(\alpha))/2)$.

In the following content, we will demonstrate that the local solution to (3.1) is global, motivated by the research **[11, 3]**. Let

$$\Phi(t) = \frac{\exp\int_0^t (a(\alpha(s)) - (1/2)\sigma_1^2(\alpha(s)))\, ds + \sigma_1(\alpha(s))\, dB_1(s)}{(1/x_0) + \int_0^t b \exp\int_0^s (a(\alpha(\tau)) - (1/2)\sigma_1^2(\alpha(\tau)))\, d\tau + \sigma_1(\alpha(\tau))\, dB_1(\tau)\, ds}. \tag{3.2}$$

Thus, $\Phi(t)$ is the unique solution of equation

$$\begin{cases} d\Phi(t) = \Phi(t)(a(\alpha(t)) - b\Phi(t))\, dt + \Phi(t)\sigma_1(\alpha(t))\, dB_1(t), \\ \Phi(0) = x_0. \end{cases}$$

Hence, the comparison theorem implies $x(t) \leq \Phi(t)$, $t \in [0, \tau_e)$ almost surely, and

$$\begin{cases} d\psi(t) = \psi(t)(r(\alpha(t)) - (f/m(\alpha))\psi(t))\,dt + \psi(t)\sigma_2(\alpha(t))\,dB_2(t), \\ \psi(0) = y_0 \end{cases}$$

has a unique solution

(3.3)

$$\psi(t) = \frac{\exp\int_0^t (r(\alpha(s)) - (1/2)\sigma_2^2(\alpha(s)))\,ds + \sigma_2(\alpha(s))\,dB_2(s)}{(1/y_0) + \int_0^t (f/m(\alpha(s)))\exp[\int_0^s (r(\alpha(\tau) - (1/2)\sigma_2^2(\alpha(\tau)))d\tau + \sigma_2(\alpha(\tau))dB_2(\tau)]ds}.$$

Obviously, $\psi(t) \le y(t)$, $t \in [0, \tau_e)$ almost surely. Moreover,

$$d\Psi(t) = \Psi(t)\left(r(\alpha(t)) - \frac{f\Psi(t)}{m(\alpha(t)) + \Phi(t)}\right)dt + \Psi(t)\sigma_2(\alpha(t))\,dB_2(t),$$

where

(3.4)

$$\Psi(t) = \frac{\exp\int_0^t (r(\alpha(s)) - (1/2)\sigma_2^2(\alpha(s)))\,ds + \sigma_2(\alpha(s))\,dB_2(s)}{1/y_0 + \int_0^t (f/m(\alpha(s)) + \Phi(s))\exp[\int_0^s (r(\alpha(\tau)) - (1/2)\sigma_2^2(\alpha(\tau)))d\tau + \sigma_2(\alpha(\tau))dB_2(\tau)]ds}.$$

Thus, we obtain $y(t) \le \Psi(t)$, $t \in [0, \tau_e)$, almost surely. In addition,

$$d\phi(t) = \phi(t)\left(a(\alpha(t)) - b\phi(t) - \frac{c\Psi(t)}{m(\alpha(t))}\right)dt + \phi(t)\sigma_1(\alpha(t)\,dB_1(t). \tag{3.5}$$

It is easy to see that we have

$$x(t) \ge \phi(t), \quad t \in [0, \tau_e) \quad \text{almost surely.}$$

Simply, we obtain

$$\phi(t) \le x(t) \le \Phi(t)$$

and

$$\psi(t) \le y(t) \le \Psi(t) \quad t \in [0, \tau_e) \quad \text{almost surely.}$$

It can easily be verified that $\phi(t)$, $\Phi(t)$, $\psi(t)$ and $\Psi(t)$ all exist on $t \ge 0$; hence, this leads to the following.

Theorem 3.1. *There is a unique positive solution $X(t) = (x(t), y(t))$ of* (3.1) *for any initial value $(x_0, y_0) > 0$, $\alpha \in S$, and the solution has the properties*:

(3.6)

$$\phi(t) \le x(t) \le \Phi(t) \quad \text{and} \quad \psi(t) \le y(t) \le \Psi(t) \quad t > 0 \text{ almost surely,}$$

where $\phi(t)$, $\Phi(t)$, $\psi(t)$ and $\Psi(t)$ are defined as (3.2), (3.3), (3.4) *and* (3.5).

