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A GENERALIZATION OF THE BÔCHER-GRACE THEOREM

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ABSTRACT. The Bôcher-Grace theorem can be stated as follows: let p be a third degree complex polynomial. Then, there is a unique inscribed ellipse interpolating the midpoints of the triangle formed from the roots of p, and the foci of the ellipse are the critical points of p. Here, we prove the following generalization: let p be an nth degree complex polynomial, and let its critical points take the form

$$\alpha + \beta \cos k\pi/n, \quad k = 1, \ldots, n-1, \ \beta \neq 0.$$

Then, there is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of p, and the foci of this ellipse are the two most extreme critical points of p: $\alpha \pm \beta \cos \pi/n$.

1. Introduction. The Bôcher-Grace theorem has been independently discovered by many mathematicians. Recently, proofs have been given by Kalman [**3**] and Minda and Phelps [**6**]. Bôcher proved the theorem in 1892, and then, Grace proved it in 1902. Surprisingly, a significant generalization was proved prior to both Bôcher and Grace by Siebeck in 1864. He showed that the critical points of an nth degree polynomial are the foci of the curve of class $n-1$ which touches each line segment, joining the roots of the polynomial at its midpoints. The Bôcher-Grace theorem is the $n = 3$ case of Siebeck's theorem. Marden's book [**4**] gives a wonderful introduction to this material and contains an extensive bibliography.

The purpose of this paper is to give a new and different generalization to Bôcher-Grace's theorem.

2. Background. A *conformal similarity transformation* in the complex plane $\mathbb{C}$ is a complex function that takes the form

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$$S(z) = \alpha + \beta\, z \tag{2.1}$$

for complex numbers α and $\beta \neq 0$. Upon writing $\beta = re^{i\theta}$, it is easy to see the action taken upon z when applying S: a rotation, a uniform scaling of the real and imaginary parts of z and a translation in the plane. It is understood that conformal similarity transformations preserve the eccentricity of ellipses, and for this reason, any ellipse in the complex plane can be mapped via a conformal similarity transformation to an ellipse with foci ± 1.

When studying relationships between the critical points of a polynomial and its roots, conformal similarity transformations play a very important, standardizing role. Specifically, if S is a conformal similarity transformation as defined in equation (2.1) and a complex polynomial $p(z) = \prod_{k=1}^{n}(z - z_k)$ has critical points r_k, $k = 1, \dots, n-1$, then the polynomial $P(z) = \prod_{k=1}^{n}(z - S(z_k))$ has critical points $S(r_k)$, $k = 1, \dots, n-1$. A proof of this fact may be found in **[3]**.

An *affine transformation* in $\mathbb{C}$ is a complex function that takes the form

$$\Phi(z) = \alpha\, z + \beta\, \overline{z} + \gamma \tag{2.2}$$

for complex numbers γ, $\alpha \neq 0$ and $\beta \neq 0$. Unlike the conformal similarity transformation, it is less clear what action is taken when applying Φ to z. However, if $|\alpha| \neq |\beta|$, then we can introduce real parameters $a > |b| > 0$ and $\varphi, \theta \in (-\pi, \pi]$ so that

$$\alpha = \frac{1}{2}(a+b)e^{i(\varphi+\theta)} \quad \text{and} \quad \beta = \frac{1}{2}(a-b)e^{i(\varphi-\theta)}.$$

With these new parameters, Φ can be decomposed into three distinct components: a rotation, a purely affine transformation and a conformal similarity transformation, that is,

$$\Phi(z) = (S \circ A \circ R)(z) \tag{2.3}$$

where

$$\begin{cases} R(z) = e^{i\theta} z, \\ A(z) = \dfrac{a+b}{2c}\, z + \dfrac{a-b}{2c}\, \overline{z}, \\ S(z) = c\, e^{i\varphi} z + \gamma, \end{cases}$$

where $c = \sqrt{a^2 - b^2}$. We refer to A as a *purely affine transformation* since it is the component of Φ that distinguishes affine from conformal similarity transformations, permitting independent scaling of the real and imaginary parts of z. It is easy to verify, from equation (2.3), that

$$A(e^{it}) = \frac{a}{c}\cos t + i\frac{b}{c}\sin t, \tag{2.4}$$

and thus, the image of the unit circle, parametrized by e^{it}, under the affine transformation A, is an ellipse with eccentricity c/a and foci ± 1. Additionally, we note that the images of the rotated roots of unity $e^{i(\theta+2k\pi/n)}$, $k = 1, \ldots, n$, under A are

$$\frac{a}{c}\cos\left(\theta + \frac{2k\pi}{n}\right) + i\frac{b}{c}\sin\left(\theta + \frac{2k\pi}{n}\right), \quad k = 1, \ldots, n. \tag{2.5}$$

We require several properties of affine transformations, properties summarized here, without proof. Affine transformations are invertible if $|\alpha| \neq |\beta|$. Thus, not only is the image of the unit circle under an affine transformation an ellipse, but ellipses in $\mathbb{C}$ can be mapped onto the unit circle via an affine transformation. Affine transformations preserve parallel lines and preserve the midpoints of line segments.

