EXISTENCE OF ROTATING-PERIODIC SOLUTIONS
FOR NONLINEAR SYSTEMS VIA
UPPER AND LOWER SOLUTIONS

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ABSTRACT. This paper concerns the existence of affine-periodic solutions for nonlinear systems with certain affine-periodic symmetry. The existence result is actually proved based on the existence of upper and lower solutions and the conditions on them. Some applications are also given.

1. Introduction. We consider the following system

$$(1.1) \qquad x' = f(t, x), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t},$$

where $f : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$. Throughout the paper, we assume:

$$(\mathrm{H1}) \qquad f : \mathbb{R}^1 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is continuous.}$$

When system (1.1) is T-periodic, i.e., $f(t + T, \cdot) = f(t, \cdot)$ for all t, the exploration of the existence of T-periodic solutions in qualitative theory is a standard topic.

In the present paper, we explore the existence of affine-periodic solutions when system (1.1) possesses affine-periodicity. We introduce more precise statements as follows.

Definition 1.1. Let $Q \in \mathrm{GL}(n)$, i.e., Q is a nonsingular matrix of order $n \times n$. System (1.1) is said to be (T, Q)-*affine-periodic* if

$$f(t + T, x) = Q f(t, Q^{-1} x) \quad \text{for all } t \in \mathbb{R}^1, \ x \in \mathbb{R}^n.$$

System (1.1) is called *Q-rotating-periodic* if $Q \in \mathrm{O}(n)$, i.e., Q is an orthogonal matrix.

Definition 1.2. A map $x : \mathbb{R}^1 \to \mathbb{R}^n$ is said to be a (T, Q)-affine-periodic solution if it is a solution of (1.1) on $\mathbb{R}^1$ and

$$x(t + T) = Qx(t) \quad \text{for all } t.$$

When $Q \in \mathrm{O}(n)$, the solution $x(t)$ is also called the *Q-rotating-periodic solution*.

Obviously, a (T, Q)-affine-periodic solution $x(t)$ corresponds, respectively, to a T-periodic, T-anti-periodic, harmonic or quasi-periodic solution when $Q = \mathrm{id}$ (the identical matrix), $-\mathrm{id}$, $Q^N = \mathrm{id}$ for some positive integer N or $Q \in \mathrm{O}(n)$.

As is well known, the upper and lower solutions method is an effective tool in the study of periodic solutions. For a survey, see, for example, **[1]**–**[3]**, **[15]**. The main idea is to reduce the existence of periodic solutions from the existence of upper and lower solutions. The main aim of this paper is to establish an existence theorem of Q-rotating-periodic solutions by employing the upper and lower solutions.

First, we introduce some notation. Let $x, y \in \mathbb{R}^n$. We denote

$$x \leq y \, (\text{or } x < y) \Longleftrightarrow x_i \leq y_i \, (x_i < y_i) \quad \text{for all } i,$$

where x_i and y_i are components of x and y, respectively.

Consider the system

(1.2) $$x' = g(t, x),$$

where $g : \mathbb{R}^1 \times \mathbb{R}^n$ is continuous.

The following definitions are standard.

Definition 1.3. The C^1 functions $\beta, \alpha : \mathbb{R}^1 \to \mathbb{R}^n$ are said to be upper and lower solutions of (1.2), respectively, if

(i) $\alpha(t) \leq \beta(t)$ for all t;

(ii) $\alpha'(t) \leq g(t, \alpha(t))$, $\beta'(t) \geq g(t, \beta(t))$ for all t.

Definition 1.4. Let

$$\Omega(t) = \{p \in \mathbb{R}^n : \alpha(t) \le p \le \beta(t)\} \quad \text{for all } t.$$

A function $g : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *Kemke-type function* relative to $\Omega(t)$ if, for all $t \in \mathbb{R}$ and $i = 1, \dots, n$,

$$g_i(t, x) \le g_i(t, y) \quad \text{for all } x \le y, \ x_i = y_i \text{ and } x, y \in \Omega(t).$$

Now, we are in a position to state our main result:

Theorem 1.5. *Let* $Q \in O(n)$, *and let system* (1.1) *be* Q-*rotating-periodic. Assume that*:

(i) *there exist* C^1 *upper and lower solutions* β *and* α *of* (1.2) *for* $g = f$ *such that* $\Omega(t)$ *is bounded on* $\mathbb{R}^1$ *and, for some constant* l_0, $\sigma > 0$ *and a* Q-*rotation-periodic function* $a(t)$ *with*

$$\alpha(t) \le a(t) - \sigma < a(t) < a(t) + \sigma \le \beta(t),$$
$$\beta'(t) > -l_0(\beta(t) - a(t)),$$
$$\alpha'(t) < l_0(a(t) - \alpha(t)) \quad \text{for all } t.$$

(ii) *The function* $f(t, x)$ *is of Kemke type relative to* $\Omega(t)$.

