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NONLOCAL INITIAL VALUE PROBLEMS FOR HADAMARD-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS

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ABSTRACT. In this paper, we study initial value problems of fractional differential equations and inclusions of Hadamard type, supplemented with nonlocal conditions. Some new existence and uniqueness results are obtained by using fixed point theorems for single valued and multi-valued maps. Examples illustrating the main results are also presented.

1. Introduction. The mathematical modeling of several phenomena in applied sciences, such as physics, biology and ecology, gives rise to the problems with non-classical boundary conditions. Such conditions, which connect the values of unknown functions on the boundary and inside of the given domain, are known as nonlocal boundary conditions. The idea of nonlocal conditions dates back to the work of Hilb [**20**]. However, the systematic investigation of a certain class of spatial nonlocal problems was carried out by Bitsadze and Samarskii [**7**]. We refer the reader to [**6, 11**] and the references cited therein for motivation regarding nonlocal conditions.

Fractional calculus (differentiation and integration of arbitrary order) naturally arises in various areas of applied science and engineering, such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular thermodynamic variation, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex media, viscoelasticity and damping, control theory, wave propagation, percolation, identification, experimental data fitting, etc. [**23, 29, 30, 31**].

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Differential equations of fractional order have attracted the attention of several researchers, see [**1, 2, 4, 14, 17, 22, 26, 28**] and the references therein. For some recent work on Hadamard-type fractional differential equations, inclusions and inequalities, we refer the reader to [**3**].

In this paper, we investigate the existence of solutions for Hadamard type fractional differential equations and inclusions equipped with nonlocal initial conditions. We first consider the following Hadamard type single-valued nonlocal nonlinear initial value problem

$$\begin{cases} {}^H D^q x(t) = f(t, x(t)) \\ x(1) + \sum_{j=1}^{m} \zeta_j \, x(t_j) = 0, \end{cases} \quad 1 < t < T, \ 0 < q \le 1, \tag{1.1}$$

where ${}^H D^q$ denotes the Hadamard fractional derivative of order q, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$, t_j, $j = 1, 2, \dots, m$, are given points with $1 \le t_1 \le \cdots \le t_m < T$, and ζ_j are real numbers such that

$$1 + \sum_{j=1}^{m} \zeta_j \neq 0.$$

Then, we extend our study to the Hadamard type multi-valued problem of the form

$$\begin{cases} {}^H D^q x(t) \in F(t, x(t)) \\ x(1) + \sum_{j=1}^{m} \zeta_j \, x(t_j) = 0, \end{cases} \quad 1 < t < T, \ 0 < q \le 1, \tag{1.2}$$

where $F : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

We emphasize that problem (1.1) was studied for $q = 1$ in [**9**], for the time scales setting in [**5**] and for Caputo fractional differential equations in [**8**]. Here, we extend the work presented in [**5, 8, 9**] to the fractional case of Hadamard type single-valued and multi-valued nonlocal initial value problems.

A variety of existence results for problem (1.1) are proven by applying fixed point theorems. In subsection 3.1, we discuss an existence result using the idea employed in [**5, 8, 9**], where the growth condition is split into two sub intervals: one containing the points involved in the nonlocal condition, while the second one deals with the rest of the interval. In subsection 3.2, we present some more existence and uniqueness results for problem (1.1). In precise terms, an existence and uniqueness result is obtained by using Banach's fixed point theorem. Leray-Schauder alternative is employed to obtain an existence result by assuming a growth condition on $f(t,x(t))$ on the whole interval. The multi-valued problem (1.2) with convex and non-convex multi-valued maps is studied in Section 4. In the case of convex-valued maps (the upper semicontinuous case), we apply the nonlinear alternative of Leray-Schauder type to obtain an existence result for problem (1.2) in subsection 4.1. When the right hand side of the inclusion in (1.2) is not necessarily convex valued (the lower semicontinuous case), the desired existence result for (1.2) is obtained in subsection 4.2 by combining the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multi-valued maps with nonempty closed and decomposable values. The last result (the Lipschitz case), concerning the existence of solutions for problem (1.2) with not necessary nonconvex valued right hand side, is proven by applying a fixed point theorem for contractive multi-valued maps due to Covitz and Nadler in subsection 4.3. Although the methods employed in the present study are well known, their exposition in the framework of problems (1.1) and (1.2) is new.

2. Preliminaries. In this section, we introduce notation and definitions which are used throughout this paper. Let $X = C([1,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ with the norm

$$\|x\| = \|x\|_{[1,T]} = \max_{t\in[1,T]} |x(t)|.$$

We denote by $L^1([1,T],\mathbb{R})$ the Banach space of measurable functions $x : [1,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_1^T |x(t)|\,dt \quad \text{for all } x \in L^1([1,T],\mathbb{R}).$$

Let us recall some definitions on fractional calculus **[3, 23, 29, 31]**.

Definition 2.1 (**[23]**)**.** The *Hadamard derivative of fractional order* q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$ {}^H D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \, \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{g(s)}{s} \, ds, $$
$$ n - 1 < q < n, \quad n = [q] + 1, $$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 (**[23]**)**.** The *Hadamard fractional integral of order* q for a function g is defined as

$$ I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{g(s)}{s} \, ds, \quad q > 0, $$

provided the integral exists.

In order to define the solution of problem (1.1), we consider the following lemma.