Theorem 3.1 states that equation (3.1) has a globally unique solution.

Now, we investigate certain asymptotic limits of the population model (3.1). Referring to **[11]**, it is not difficult to see that

$$\lim_{t\to\infty} \frac{\ln \Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\ln \psi(t)}{t} = 0 \quad \text{almost surely.} \tag{3.7}$$

Next, we give the following essential theorems which will be used.

Theorem 3.2. *Assume that condition* (A_1) *holds. We have the property*:

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = 0 \quad \textit{almost surely.} \tag{3.8}$$

Proof. From (3.6) and (3.7),

$$0 = \liminf_{t\to\infty} \frac{\ln \psi(t)}{t} \le \liminf_{t\to\infty} \frac{\ln y(t)}{t} \quad \text{almost surely.}$$

Thus, it remains to show that $\limsup_{t\to\infty}(\ln y(t))/t \le 0$. Note that the quadratic variation of the stochastic integral

$$\int_0^t \sigma_i(\alpha(s))\, dB_i(s)$$

is

$$\int_0^t \sigma_i^2(\alpha(s))\, ds \le Kt;$$

thus, the strong law of large numbers for local martingales yields that

$$\frac{\int_0^t \sigma_i(\alpha(s))\, dB_i(s)}{t} \longrightarrow 0 \quad \text{almost surely} \quad t \to \infty.$$

Therefore, for any $\epsilon > 0$, there exists some positive constant $T < \infty$ such that

$$\left| \int_0^t \sigma_i(\alpha(s))\, dB_i(s) \right| < \epsilon t \quad \text{almost surely for any } t \ge T.$$

Then, for any $t > s \geq T$, we have

(3.9)

$$\left| \int_s^t \sigma_i(\alpha(\tau))\, dB_i(\tau) \right| \leq \left| \int_0^t \sigma_i(\alpha(\tau))\, dB_i(\tau) \right| + \left| \int_0^s \sigma_i(\alpha(\tau))\, dB_i(\tau) \right| \\ \leq \epsilon(t+s) \quad \text{almost surely.}$$

Moreover, it follows from (3.7) that, for the above ϵ and T, we get

$$-\epsilon t \leq \ln \Phi(t) \leq \epsilon t \text{ almost surely } t \geq T. \tag{3.10}$$

By (3.4), (3.6), (3.9) and (3.10), for $t > s \geq T$, we derive

$$\begin{aligned} \frac{1}{y(t)} \geq \frac{1}{\Psi(t)} &= \exp - \left[\int_T^t \left(r(\alpha(s)) - \frac{1}{2}\sigma_2^2(\alpha(s)) \right) ds + \sigma_2(\alpha(s))\, dB_2(s) \right] \\ &\quad \cdot \left[\frac{1}{y(T)} + \int_T^t \frac{f}{m(\alpha(s)) + \Phi(s)} \right. \\ &\qquad \cdot \exp \left[\int_T^s \left(r(\alpha(\tau)) - \frac{1}{2}\sigma_2^2(\alpha(\tau)) \right) d\tau \right. \\ &\qquad\qquad \left. \left. + \sigma_2(\alpha(\tau)) dB_2(\tau) \right] ds \right] \\ &\geq \int_T^t \frac{f}{m(\alpha(s)) + \Phi(s)} \exp - \left[\int_s^t \left(r(\alpha(\tau)) - \frac{1}{2}\sigma_2^2(\alpha(\tau)) \right) d\tau \right. \\ &\qquad\qquad \left. + \sigma_2(\alpha(\tau)) dB_2(\tau) \right] ds \\ &\geq \int_T^t \frac{f}{m(\alpha(s)) + e^{\epsilon s}} \exp - \left[\int_s^t \left(r(\alpha(\tau)) - \frac{1}{2}\sigma_2^2(\alpha(\tau)) \right) d\tau \right. \\ &\qquad\qquad \left. + \sigma_2(\alpha(\tau)) dB_2(\tau) \right] ds \\ &\geq \int_T^t \frac{f}{m(\alpha(s)) + 1} e^{-\epsilon s} e^{-A(t-s)} e^{-\epsilon(t+s)} ds \\ &\geq \frac{f}{\check{m} + 1} e^{-(A+\epsilon)t} \int_T^t e^{(A-2\epsilon)s} ds. \end{aligned}$$