Then (1.1) *admits a* Q-*rotating-periodic solution* $x_*(t)$ *with* $\alpha(t) \le x_*(t) \le \beta(t)$ *for all* t.

Remark 1.6. In Theorem 1.5, we do not add any "periodicity" on upper and lower solutions β and α other than boundedness. This is also an explicit improvement to some classical results for periodic solutions when $Q = \text{id}$.

Corollary 1.7. *Let* $Q \in O(n)$, *and let system* (1.1) *be* Q-*rotating-periodic. Assume that*:

(i) *there exist* C^1 *upper and lower solutions* β *and* α *of* (1.2) *such that* $\Omega(t)$ *is bounded on* $\mathbb{R}^1$ *and, for some constant* l_0, $\sigma > 0$ *and a*

Q-rotation-periodic function $a(t)$ *with*

$$\alpha(t) \leq a(t) - \sigma < a(t) < a(t) + \sigma \leq \beta(t),$$
$$\beta'(t) > -l_0(\beta(t) - a(t)),$$
$$\alpha'(t) < l_0(a(t) - \alpha(t)) \quad for \ all \ t.$$

(ii) *The function* $g(t, x)$ *is of Kemke type relative to* $\Omega(t)$.

(iii) *The following hold*:

$$f_i(t, [\alpha_i(t)]) \geq g_i(t, [\alpha_i(t)]), \qquad f_i(t, [\beta_i(t)]) \leq g_i(t, [\beta_i(t)]),$$
$$for \ all \ [\alpha_i(t)] = (x_1, \dots, x_{i-1}, \alpha_i(t), x_{i+1}, \dots, x_n)^\top,$$
$$[\beta_i(t)] = (x_1, \dots, x_{i-1}, \beta_i(t), x_{i+1}, \dots, x_n)^\top \in \Omega(t),$$
$$t \in \mathbb{R}^1, \quad i = 1, \dots, n.$$

Then, (1.1) *admits a Q-rotating-periodic solution* $x_*(t)$ *with* $\alpha(t) \leq x_*(t) \leq \beta(t)$ *for all* t.

Here, $x^\top$ denotes the transpose of x.

Remark 1.8. Corollary 1.7 (iii) is more flexible in applications, which is also an improvement to classical results when $Q = \mathrm{id}$.

Theorem 1.5 extends some classical results on periodic or anti-periodic, when $Q = \mathrm{id}$ or $-\mathrm{id}$, see [1]–[3], [15]. In particular, it extends them to harmonic or quasi-periodic cases.

It should be pointed that some results have been obtained on the existence of affine-periodic solutions, see [5, 13, 14, 16], which are based upon topological degree theory and some asymptotical fixed point theorems. Application of these results, including ours, can lead to the existence of quasi-periodic solutions with large amplitude under certain symmetry.

The plan of the paper is as follows. In Section 2, we first give a Massera-type criterion on (T, Q)-affine-periodic solutions by using Brouwer's fixed point theorem. The advantage of the argument is that it can tell us where the initial value of a (T, Q)-affine-periodic solution stays. Then, combining Massera's criterion with the topological degree

theory, we give the proof of Theorem 1.5. Finally, in Section 3, we give some applications.

2. Massera's criterion and the proof of Theorem 1.5. Before proving Theorem 1.5, we first give a Massera-type criterion on the existence of (T, Q)-affine-periodic solutions for the following (T, Q)-affine-periodic linear system:

$$(2.1) \qquad\qquad x' = A(t)x + g(t),$$

i.e., $A : \mathbb{R}^1 \to \mathbb{R}^{n \times n}$ and $g : \mathbb{R}^1 \to \mathbb{R}^n$ are continuous and satisfy

$$(2.2) \qquad A(t + T) = QA(t)Q^{-1}, \qquad g(t + T) = Qg(t).$$

Here, $Q \in \mathrm{GL}(n)$.