Lemma 2.3. *Assume that* $1 + \sum_{j=1}^m \zeta_j \neq 0$. *For a given* $y \in X$, *the unique solution of the initial value problem*

$$ (2.1) \qquad \begin{cases} {}^H D^q x(t) = y(t) & 1 < t < T, \ 0 < q \le 1, \\ x(1) + \displaystyle\sum_{j=1}^m \zeta_j \, x(t_j) = 0 & 1 \le t_1 \le \cdots \le t_m < T, \end{cases} $$

is given by

$$ (2.2) \qquad x(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{y(s)}{s} \, ds - \frac{1}{1 + \displaystyle\sum_{j=1}^m \zeta_j} \sum_{j=1}^m \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{q-1} \frac{y(s)}{s} \, ds. $$

Proof. For some constant $x_0 \in \mathbb{R}$, we have

$$ (2.3) \qquad x(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{y(s)}{s} \, ds - x_0. $$

Then, we obtain

$$x(t_j) = \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{q-1} \frac{y(s)}{s} \, ds - x_0,$$

which, together with the initial condition in (2.1), yields

$$x_0 = \frac{1}{1 + \sum_{j=1}^{m} \zeta_j} \sum_{j=1}^{m} \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{q-1} \frac{y(s)}{s} \, ds.$$

Substituting the value of x_0 in (2.3), we obtain a unique solution of problem (2.1) given by (2.2). This completes the proof. □

In the sequel, we set

$$A = 1 + |\beta| \sum_{j=1}^{m} |\zeta_j|, \qquad \beta = \left(1 + \sum_{j=1}^{m} \zeta_j \right)^{-1}, \tag{2.4}$$

with $1 + \sum_{j=1}^{m} \zeta_j \neq 0$.

3. Main results for problem (1.1). In this section, we establish the existence and uniqueness results for problem (1.1).

3.1. Existence result with the mixed growth condition. We assume that $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and prove an existence result with mixed growth condition.

Definition 3.1. The map $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is said to be L^1-*Carathéodory* if

(i) $t \mapsto f(t, x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \mapsto f(t, x)$ is continuous for almost all $t \in [1, T]$;
(iii) For each $\rho > 0$, there exists a $\phi_\rho \in L^1([1, T], \mathbb{R}_+)$ such that

$$|f(t, x)| \leq \phi_\rho(t) \quad \text{for all } \|x\| \leq \rho \text{ and for almost all } t \in [1, T].$$

In view of Lemma 2.3, solutions of problem (1.1) are fixed points of the operator $\mathbf{F} : X \to X$, defined by

$$(3.1)\qquad (\mathbf{F}x)(t) = \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds - \beta\sum_{j=1}^m \zeta_j\,\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds.$$

Note that the operator $\mathbf{F}$ given by (3.1) appears as a sum of two integral operators: Fredholm type S_F (say), whose values depend on the restriction of function values to $[1, t_m]$, and is given by

$$S_F x(t) = \begin{cases} \dfrac{1}{\Gamma(q)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds \\ \quad -\beta\displaystyle\sum_{j=1}^m \zeta_j\,\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds, & t < t_m, \\ \dfrac{1}{\Gamma(q)}\displaystyle\int_1^{t_m} \left(\log\frac{t_m}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds \\ \quad -\beta\displaystyle\sum_{j=1}^m \zeta_j\,\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds, & t \geq t_m, \end{cases}$$

and Volterra type S_V (say), defined by

$$S_V x(t) = \begin{cases} 0 & t < t_m, \\ \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_m}^t \left(\log\frac{t}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds, & t \geq t_m, \end{cases}$$

depending on the restriction of function to $[t_m, T]$. This allows us to split the growth condition on the nonlinear term $f(t,x)$ into two parts, namely, for $t \in [1, t_m]$ and $t \in [t_m, T]$.

Theorem 3.2. *Assume that*

(H_1) $f : [1,T]\times\mathbb{R}\to\mathbb{R}$ *is an* L^1*-Carathéodory function.*

(H_2) *There exist a continuous function* ω *nondecreasing in its second argument,* $p \in L^1[t_m, T]$ *and a nondecreasing function* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|f(t,x)| \leq \begin{cases} \omega(t, |x|) & t \in [1, t_m], \\ p(t)\Psi(|x|) & t \in [t_m, T]. \end{cases}$$

(H_3) *There exists an* $R_0 > 0$ *such that*

$$\rho > R_0 \Longrightarrow \frac{1}{\rho\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s}\right)^{q-1} \frac{\omega(s,\rho)}{s}\, ds < \frac{1}{A}.$$

(H_4) $\limsup_{R\to\infty} \dfrac{R}{Q_1 + \Psi(R)Q_2} > 1$, *where*

$$Q_1 = \frac{A}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s}\right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds$$

and

$$Q_2 = \frac{1}{\Gamma(q)} \int_{t_m}^{T} \left(\log \frac{T}{s}\right)^{q-1} \frac{p(s)}{s}\, ds.$$

Then, problem (1.1) *has at least one solution on* $[1, T]$.