Therefore,

$$\frac{e^{(A+\epsilon)t}}{\Psi(t)} \geq \frac{f}{(\check{m}+1)(A - 2\epsilon)} \left(e^{(A-2\epsilon)t} - e^{(A-2\epsilon)T} \right).$$

Then,

$$\lim_{t\to\infty}\frac{(A+\epsilon)t}{t}\geq\limsup_{t\to\infty}\frac{\ln\Psi(t)}{t}+\lim_{t\to\infty}\frac{\ln f/[(\check{m}+1)(A-2\epsilon)]}{t}$$
$$+\lim_{t\to\infty}\frac{\ln[e^{(A-2\epsilon)t}-e^{(A-2\epsilon)T}]}{t},$$

that is,

$$A+\epsilon\geq\limsup_{t\to\infty}\frac{\ln\Psi(t)}{t}+A-2\epsilon.$$

Thus,

$$\limsup_{t\to\infty}\frac{\ln\Psi(t)}{t}\leq 3\epsilon.$$

Using the fact that $\epsilon>0$ is arbitrary, we obtain that

$$\limsup_{t\to\infty}\frac{\ln\Psi(t)}{t}\leq 0.$$

Consequently,

$$\limsup_{t\to\infty}\frac{\ln y(t)}{t}\leq\limsup_{t\to\infty}\frac{\ln\Psi(t)}{t}\leq 0.$$

The proof is complete. □

We now consider the long time behavior of the Leslie-Gower term $y(t)/(m(\alpha(t))+x(t))$. Denote $V(y,\alpha)=\ln y$. Using the generalized Itô lemma, we conclude that

$$d(\ln y(t))=\left(r(\alpha(t))-\frac{\sigma_2^2(\alpha(t))}{2}-\frac{fy(t)}{m(\alpha(t))+x(t)}\right)dt+\sigma_2(\alpha(t))\,dB_2(t).$$

Hence,

$$\ln y(t)-\ln y_0=\int_0^t r(\alpha(s))-\frac{\sigma_2^2(\alpha(s))}{2}\,ds-\frac{fy(s)}{m(\alpha(s))+x(s)}\,ds$$
$$+\int_0^t\sigma_2(\alpha(s))\,dB_2(s).$$

By virtue of (3.8), the strong law of large numbers for local martingales and the ergodic properties of Markov chains, we obtain

$$\lim_{t\to\infty}\frac{\int_0^t (y(s))/(m(\alpha(s))+x(s)ds)}{t}=\frac{1}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right).$$

Theorem 3.3. *Assume that* (A_1) *holds. Then, the positive solution* $X(t)=(x(t),y(t))$ *to* (3.1) *with initial value* $(x_0,y_0)>0$, $\alpha\in S$, *satisfies*

(3.11)

$$\lim_{t\to\infty}\frac{\int_0^t (y(s))/(m(\alpha(s))+x(s))\,ds}{t}=\frac{1}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)=: P.$$

Theorem 3.4. *Let condition* (A_2) *hold. Then,* $x(t)$ *has the property*

$$\lim_{t\to\infty}\frac{\ln x(t)}{t}=0 \quad \textit{almost surely.} \tag{3.12}$$

Proof. From (3.6) and (3.7), we know that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t}\le\limsup_{t\to\infty}\frac{\ln\Phi(t)}{t}=0 \quad \text{almost surely.}$$