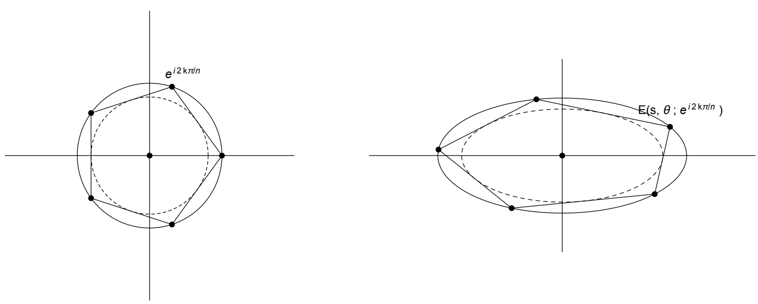

FIGURE 1. Rotated roots of unity ($n = 5$); the affine image under A.

We shall call a convex n-gon $\mathcal{P}$ *affinely regular* if it is the affine image of the regular convex polygon formed by the n roots of unity. In Figure 1, an affinely regular polygon is illustrated in the case when $n = 5$. At left is the application of $R(z) = e^{i\theta}z$ to the roots of unity $e^{i2k\pi/n}$, $k = 1, \ldots, n$; at right is the application of A to these rotated roots of unity; here, $S(z) = z$. We also draw attention to the circumscribing and inscribing circles at left and the corresponding circumscribing and inscribing ellipses at right. The circles have radii 1 and $\cos \pi/n$, while the ellipses have foci ± 1 and $\pm \cos \pi/n$, respectively. In light of the summarized properties of affine transformations, we note that, since the inscribing circle is tangent at the midpoints of the regular polygon pictured, the inscribed ellipse is also tangent at the midpoints of the corresponding affinely regular polygon.

In fact, the existence of an inscribed ellipse interpolating the midpoints of a convex polygon can be shown to characterize affinely regular polygons. In order to see this, let $\mathcal{P}$ be a polygon with an inscribed ellipse interpolating its midpoints, and let $\mathcal{P}'$ denote the convex polygon formed by the midpoints. Let Φ be an affine transformation that maps the inscribed ellipse to the unit circle. The images of the vertices of $\mathcal{P}'$ under Φ lie on the unit circle, and the sides of $\Phi(\mathcal{P})$ are tangent to the unit circle. There are $2n$ right triangles that can be formed using the origin, a vertex from $\Phi(P')$ and an adjacent vertex from $\Phi(\mathcal{P})$. It is not difficult to see that each of these triangles is congruent to all other such triangles. Hence, the vertices of $\Phi(\mathcal{P})$ all lie on a common circle, implying that $\mathcal{P}$ is an affinely regular polygon.

Here, we list some useful properties of the Chebyshev polynomials that can be found in **[5]**. The *Chebyshev polynomials* $T_n(x)$ of the first kind and $U_n(x)$ of the second kind can be defined by

$$(2.6) \qquad T_n(\cos\theta) = \cos(n\theta) \quad \text{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \ n \geq 0.$$

From equation (2.6), we see that $T_n'(x) = nU_{n-1}(x)$ and that the $n-1$ roots of $U_{n-1}(x)$ are $\{\cos k\pi/n : k = 1, \ldots, n-1\}$. For complex z, the formula for T_n is given by:

$$(2.7) \qquad T_n(z) = \frac{1}{2}\big((z + \sqrt{z^2-1})^n + (z - \sqrt{z^2-1})^n\big).$$

From our decomposition of affine transformations in equation (2.3), a general affine transformation can be thought of first as a mapping of the unit circle to an ellipse with foci ± 1, followed by a conformal similarity transformation. There is effectively a one-to-one correspondence between the functions A and the family of ellipses with foci ± 1. This family, parametrized by $s > 0$, is given by:

$$\cosh s \cos t + i \sinh s \sin t, \quad 0 \le t < 2\pi. \tag{2.8}$$

The foci of each ellipse is ± 1 since $\cosh^2 s - \sinh^2 s = 1$. The eccentricity of each ellipse is $0 < \operatorname{sech} s < 1$. Since conformal similarities preserve the eccentricity of an ellipse, it follows that any ellipse in the plane can be mapped by a conformal similarity transformation to a member of this family.

A property of the Chebyshev polynomials $T_n(z)$ that deserves to be better known is that they are periodic on the family of ellipses parametrized in equation (2.8). We explain this assertion in the next lemma.

Lemma 2.1. *Let $a > b > 0$ with $c^2 = a^2 - b^2$. The complex function*

$$f(t) = T_n\left(\frac{a}{c}\cos t + i\frac{b}{c}\sin t\right) \tag{2.9}$$

is periodic for real t, with period $2\pi/n$. Stated another way, the Chebysev polynomial $T_n(z)$ is periodic on the ellipse:

$$\frac{a}{c}\cos t + i\frac{b}{c}\sin t, \tag{2.10}$$

taking on the value

$$T_n\left(\frac{a}{c}\cos\theta + i\frac{b}{c}\sin\theta\right)$$

exactly n times.