Proof. We show that the solutions of (1.1) are a priori bounded. Let x be a solution. Then, for $t \in [1, t_m]$, we have

$$\begin{aligned}
|x(t)| &= \lambda \left| \frac{1}{\Gamma(q)} \int_1^{t} \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \right. \\
&\qquad \left. - \beta \sum_{j=1}^{m} \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \right| \\
&\le \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s}\right)^{q-1} \frac{|f(s,x(s))|}{s}\, ds \\
&\quad + |\beta| \sum_{j=1}^{m} |\zeta_j| \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{|f(s,x(s))|}{s}\, ds \\
&\le A \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s}\right)^{q-1} \frac{\omega(s,|x(s)|)}{s}\, ds,
\end{aligned}$$

which, on taking the supremum for $t \in [1, t_m]$, yields

$$\|x\|_{[1,t_m]} \le A \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s}\right)^{q-1} \frac{\omega(s,\|x\|_{[1,t_m]})}{s}\, ds.$$

Then, assumption (H_3) guarantees that

$$\|x\|_{[1,t_m]} \le R_0.$$

Next, we let $t \in [t_m, T]$. Then

$$\begin{aligned}
|x(t)| &= \lambda \left| \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{f(s,x(s))}{s}\, ds \right. \\
&\qquad \left. - \beta \sum_{j=1}^{m} \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{q-1} \frac{f(s,x(s))}{s}\, ds \right| \\
&\le \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t}{s} \right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds \\
&\quad + |\beta| \sum_{j=1}^{m} \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_m}^{t} \left(\log \frac{t}{s} \right)^{q-1} \frac{p(s)\Psi(|x(s)|)}{s}\, ds \\
&\le A \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t}{s} \right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_m}^{t} \left(\log \frac{t}{s} \right)^{q-1} \frac{p(s)\Psi(|x(s)|)}{s}\, ds \\
&\le A \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t}{s} \right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds \\
&\quad + \Psi(\|x\|_{[t_m,T]}) \frac{1}{\Gamma(q)} \int_{t_m}^{t} \left(\log \frac{t}{s} \right)^{q-1} \frac{p(s)}{s}\, ds \\
&\le A \frac{1}{\Gamma(q)} \int_1^{t_m} \left(\log \frac{t_m}{s} \right)^{q-1} \frac{\omega(s,R_0)}{s}\, ds \\
&\quad + \Psi(\|x\|_{[t_m,T]}) \frac{1}{\Gamma(q)} \int_{t_m}^{T} \left(\log \frac{T}{s} \right)^{q-1} \frac{p(s)}{s}\, ds.
\end{aligned}$$

Consequently, we have

$$\frac{\|x\|_{[t_m,T]}}{Q_1 + \Psi(\|x\|_{[t_m,T]})Q_2} \le 1. \tag{3.2}$$

Now, (H_4) implies that there exists an $R^* > 0$ such that, for all $R > R^*$, we have

$$\frac{R}{Q_1 + \Psi(R)Q_2} > 1. \tag{3.3}$$

Comparing inequalities (3.2) and (3.3), we find that

$$\|x\|_{[t_m,T]} \leq R^*.$$

Let $\gamma = \max\{R_0, R^*\}$. Then, we have $\|x\|_{[1,T]} \leq \gamma$. It follows from (H_1) that there exists a $\phi_\gamma \in L^1([1,T], \mathbb{R}_+)$ such that

$$|f(t,x(t))| \leq \phi_\gamma(t) \quad \text{for almost every } t \in [1,T].$$

The operator $\mathbf{F} : \overline{B}_\gamma \to X$, defined by (3.1), is continuous and completely continuous. Indeed, $\mathbf{F}$ is continuous in view of (H_1), see **[19]**, and, for completely continuous, we remark that it is uniformly bounded as

$$\begin{aligned}
|\mathbf{F}x(t)| = \bigg| \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \\
- \beta \sum_{j=1}^m \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \bigg| \\
\leq \frac{1}{\Gamma(q)} \int_1^T \left(\log \frac{T}{s}\right)^{q-1} \frac{\phi_\gamma(s)}{s}\, ds \\
+ |\beta| \sum_{j=1}^m |\zeta_j| \frac{1}{\Gamma(q)} \int_1^T \left(\log \frac{T}{s}\right)^{q-1} \frac{\phi_\gamma(s)}{s}\, ds \\
= A \frac{1}{\Gamma(q)} \int_1^T \left(\log \frac{T}{s}\right)^{q-1} \frac{\phi_\gamma(s)}{s}\, ds,
\end{aligned}$$

and equicontinuous, since

$$\begin{aligned}
|\mathbf{F}x(\nu_2) - \mathbf{F}x(\nu_1)| = \bigg| \frac{1}{\Gamma(q)} \int_1^{\nu_1} \left[\left(\log \frac{\nu_2}{s}\right)^{q-1} - \left(\log \frac{\nu_1}{s}\right)^{q-1}\right] \frac{f(s,x(s))}{s}\, ds \\
+ \frac{1}{\Gamma(q)} \int_{\nu_1}^{\nu_2} \left(\log \frac{\nu_2}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \bigg| \\
\leq \bigg| \frac{1}{\Gamma(q)} \int_1^{\nu_1} \left[\left(\log \frac{\nu_2}{s}\right)^{q-1} - \left(\log \frac{\nu_1}{s}\right)^{q-1}\right] \frac{\phi_\gamma(s)}{s}\, ds \\
+ \frac{1}{\Gamma(q)} \int_{\nu_1}^{\nu_2} \left(\log \frac{\nu_2}{s}\right)^{q-1} \frac{\phi_\gamma(s)}{s}\, ds \bigg|,
\end{aligned}$$

where $1 < \nu_1 < \nu_2 < T$. Hence, by the Leray-Schauder alternative, we deduce that the operator $\mathbf{F}$ has a fixed point in B_γ, which is a solution of problem (1.1). $\square$

3.2. Further existence and uniqueness results. In the next theorem, we prove the uniqueness of solutions for problem (1.1) via Banach's fixed point theorem.