Now, we merely must show that

$$\liminf_{t\to\infty}\frac{\ln x(t)}{t}\ge 0.$$

From the proof of Theorem 3.2, we obtain that, for any $\epsilon>0$, there exists some positive constant $T<\infty$ such that, for any $t>s\ge T$,

$$\left|\int_s^t \sigma_i(\alpha(\tau)0)\,dB_i(\tau)\right|\le\epsilon(t+s) \quad \text{almost surely,}$$

and

$$|\ln\Phi(t)|\le\epsilon t \quad \text{almost surely.}$$

From the generalized Itô lemma, we have

$$\begin{aligned}\frac{1}{x(t)} &= \frac{1}{x(T)} \exp\Big[-\int_T^t a(\alpha(s)) - \frac{\sigma_1^2(\alpha(s))}{2} - \int_T^t \frac{cy(s)}{m(\alpha(s)) + x(s)}\, ds \\ &\qquad - \int_T^t \sigma_1(\alpha(s))\, dB_1(s)\Big] \\ &\quad + \int_T^t b \exp\Big[-\int_s^t a(\alpha(\tau)) - \frac{\sigma_1^2(\alpha(\tau))}{2} - \int_s^t \frac{cy(\tau)}{m(\alpha(\tau)) + x(\tau)}\, d\tau \\ &\qquad - \int_s^t \sigma_1(\alpha(\tau))\, dB_1(\tau)\Big] \\ &=: I_1 + I_2.\end{aligned}$$

It follows from the result of Theorem 3.3 that, for any $\epsilon > 0$, there exists some positive constant $T < \infty$ such that

$$\int_T^t \frac{y(s)}{m(\alpha(s)) + x(s)}\, ds < (P+\epsilon)t - (P-\epsilon)T \text{ almost surely } t > s > T.$$

Similarly, we have

$$\int_s^t \frac{y(\tau)}{m(\alpha(\tau)) + x(\tau)}\, d\tau < (P+\epsilon)t - (P-\epsilon)T \quad \text{almost surely.}$$

Therefore,

$$\begin{aligned} I_1 &\le \frac{1}{x(T)} e^{-\Delta(t-T)+c(P+\epsilon)t-c(P-\epsilon)T+\epsilon(t+T)} \\ &= \frac{1}{x(T)} e^{-[\Delta - c(P+\epsilon)](t-T)+\epsilon[t+(2c+1)T]} \\ &\le K e^{\epsilon[t+(2c+1)T]} \quad \text{almost surely}\end{aligned}$$

and

$$\begin{aligned} I_2 &\le b\int_T^t e^{-\Delta(t-s)+c(P+\epsilon)t-c(P-\epsilon)s+\epsilon(t+s)} ds \\ &= b\int_T^t e^{-[\Delta-c(P+\epsilon)](t-s)+\epsilon[t+(2c+1)s]} ds \\ &\le K\int_T^t e^{\epsilon[t+(2c+1)s]} ds \le K e^{\epsilon(2c+2)t} \quad \text{almost surely.}\end{aligned}$$

Thus,

$$\frac{1}{x(t)} \leq Ke^{\epsilon[t+(2c+1)T]} + Ke^{\epsilon(2c+2)t} \quad \text{almost surely.}$$

It is easy to conclude that

$$e^{-\epsilon(2c+2)t}\frac{1}{x(t)} \leq K \quad \text{almost surely,}$$

that is,

$$\frac{1}{x(t)} \leq Ke^{\epsilon(2c+2)t} \quad \text{almost surely.}$$

Hence,

$$\limsup_{t\to\infty}(-\frac{\ln x(t)}{t}) \leq \epsilon(2c+2) \quad \text{almost surely.}$$

Hence,

$$\liminf_{t\to\infty}\frac{\ln x(t)}{t} \geq -\epsilon(2c+2) \quad \text{almost surely.}$$

Since $\epsilon > 0$ is arbitrary, it is inferred that

$$\liminf_{t\to\infty}\frac{\ln x(t)}{t} \geq 0 \quad \text{almost surely.}$$

This completes the proof. $\square$

Next, we proceed with stochastic persistent conditions for the system (3.1).