Proof. Key to this result is the observation that, if $a^2 - b^2 = 1$ and if $z = a\cos t + ib\sin t$, then

$$\sqrt{z^2 - 1} = b\cos t + ia\sin t \tag{2.11}$$

(see [**5,** page 17, Exercise 5]). Equation (2.11) implies that

$$z \pm \sqrt{z^2 - 1} = (a \pm b)\cos t + i(a \pm b)\sin t. \tag{2.12}$$

Substituting equation (2.12) into the formula (2.7) yields

$$T_n(a\cos t + ib\sin t) = \frac{1}{2}\big((a+b)^n e^{\text{int}} + (a-b)^n e^{\text{int}}\big).$$

Of course, e^{int} is periodic with period $2\pi/n$.

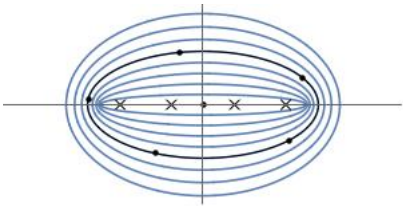

FIGURE 2. Ellipses with foci ± 1; affine image of rotated roots of unity.

More generally, if $a^2 - b^2 = c^2 \neq 1$, then

$$T_n\left(\frac{a}{c}\cos\left(\theta+\frac{2k\pi}{n}\right)+i\frac{b}{c}\sin\left(\theta+\frac{2k\pi}{n}\right)\right) = \frac{1}{2}\left(\left(\frac{a+b}{c}\right)^n e^{in\theta}+\left(\frac{a-b}{c}\right)^n e^{-in\theta}\right), \tag{2.13}$$

for $k = 1, \ldots, n$. $\square$

At this stage, we invite the reader to compare the formula for the affine image of the rotated roots of unity under the affine transformation A in equation (2.5) with the argument of T_n in equation (2.13). They are identical. This means that T_n is constant on the affine images of the rotated roots of unity. This is true for every affine transformation A, as defined in equation (2.3).

In Figure 2, we illustrate some members of the family (2.8). In addition, we plot the $n = 5$ images of a particular rotation of the n roots of unity under a particular affine transformation A. The polynomial T_n is constant on these n points. Along the real axis, we also mark the $n-1$ roots of $T_n'(z) = nU_{n-1}(z)$.

3. Main result.

Theorem 3.1 (Bôcher-Grace theorem for polygons)**.** *Let p be an nth degree complex polynomial, and let its critical points take the form*

$$\alpha+\beta\cos k\pi/n, \quad k = 1, \ldots, n-1, \ \beta \neq 0. \tag{3.1}$$

There is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of p*, and the foci of this ellipse are the two most extreme critical points of* p*:* $\alpha \pm \beta \cos \pi/n$*.*

Proof. Assume that the critical points of p take the form in expression (3.1). Without loss of generalization, exploiting properties of conformal similarity transformations, we may assume $\alpha = 0$ and $\beta = 1$. Thus, $p'(z) = \gamma U_{n-1}(z)$ and $p(z) = \gamma/nT_n(z)+\delta$, for complex constants $\gamma \neq 0$ and δ. If we designate by z_k, $k = 1, \dots, n$, the roots of p, then $T_n(z_k) = -n\delta/\gamma$, $k = 1, \dots, n$, that is to say, T_n is constant on the z_k, $k = 1, \dots, n$. As noted earlier, $T_n(z)$ can only be constant for n distinct points in $\mathbb{C}$, all of which lie on a single ellipse from the family (2.8). Thus, the z_k must take the form (2.5). In turn, this implies that the roots of p form an affinely regular polygon $\mathcal{P}$, and hence, admit an inscribed ellipse interpolating the midpoints of $\mathcal{P}$. In our discussion of affinely regular polygons, we noted that the foci of the inscribed ellipse are $\pm \cos \pi/n$, consistent with the assertion of the theorem. $\square$

When $n = 3$, the critical points *always* take the form $\alpha \pm \beta \cos \pi/3$, that is, all triangles are affinely regular; all triangles can be mapped to an equilateral triangle with an affine transformation. This is the original Bôcher-Grace theorem. When $n = 4$, the critical points take the form $\alpha \pm \beta \cos k\pi/4$, $k = 1, 2, 3$, if and only if the roots of p form a parallelogram. Stated another way, the only quadrilaterals that are affine images of squares are parallelograms.

The form of the critical points in expression (3.1) can be used to characterize affinely regular polygons, that is, a polygon $\mathcal{P}$ formed from the roots of a polynomial p is affinely regular if and only if the critical points of p take the form of the expression (3.1). With this observation, we conclude this article with a stronger statement than the generalized Bôcher-Grace theorem above.

Theorem 3.2 (Bôcher-Grace characterization theorem). *Let* p *be an* n*th degree polynomial, and let* $\mathcal{P}$ *denote the convex polygon formed by the roots of* p*. The polygon* $\mathcal{P}$ *admits an inscribed ellipse interpolating its midpoints if and only if the critical points of* p *take the form of the expression* (3.1)*.*

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