Theorem 3.3. *Assume that* $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ *is a jointly continuous function and satisfies the assumption*:

(A_1) $|f(t,x)-f(t,y)|\le L\|x-y\|$ *for all* $t\in[1,T]$, $L>0$, $x,y\in\mathbb{R}$.

Then, problem (1.1) *has a unique solution on* $[1,T]$ *if* $L<\Gamma(q+1)/(A(\log T)^q)$.

Proof. Consider the operator $\mathbf{F}$ defined by (3.1), and show that $\mathbf{F}B_r\subset B_r$, where

$$B_r=\{x\in X:\|x\|\le r\}$$

with

$$r\ge\frac{A(\log T)^qM}{\Gamma(q+1)-A(\log T)^qL},\quad \sup_{t\in[1,T]}|f(t,0)|=M.$$

For $x\in B_r$, we have

$$\begin{aligned}
|(\mathbf{F}x)(t)| &= \bigg|\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds\\
&\qquad -\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{f(s,x(s))}{s}\,ds\bigg|\\
&\le\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)\,\frac{ds}{s}\\
&\quad+|\beta|\sum_{j=1}^m|\zeta_j|\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\\
&\quad\times(|f(s,x(s))-f(s,0)|+|f(s,0)|)\,\frac{ds}{s}\\
&\le(Lr+M)\frac{1}{\Gamma(q)}\bigg[\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{ds}{s}+|\beta|\sum_{j=1}^m|\zeta_j|\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{ds}{s}\bigg]
\end{aligned}$$

$$\leq (Lr + M)\frac{1}{\Gamma(q+1)}\left[(\log T)^q + |\beta|(\log T)^q \sum_{j=1}^{m} |\zeta_j|\right]$$
$$= \frac{A(\log T)^q(Lr + M)}{\Gamma(q+1)} \leq r.$$

In consequence, $\|\mathbf{F}x\| \leq r$, for any $x \in B_r$, which shows that $\mathbf{F}B_r \subset B_r$. Now, for $x, y \in X$ and for each $t \in [1, T]$, we obtain

$$\begin{aligned}
&|(\mathbf{F}x)(t) - (\mathbf{F}y)(t)| \\
&\quad \leq \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \\
&\qquad + |\beta| \sum_{j=1}^{m} |\zeta_j| \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s}\, ds \\
&\quad \leq L\|x-y\| \frac{1}{\Gamma(q)} \left[\int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{ds}{s} + |\beta| \sum_{j=1}^{m} |\zeta_j| \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{ds}{s}\right] \\
&\quad \leq \frac{L(\log T)^q A}{\Gamma(q+1)} \|x - y\|.
\end{aligned}$$

By the given condition $L(\log T)^q A/(\Gamma(q+1)) < 1$, it follows that the operator $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (the Banach fixed point theorem). The proof is complete. $\square$

The following existence result is based on the Leray-Schauder nonlinear alternative.

Theorem 3.4. *Let $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Assume that:*

(A_2) *there exist a function $p \in L^1([1, T], \mathbb{R}^+)$ and a nondecreasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, x)| \leq p(t)\Omega(\|x\|)$, for all $(t, x) \in [1, T] \times \mathbb{R}$.*

(A_3) *There exists a constant $K > 0$ such that*

$$\frac{K}{A\Omega(K)(1/\Gamma(q)) \int_1^T (\log T/s)^{q-1}(p(s)/s)\, ds} > 1.$$

Then problem (1.1) *has at least one solution on* $[1, T]$.

Proof. We show the boundedness of the set of all solutions to equations $x = \lambda \mathbf{F} x$ for $\lambda \in [0,1]$. For that, let x be a solution of $x = \lambda \mathbf{F} x$ for $\lambda \in [0,1]$. Then, for $t \in [1,T]$, we have

$$\begin{aligned}
|x(t)| = |\lambda(\mathbf{F}x)(t)| &\le \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \\
&\quad + |\beta| \sum_{j=1}^m |\zeta_j| \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{f(s,x(s))}{s}\, ds \\
&\le \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{p(s)\Omega(\|x\|)}{s}\, ds \\
&\quad + |\beta| \sum_{j=1}^m |\zeta_j| \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{q-1} \frac{p(s)\Omega(\|x\|)}{s}\, ds \\
&= A\Omega(\|x\|) \frac{1}{\Gamma(q)} \int_1^T \left(\log \frac{T}{s}\right)^{q-1} \frac{p(s)}{s}\, ds,
\end{aligned}$$

which implies that

$$\frac{\|x\|}{A\Omega(\|x\|)(1/\Gamma(q)) \int_1^T (\log T/s)^{q-1} (p(s)/s)\, ds} \le 1.$$

In view of (A_3), there is no solution x such that $\|x\| \neq K$. We set

$$U = \{x \in X : \|x\| < K\}.$$

As in the proof (last step) of Theorem 3.2, it can be shown that the operator $\mathbf{F} : \overline{U} \to X$ defined by (3.1) is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathbf{F}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type **[18]**, we deduce that $\mathbf{F}$ has a fixed point $u \in \overline{U}$, which is a solution of problem (1.1). This completes the proof. □

As a special case, when $p(t) = 1$ and $\Omega(\|x\|) = \kappa\|x\| + N$, we have the following corollary.

Corollary 3.5. *Let $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Assume that*:

(A_4) *there exist constants* $0 \leq \kappa < \Gamma(q+1)/(A(\log T)^q)$ *and* $N > 0$ *such that*

$$|f(t,x)| \leq \kappa|x| + N \quad \textit{for all } t \in [1,T],\ x \in \mathbb{R}.$$

Then, problem (1.1) *has at least one solution on* $[1,T]$.