Theorem 3.5. *Assume that* (A_1) *and* (A_2) *hold. Then, the system* (3.1) *is stochastically persistent in mean, that is, the positive solution* $X(t) = (x(t), y(t))$ *of equation* (3.1) *with initial value* $(x_0, y_0) > 0$ *obeys*:

$$\liminf_{t\to\infty}\frac{\int_0^t x(s)\,ds}{t} > 0 \quad \text{and} \quad \liminf_{t\to\infty}\frac{\int_0^t y(s)\,ds}{t} > 0 \text{ almost surely.}$$

Proof. Firstly, we show that

$$\liminf_{t\to\infty}\frac{\int_0^t x(s)\,ds}{t} > 0 \quad \text{almost surely.}$$

Using the generalized Itô lemma, we have

$$
\begin{aligned}
\ln x(t) - \ln x_0 = &\int_0^t a(\alpha(s)) - \frac{\sigma_1^2(\alpha(s))}{2}\, ds \\
&- \int_0^t \left(bx(s) + \frac{cy(s)}{m(\alpha(s)) + x(s)} \right) ds \\
&+ \int_0^t \sigma_1(\alpha(s)) dB_1(s).
\end{aligned}
\tag{3.13}
$$

Note the quadratic variation of the stochastic integral $\int_0^t \sigma_1(\alpha(s))\, dB_1(s)$ is $\int_0^t \sigma_1^2(\alpha(s))\, ds \le Kt$; thus, the strong law of large numbers for local martingales yields that

$$
\frac{\int_0^t \sigma_1(\alpha(s))\, dB_1(s)}{t} \longrightarrow 0 \quad \text{almost surely} \quad t \to \infty. \tag{3.14}
$$

By virtue of (3.11) and the ergodic properties of Markov chains, we conclude:

$$
\lim_{t\to\infty} \frac{\int_0^t [y(s)/m(\alpha(s)) + x(s)]\, ds}{t} = \frac{1}{f} \sum_{\alpha=1}^{N} \pi_\alpha \left(r(\alpha) - \frac{\sigma_2^2(\alpha)}{2} \right) \tag{3.15}
$$

and

$$
\lim_{t\to\infty} \frac{\int_0^t (a(\alpha(s)) - [\sigma_1^2(\alpha(s))]/2)\, ds}{t} = \sum_{\alpha=1}^{N} \pi_\alpha \left(a(\alpha) - \frac{\sigma_1^2(\alpha)}{2} \right). \tag{3.16}
$$

It follows from (3.12)–(3.16) that

$$
\begin{aligned}
\lim_{t\to\infty} &\frac{b \int_0^t x(s)\, ds}{t} \\
&= \sum_{\alpha=1}^{N} \pi_\alpha \left(a(\alpha) - \frac{\sigma_1^2(\alpha)}{2} \right) - \frac{c}{f} \sum_{\alpha=1}^{N} \pi_\alpha \left(r(\alpha) - \frac{\sigma_2^2(\alpha)}{2} \right).
\end{aligned}
\tag{3.17}
$$

Secondly, we show that

$$
\liminf_{t\to\infty} \frac{\int_0^t y(s) ds}{t} > 0 \quad \text{almost surely.}
$$

Obviously,

$$\frac{1}{\widehat{m}}\liminf_{t\to\infty}\frac{\int_0^t y(s)\,ds}{t} \ge \lim_{t\to\infty}\frac{\int_0^t [y(s)/m(\alpha(s))+x(s)]\,ds}{t} = \frac{1}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right) > 0 \text{ almost surely.}$$

Thus,

$$\liminf_{t\to\infty}\frac{\int_0^t y(s)\,ds}{t} \ge \frac{\widehat{m}}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right) > 0 \text{ almost surely,}$$

as required. The proof is complete. □

4. Harvesting policy for the prey-predator model with Leslie-Gower and Holling-type II schemes with regime switching. When the harvesting problems of population resources is discussed, we aim to gain the optimal harvesting effort and corresponding maximum sustainable yield. Optimal harvesting policy for stochastic prey-predator population model with Markovian switching (1.5) will be taken into account. Here, we mainly consider the optimal harvesting policy of the prey $x(t)$. We make use of the ergodic property of Markov chain in order to proceed with our study.