Definition 2.1. A solution $x(t)$ of system (1.1) is said to be (T, Q)-affine-bounded in forward if $\{Q^{-m}x(t + mT)\}_{m=0}^\infty$ is bounded in $\mathbb{R}_+^1 = [0, \infty)$.

This yields:

Theorem 2.2 (Massera-type criterion). *Assume that $x_0(t)$ is a Q-affine-bounded solution of (T, Q)-affine-periodic system (2.1) in forward. Then, (2.1) has a (T, Q)-affine-periodic solution $x_*(t)$ with $x_*(0) \in \overline{\mathrm{co}}\{Q^{-m}x_0(mT)\}$, where $\overline{\mathrm{co}}$ denotes the usual convex closure of the set.*

Remark 2.3. When $Q = \mathrm{id}$, Theorem 2.2 is the well-known Massera's criterion [**7**]. For further developments, see [**4, 6**], [**8**]–[**12**].

Remark 2.4. The following argument is different from some standard proofs of Massera's criterion and is inherently useful since it yields a (T, Q)-affine-periodic-solution $x_*(t)$ with the initial value $x_*(0) \in \overline{\mathrm{co}}\{Q^{-m}x_0(mT)\}$.

Proof of Theorem 2.2. Set

$$S_0 = \{Q^{-m}x_0(mT)\}_{m=0}^\infty, \qquad S = \mathrm{co}S_0,$$

where $\mathrm{co}S_0$ denotes the convex hull of S_0. Define a map

$$P : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

by

$$P(p) = Q^{-1}x(T, p) \quad \text{for all } p \in \mathbb{R}^n,$$

where $x(t, p)$ denotes the solution of (2.1) with the initial value condition $x(0) = p$. We claim that $P : \overline{S} \to \overline{S}$, where $\overline{S}$ denotes the closure of S. First, we prove

$$P : S_0 \longrightarrow S_0.$$

Indeed, since $x(t, x_0(0)) = x_0(t)$, by uniqueness, we have

$$P(x_0(0)) = Q^{-1}x(T, x_0(0)) = Q^{-1}x_0(T) \in S_0.$$

Note that

$$
\begin{aligned}
\frac{\mathrm{d}x(t+T, p)}{\mathrm{d}t} &= \frac{\mathrm{d}x(t+T, p)}{\mathrm{d}(t+T)} \\
&= A(t+T)x(t+T, p) + g(t+T) \\
&= QA(t)Q^{-1}x(t+T, p) + Qg(t) \\
&= Q(A(t)Q^{-1}x(t+T, p) + g(t)),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}x_0(t+mT)}{\mathrm{d}t} &= \frac{\mathrm{d}x_0(t+mT)}{\mathrm{d}(t+mT)} \\
&= A(t+mT)x_0(t+mT) + g(t+mT) \\
&= QA(t+(m-1)T)Q^{-1}x_0(t+mT) + Qg(t+(m-1)T) \\
&= \cdots \\
&= Q^m A(t)Q^{-m}x_0(t+mT) + Q^m g(t) \\
&= Q^m(A(t)Q^{-m}x_0(t+mT) + g(t)) \quad \text{for all } m \geq 1.
\end{aligned}
$$

Thereby, $Q^{-1}x(t+T, p)$ and $Q^{-m}x_0(t+mT)$, $m = 1, 2, \ldots$, are all solutions of (2.1).

Since

$$
\begin{aligned}
Q^{-1}x(t+T, Q^{-(m-1)}x_0((m-1)T))\big|_{t=-T} &= Q^{-m}x_0((m-1)T) \\
&= Q^{-m}x_0(t+mT)\big|_{t=-T},
\end{aligned}
$$

by uniqueness, we obtain

$$Q^{-1}x(t+T, Q^{-(m-1)}x_0((m-1)T)) \equiv Q^{-m}x_0(t+mT) \quad \text{for all } t$$
$$\implies P(Q^{-(m-1)}x_0((m-1)T))$$
$$= Q^{-1}x(T, Q^{-(m-1)}x_0((m-1)T)) = Q^{-m}x_0(mT) \in S_0$$
$$\implies P : S_0 \longrightarrow S_0.$$