4. Main results for problem (1.2). This section is devoted to the existence of solutions for problem (1.2). We first outline some basic concepts of multi-valued analysis [**15, 21**].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{\mathrm{cl,b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{\mathrm{cp,c}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$ is

(i) *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$.

(ii) *Bounded* on bounded sets if $G(Y) = \cup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$, i.e., $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$.

(iii) *Upper semi-continuous* (usc) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.

(iv) *Lower semi-continuous* (lsc) if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) *Completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is usc if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) *Measurable* if, for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y-z| : z \in G(t)\}$$

is measurable. Recall that G *has a fixed point* if there is an $x \in X$ such that $x \in G(x)$. The fixed point set of the multi-valued operator G will be denoted by $\operatorname{Fix} G$.

4.1. The Carathéodory case. In this subsection, we consider the case when F has convex values and is of Carathéodory type and prove an existence result for problem (1.2) by applying nonlinear alternative of Leray-Schauder type.

For each $y \in C([1,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([1,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [1,T]\}.$$

We define the graph of G to be the set $\mathrm{Gr}(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and state a known result for closed graphs and upper-semicontinuity.

Lemma 4.1 ([**15,** Proposition 1.2])**.** *If $G : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is* usc*, then* Gr(G) *is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ as $n \to \infty$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

We also need the following lemmas in the sequel.

Lemma 4.2 ([**27**])**.** *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp,c}}(X)$ be an L^1-Carathéodory multi-valued map, and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then, the operator*

$$\Theta \circ S_F : C(J,X) \longrightarrow \mathcal{P}_{\mathrm{cp,c}}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}),$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 4.3 (Nonlinear alternative for Kakutani maps [**18**])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{\mathrm{cp,c}}(C)$ is an upper semicontinuous compact map. Then, either*:

(i) *F has a fixed point in $\overline{U}$; or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Definition 4.4. A function $x \in C^1([1,T],\mathbb{R})$ is said to be a *solution of initial value problem* (1.2) if

$$x(1) + \sum_{j=1}^{m} \zeta_j x(t_j) = 0$$

and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t,x(t))$ almost everywhere on $[1,T]$ and

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{v(s)}{s}\,ds \\ &-\beta\sum_{j=1}^m \zeta_j \frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v(s)}{s}\,ds. \end{aligned}$$

Theorem 4.5. *Assume that* (A_5) *holds. In addition, we suppose that*:

(B_1) $F:[1,T]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory*;

(B_2) *there exist a continuous nondecreasing function* $\Phi:[0,\infty)\to(0,\infty)$ *and a function* $p\in L^1([1,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y\in F(t,x)\} \le p(t)\Phi(\|x\|)$$
$$\text{for each } (t,x)\in[1,T]\times\mathbb{R};$$

(B_3) *there exists a constant* $\widehat{K}>0$ *such that*

$$\frac{\widehat{K}}{A\Phi(\widehat{K})(1/\Gamma(q))\int_1^T(\log T/s)^{q-1}(p(s)/s)\,ds} > 1,$$

where A *is given by* (2.4). *Then, problem* (1.2) *has at least one solution on* $[1,T]$.

Proof. In order to transform problem (1.2) into a fixed point problem, we define an operator $\mathcal{F}: C([1,T],\mathbb{R})\to\mathcal{P}(C([1,T],\mathbb{R}))$ by

$$\mathcal{F}(x) = \{h\in C([1,T],\mathbb{R}) : h(t) = N(x)(t)\}, \tag{4.1}$$

where

$$N(x)(t) = \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v(s)}{s}\,ds - \beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v(s)}{s}\,ds,$$

for $v\in S_{\mathrm{F},x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of problem (1.2).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 4.3) in several steps.

Step 1. *$\mathcal{F}(x)$ is convex for each $x\in C([1,T],\mathbb{R})$.* This step is obvious since $S_{F,x}$ is convex (F has convex values).

Step 2. *$\mathcal{F}$ maps bounded sets* (*balls*) *into bounded sets in* $C([1,T],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([1,T],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([1,T],\mathbb{R})$. Then, for each $h \in \mathcal{F}(x)$, $x \in B_r$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{v(s)}{s}\,ds - \beta\sum_{j=1}^m \zeta_j \frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v(s)}{s}\,ds.$$

Then, for $t \in [1,T]$, we have

$$\begin{aligned}
|h(t)| &\le \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{|v(s)|}{s}\,ds \\
&\quad + |\beta|\sum_{j=1}^m |\zeta_j| \frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{|v(s)|}{s}\,ds \\
&\le \frac{\Phi(\|x\|)}{\Gamma(q)}\Bigg[\int_1^T \left(\log\frac{t}{s}\right)^{q-1}\frac{p(s)}{s}\,ds \\
&\qquad\qquad + |\beta|\sum_{j=1}^m |\zeta_j|\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{p(s)}{s}\,ds\Bigg],
\end{aligned}$$

and, consequently,

$$\|h\| \le \frac{A\Phi(r)}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{p(s)}{s}\,ds.$$