Here, we present the following assumptions.

(A_1) $0 < \min\{r(\alpha)-(1/2)\sigma_2^2(\alpha),\ \alpha\in S\} \le \max\{r(\alpha)-(1/2)\sigma_2^2(\alpha),\ \alpha\in S\} = A$;

(A_3) $\min\{a(\alpha)-E-(1/2)\sigma_1^2(\alpha),\ \alpha\in S\} = \Delta > 0$, $\Delta - c(P+\epsilon) > 0$, where ϵ is a sufficiently small positive constant and

$$P =: \frac{1}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right).$$

In the same manner as the proofs of Theorems 3.1–3.4, we derive Theorems 4.1 and 4.2. Here we don't list the corresponding arguments in detail, only give the essential results.

Theorem 4.1. *Equation* (1.5) *has a globally unique solution* $X(t) = (x(t), y(t))$ *for any initial value* $(x_0, y_0) > 0$. *In addition, under the* *conditions* (A_1) *and* (A_3)*, we have the properties*

$$\lim_{t\to\infty}\frac{\ln x(t)}{t}=0, \qquad \lim_{t\to\infty}\frac{\ln y(t)}{t}=0 \text{ almost surely.}$$

When harvesting problems are considered, the corresponding average population level is derived below.

Theorem 4.2. *Assume that the conditions of Theorem* 4.1 *hold. Then, the solution* $X(t)=(x(t),y(t))$ *to* (1.5) *obeys*

$$\lim_{t\to\infty}\frac{\int_0^t x(s)\,ds}{t} = \frac{1}{b}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right)-\frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)-E\right] \tag{4.1}$$

almost surely, where $\alpha\in S=\{1,2,\dots,N\}$.

Proof. The proof is similar to the argument of (3.17); hence, it is omitted here. □

When the species $x(t)$ is subjected to exploitation, it is important and necessary to consider the corresponding maximum sustainable revenue.

Theorem 4.3. *Let the conditions of Theorem* 4.1 *hold. Then, the optimal harvesting effort of* $x(t)$ *is*

$$E^*=\frac{1}{2}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right)-\frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)\right]. \tag{4.2}$$

The optimal sustainable harvesting yield reads

$$\lim_{t\to\infty}\frac{\int_0^t E^*x(s)ds}{t}=\frac{1}{4b}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right)-\frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)\right]^2. \tag{4.3}$$

Proof. It follows from (4.1) that

$$\lim_{t\to\infty}\frac{\int_0^t x(s)ds}{t} = \frac{1}{b}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right) - \frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)-E\right].$$

Thus,

(4.4)

$$\begin{aligned}\lim_{t\to\infty}\frac{E\int_0^t x(s)\,ds}{t} &= \lim_{t\to\infty}\frac{\int_0^t Ex(s)\,ds}{t}\\ &= \frac{1}{b}\left[E\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right) - \frac{c}{f}E\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)-E^2\right]\\ &=: F(E).\end{aligned}$$

Equation (4.4) indicates that $F(E)$ is not a stochastic function. Letting $F'(E)=0$, there exists a unique extreme value point

$$E^* = \frac{1}{2}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right) - \frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)\right].$$

Thus, the optimal harvesting effort of $x(t)$ is as required. Substituting (4.2) into (4.4) implies

$$\begin{aligned}F(E^*) &= \lim_{t\to\infty}\frac{\int_0^t E^*x(s)\,ds}{t}\\ &= \frac{1}{4b}\left[\sum_{\alpha=1}^{N}\pi_\alpha\left(a(\alpha)-\frac{\sigma_1^2(\alpha)}{2}\right) - \frac{c}{f}\sum_{\alpha=1}^{N}\pi_\alpha\left(r(\alpha)-\frac{\sigma_2^2(\alpha)}{2}\right)\right]^2.\end{aligned}$$

We have the optimal sustainable yield (4.3), as desired. □

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