Step 3. *$\mathcal{F}$ maps bounded sets into equicontinuous sets of* $C([1,T],\mathbb{R})$. Let $1 < \nu_1 < \nu_2 < T$ and $x \in B_r$. Then, for each $h \in \mathcal{F}(x)$, we obtain

$$\begin{aligned}
|h(\nu_2) - h(\nu_1)| &\le \frac{\Phi(r)}{\Gamma(q)}\int_1^{\nu_1}\left|\left(\log\frac{\nu_2}{s}\right)^{q-1} - \left(\log\frac{\nu_1}{s}\right)^{q-1}\right|\frac{p(s)}{s}\,ds \\
&\quad + \frac{\Phi(r)}{\Gamma(q)}\int_{\nu_1}^{\nu_2}\left(\log\frac{\nu_2}{s}\right)^{q-1}\frac{p(s)}{s}\,ds.
\end{aligned}$$

Obviously, the right hand side of the previous inequality tends to zero independently of $x \in B_r$ as $\nu_2 - \nu_1 \to 0$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{F} : C([1,T],\mathbb{R}) \to \mathcal{P}(C([1,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\mathcal{F}$ is usc Since $\mathcal{F}$ is completely continuous, it is sufficient to establish that it has a closed graph.

Step 4. *$\mathcal{F}$ has a closed graph.* Let $x_n \to x_*$, $h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists a $v_n \in S_{F,x_n}$ such that, for each $t \in [1,T]$,

$$h_n(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v_n(s)}{s}\,ds-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v_n(s)}{s}\,ds.$$

Thus, it suffices to show that there exists a $v_* \in S_{F,x_*}$ such that, for each $t \in [1,T]$,

$$h_*(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v_*(s)}{s}ds-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v_*(s)}{s}ds.$$

Let us consider the linear operator $\Theta : L^1([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$, given by

$$\begin{aligned}f \longmapsto \Theta(v)(t) &= \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v(s)}{s}\,ds\\&\quad-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v(s)}{s}\,ds.\end{aligned}$$

Observe that $\|h_n(t)-h_*(t)\| \to 0$ as $n \to \infty$; thus, it follows from Lemma 4.2 that $\Theta \circ S_F$ is a closed graph operator. Furthermore, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have $h_*(t) = 1/\Gamma(q)\int_1^t(\log(t/s))^{q-1}(v_*(s)/s)\,ds - \beta\sum_{j=1}^m\zeta_j 1/\Gamma(q)\int_1^{t_j}(\log(t_j/s))^{q-1}$ $(v_*(s)/s)\,ds$, for some $v_* \in S_{F,x_*}$.

Step 5. *We show that there exists an open set $U \subseteq C([1,T],\mathbb{R})$ with $x \notin \lambda\mathcal{F}(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$.* Let $\lambda \in (0,1)$ and $x \in \lambda\mathcal{F}(x)$. Then, there exists a $v \in L^1([1,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1,T]$, we have

$$x(t)=\lambda\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v(s)}{s}\,ds-\beta\lambda\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v(s)}{s}\,ds.$$

Similarly to the second step, we can obtain

$$|x(t)| \le A\Phi(\|x\|)\frac{1}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{p(s)}{s}\,ds,$$

which implies that

$$\frac{\|x\|}{A\Phi(\|x\|)(1/\Gamma(q))\int_1^T(\log T/s)^{q-1}(p(s)/s)\,ds}\le 1.$$

In view of (B_3), there exists a $\widehat{K}$ such that $\|x\|\neq\widehat{K}$. Let us set

$$U=\{x\in C(I,\mathbb{R}):\|x\|<\widehat{K}\}.$$

Note that the operator $\mathcal{F}:\overline{U}\to\mathcal{P}(C(I,\mathbb{R}))$ is a compact multi-valued map, usc with convex closed values. From the choice of U, there is no $x\in\partial U$ such that $x\in\lambda\mathcal{F}(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.3), we deduce that $\mathcal{F}$ has a fixed point $x\in\overline{U}$ which is a solution of problem (1.2). This completes the proof. □

4.2. The lower semicontinuous case. Next, we deal with the case where F is not necessarily convex valued. Our strategy to deal with this situation is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo **[10]** for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G:X\to\mathcal{P}(E)$ be a multi-valued operator with nonempty closed values. G is lower semi-continuous (lsc) if the set $\{y\in X:G(y)\cap B\neq\emptyset\}$ is open for any open set B in E. Let A be a subset of $[1,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[1,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([1,T],\mathbb{R})$ is decomposable if for all $u,v\in\mathcal{A}$ and measurable $\mathcal{J}\subset[1,T]=J$, the function $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 4.6. Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([1,T],\mathbb{R}))$ be a multi-valued operator. We say N has a property (BC) if N is lower semi-continuous (lsc) and has nonempty closed and decomposable values.

Let $F:[1,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multi-valued map with nonempty compact values. Define a multi-valued operator

$$\mathcal{F}:C([1,T]\times\mathbb{R})\longrightarrow\mathcal{P}(L^1([1,T],\mathbb{R}))$$

associated with F as

$$\mathcal{F}(x)=\{w\in L^1([1,T],\mathbb{R}):w(t)\in F(t,x(t)) \text{ for almost every } t\in[1,T]\},$$

which is called the *Nemytskii operator* associated with F.

Definition 4.7. Let $F:[1,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multi-valued function with nonempty compact values. We say F is of lower semi-continuous type (lsc type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 4.8 (**[16]**). *Let Y be a separable metric space, and let $N:Y\to\mathcal{P}(L^1([1,T],\mathbb{R}))$ be a multi-valued operator satisfying the property* (BC). *Then, N has a continuous selection, that is, there exists a continuous function* (*single-valued*) *$g:Y\to L^1([1,T],\mathbb{R})$ such that $g(x)\in N(x)$ for every $x\in Y$.*

Theorem 4.9. *Assume that* (B_2), (B_3) *and the following condition hold*:

(B_4) *$F:[1,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multi-valued map such that*

(a) *$(t,x)\mapsto F(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable*;

(b) *$x\mapsto F(t,x)$ is lower semicontinuous for each $t\in[1,T]$.*

Then, problem (1.2) *has at least one solution on* $[1,T]$.

Proof. It follows from (B_2) and (B_4) that F is of lsc type. Then, by Lemma 4.8, there exists a continuous function $f:C^1([1,T],\mathbb{R})\to L^1([1,T],\mathbb{R})$ such that $f(x)\in\mathcal{F}(x)$ for all $x\in C([1,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t)=f(x(t)) \\ x(1)+\displaystyle\sum_{j=1}^m \zeta_j\, x(t_j)=0. \end{cases} \qquad 1<t<T,\ 0<q\le 1, \tag{4.2}$$

Obviously, if $x\in C^1([1,T],\mathbb{R})$ is a solution of (4.2), then x is a solution to problem (1.2). In order to transform problem (4.2) into a fixed point problem, we define the operator $\overline{\mathcal{F}}$ as

$$\begin{aligned}\overline{\mathcal{F}}x(t) &= \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{f(x(s))}{s}\,ds \\ &\quad - \beta\sum_{j=1}^{m}\zeta_j\,\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{f(x(s))}{s}\,ds, \quad t\in J.\end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.5. Thus, we omit it. This completes the proof. □

4.3. The Lipschitz case. In this subsection, we prove the existence of solutions for problem (1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multi-valued maps due to Covitz and Nadler [**13**].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then, $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space, see [**24**].

Definition 4.10. A multi-valued operator $N : X\to\mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists a $\gamma>0$ such that

$$H_d(N(x),N(y)) \le \gamma\, d(x,y) \quad \text{for each } x,y\in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$.

Lemma 4.11 ([**13**]). *Let (X,d) be a complete metric space. If $N : X\to\mathcal{P}_{\rm cl}(X)$ is a contraction, then* $\mathrm{Fix}N\neq\emptyset$.

Theorem 4.12. *Assume that*:

(C_1) $F : [1,T]\times\mathbb{R}\to\mathcal{P}_{\rm cp}(\mathbb{R})$ *is such that* $F(\cdot,x) : [1,T]\to\mathcal{P}_{\rm cp}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$;

(C_2) *$H_d(F(t,x),F(t,\overline{x})) \le m(t)|x-\overline{x}|$ for almost all $t \in [1,T]$ and $x,\overline{x} \in \mathbb{R}$ with $m \in L^1([1,T],\mathbb{R}^+)$ and $d(0,F(t,0)) \le m(t)$ for almost all $t \in [1,T]$.*

Then, problem (1.2) *has at least one solution on* $[1,T]$ *if*

$$\delta := \frac{A}{\Gamma(q)} \int_1^T \left(\log \frac{t}{s}\right)^{q-1} \frac{m(s)}{s}\, ds < 1. \tag{4.3}$$

Proof. Note that the set $S_{F,x}$ is nonempty for each $x \in C([1,T],\mathbb{R})$ by assumption (C_1). Thus, F has a measurable selection, see **[12,** Theorem III.6]. Now, we show that the operator $\mathcal{F}$ defined by (4.1) satisfies the assumptions of Lemma 4.11. First, we show that $\mathcal{F}(x) \in \mathcal{P}_{\mathrm{cl}}((C[1,T],\mathbb{R}))$ for each $x \in C([1,T],\mathbb{R})$. Let $\{u_n\}_{n\ge 0} \in \mathcal{F}(x)$ be such that $u_n \to u$, $n \to \infty$, in $C([1,T],\mathbb{R})$. Then, $u \in C([1,T],\mathbb{R})$, and there exists a $v_n \in S_{F,x_n}$ such that, for each $t \in [1,T]$,

$$u_n(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v_n(s)}{s}\,ds-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v_n(s)}{s}\,ds.$$

Since F has compact values, we pass onto a subsequence, if necessary, to obtain that v_n converges to v in $L^1([1,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and, for each $t \in [1,T]$, we have

$$\begin{aligned} v_n(t) \longrightarrow v(t) = {} & \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{v(s)}{s}\, ds \\ & - \beta \sum_{j=1}^m \zeta_j \frac{1}{\Gamma(q)} \int_1^{t_j} \left(\log\frac{t_j}{s}\right)^{q-1} \frac{v(s)}{s}\, ds. \end{aligned}$$

Hence, $u \in \mathcal{F}(x)$.

Next, we show that there exists a $\delta < 1$, defined by (4.3), such that

$$H_d(\mathcal{F}(x),\mathcal{F}(\overline{x})) \le \delta\|x-\overline{x}\| \quad \text{for each } x,\ \overline{x} \in C^1([1,T],\mathbb{R}).$$

Let $x,\overline{x} \in C^1([1,T],\mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then, there exists a $v_1(t) \in F(t,x(t))$ such that, for each $t \in [1,T]$,

$$h_1(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v_1(s)}{s}\,ds-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v_1(s)}{s}\,ds.$$

From (C_2), we have

$$H_d(F(t,x),F(t,\overline{x}))\le m(t)|x(t)-\overline{x}(t)|.$$

Thus, there exists a $w\in F(t,\overline{x}(t))$ such that

$$|v_1(t)-w|\le m(t)|x(t)-\overline{x}(t)|,\quad t\in[1,T].$$

Define $U:[1,T]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le m(t)|x(t)-\overline{x}(t)|\}.$$

Since the multi-valued operator $U(t)\cap F(t,\overline{x}(t))$ is measurable, **[12,** Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for U. Therefore, $v_2(t)\in F(t,\overline{x}(t))$, and, for each $t\in[1,T]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\overline{x}(t)|$.

For each $t\in[1,T]$, we define

$$h_2(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{v_2(s)}{s}ds-\beta\sum_{j=1}^m\zeta_j\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{v_2(s)}{s}ds.$$

Then,

$$\begin{aligned}|h_1(t)-h_2(t)|&\le\frac{1}{\Gamma(q)}\int_1^T\left(\log\frac{t}{s}\right)^{q-1}\frac{|v_1(s)-v_2(s)|}{s}\,ds\\&\quad+|\beta|\sum_{j=1}^m|\zeta_j|\frac{1}{\Gamma(q)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{q-1}\frac{|v_1(s)-v_2(s)|}{s}\,ds\\&\le\frac{A\|x-\overline{x}\|}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{m(s)}{s}\,ds.\end{aligned}$$

Hence,

$$\|h_1-h_2\|\le\frac{A\|x-\overline{x}\|}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{m(s)}{s}\,ds.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\mathcal{F}(x),\mathcal{F}(\overline{x}))\le\frac{A\|x-\overline{x}\|}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{m(s)}{s}\,ds.$$

Thus, $\mathcal{F}$ is a contraction. Therefore, it follows from Lemma 4.11 that $\mathcal{F}$ has a fixed point x which is a solution of (1.2). This completes the proof. □

5. Examples.

(a) Consider the following Hadamard type nonlocal problem:

$$
\begin{cases} {}^H D^{3/4} x(t) = f(t, x(t)) & 1 < t < e, \\ x(1) + (2/3)x(6/5) + (2/3)^2 x(8/5) + (2/3)^3 x(2) + (2/3)^4 x(12/5) = 0. \end{cases} \tag{5.1}
$$

Here, $q = 3/4$, $T = e$, $\zeta_1 = 2/3$, $\zeta_2 = 4/9$, $\zeta_3 = 8/27$, $\zeta_4 = 16/81$, $t_1 = 6/5$, $t_2 = 8/5$, $t_3 = 2$ and $t_4 = 12/5$. With the given data, the values of β and A defined by (2.4) are found to be $\beta = 81/211$ and $A = 341/211$. In order to illustrate Theorem 3.3, we take

$$
f(t, x) = \frac{1}{\sqrt{t^2 + 24}} \frac{|x|}{(1 + |x|)} + \frac{1}{t + 9} \tan^{-1} x(t) + \log t. \tag{5.2}
$$

Clearly, $L = 3/10$ as $|f(t, x) - f(t, y)| \leq (3/10)|x - y|$ and $\Gamma(q + 1)/(A(\log T)^q) = 211\Gamma(7/4)/341 \approx 0.56868678$. With $L < \Gamma(q + 1)/(A(\log T)^q)$, all the conditions of Theorem 3.3 are satisfied. Therefore, by the conclusion of Theorem 3.3, problem (5.1) with $f(t, x)$ given by (5.2) has a unique solution on $[1, e]$.

In order to demonstrate the application of Theorem 3.4, we choose

$$
f(t, x) = (\log e/t)^2 (1 + \sin x(t)), \tag{5.3}
$$

and note that $|f(t, x)| \leq p(t)\Omega(\|x\|)$ with $p(t) = (\log e/t)^2$ and $\Omega(\|x\|) = 1 + \|x\|$. Condition (A_5) is satisfied for $K > K_1 \approx 0.92150198$. Clearly, the hypothesis of Theorem 3.4 is satisfied, and consequently, problem (5.1) with $f(t, x)$ given by (5.3) has at least one solution on $[1, e]$.

(b) Consider the nonlocal multi-valued (inclusion) problem given by

$$
\begin{cases} {}^H D^{3/4} x(t) \in F(t, x(t)) & 1 < t < e, \\ x(1) + (2/3)x(6/5) + (2/3)^2 x(8/5) + (2/3)^3 x(2) + (2/3)^4 x(12/5) = 0. \end{cases} \tag{5.4}
$$

For the illustration of Theorem 4.5, consider

$$F(t,x(t)) = [\log(e/t)x(t) + 1/10, (\log(e/t))^3(\sin x(t) + 2)] \\ \leq (\log(e/t))^3(\|x\| + 2). \tag{5.5}$$

Letting $p(t) = (\log(e/t))^3, \Phi(\|x\|) = (\|x\| + 2)$, condition ($B_3$) is satisfied with $\widehat{K} > \widehat{K}_1 \approx 1.08493176$. Since all of the conditions of Theorem 4.5 are satisfied, there exists at least one solution for problem (5.4) with F given by (5.5) on $[1, e]$.

In order to explain Theorem 4.12, we take

$$F(t,x(t)) = [0,\ (5/2)(\log(e/t))^4 \tan^{-1}(x(t) + 1/4]. \tag{5.6}$$

Fixing $m(t) = (5/2)(\log(e/t))^4$, condition ($C_2$) is satisfied. From condition (4.3), $\delta \approx 0.69412000 < 1$. In consequence, the conclusion of Theorem 4.5 applies, and hence, problem (5.4) with F given by (5.6) has at least one solution on $[1, e]$.

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