Note that

$$x(T,p) = U(T)\left(p + \int_0^T U^{-1}(s)g(s)\,\mathrm{ds}\right),$$

$$x(T,q) = U(T)\left(q + \int_0^T U^{-1}(s)g(s)\,\mathrm{ds}\right)$$

$$\implies x\left(T, \sum_{i=1}^k \lambda_i p_i\right) = U(T)\left(\sum_{i=1}^k \lambda_i p_i + \int_0^T U^{-1}(s)g(s)\,\mathrm{ds}\right)$$

$$= \sum_{i=1}^k \lambda_i U(T)\left(p_i + \int_0^T U^{-1}(s)g(s)\,\mathrm{ds}\right)$$

$$= \sum_{i=1}^k \lambda_i x(T, p_i)$$

$$\text{for all } \lambda_i \in [0,1], \ \sum_{i=1}^k \lambda_i = 1, p, q \in \mathbb{R}^n,$$

where $U(t)$ denotes the fundamental solution matrix of (2.1) with the initial value $U(0) = \mathrm{id}$. Also,

$$p_0 \in S$$

$$\iff \text{ there exists } p_i \in S_0, \ \lambda_i \in [0,1], \ \sum_{i=1}^k \lambda_i = 1 \text{ such that } p_0 = \sum_{i=1}^k \lambda_i p_i$$

$$\implies x(T, p_0) = x\left(T, \sum_{i=1}^k \lambda_i p_i\right) = \sum_{i=1}^k \lambda_i x(T, p_i) \in S$$

$$\implies P : S \longrightarrow S.$$

By continuity, we have that $P : \overline{S} \to \overline{S}$, as desired. Now, by Brouwer's fixed point theorem, P has a fixed point $p_* \in \overline{S}$, i.e., $Q^{-1}x(T, p_*) = p_*$.

Again, by the uniqueness of the solution to the initial value problem, we obtain

$$Q^{-1}x(t + T, p_*) = x(t, p_*) \text{ for all } t$$
$$\iff x(t + T, p_*) = Qx(t, p_*) \quad \text{for all } t,$$

which shows that $x(t, p_*)$ is a (T, Q)-affine-periodic solution of (2.1). The proof is complete. $\square$

Proof of Theorem 1.5. Consider the auxiliary system

$$(2.3) \qquad\qquad x' = -lx + F(t, x, \lambda),$$

where $l > l_0$ and

$$F(t, x, \lambda) = \lambda lx + (1 - \lambda)la(t) + \lambda f(t, x), \quad \lambda \in [0, 1].$$

Let

$$C = \{x : \mathbb{R}^1 \to \mathbb{R}^n : x(t) \text{ is continuous and } Q\text{-rotating-periodic}\},$$

with the usual norm

$$\|x\| = \sup_{\mathbb{R}^1} |x(t)|,$$

where $|\cdot|$ denotes the usual Euclidean norm. Note that

$$|x(t + T)| = |Q^{-1}x(t + T)| = |x(t)|.$$

Therefore, each $x \in C$ is bounded. It is obvious that C is a Banach space.

For each $\varphi \in C$, consider the equation

$$(2.4) \qquad\qquad x' = -lx + F(t, \varphi(t), \lambda).$$

Let $x_0(t)$ be the solution of (2.4) with initial value $x(0) = 0$. Then,

$$x_0(t) = e^{-lt} \int_0^t e^{ls} F(s, \varphi(s), \lambda) \, \mathrm{d}s$$
$$\implies |x_0(t)| \leq 2e^{-lt}(e^{lt} - 1)L$$
$$\text{for all } t \geq 0 \text{ for some } L > 0$$
$$\implies |Q^{-m}x_0(t + mT)| = |x_0(t + mT)|$$
$$\leq 2e^{-l(t+mT)}(e^{l(t+mT)} - 1)L \leq 2L \quad \text{for all } t \geq 0.$$

According to Theorem 2.2, (2.4) has Q-rotating-periodic solutions. Let $x(t)$ be a Q-rotating-periodic solution of (2.4). Then,

$$x(t) = e^{-lt}\left(x(0) + \int_0^t e^{ls}F(s,\varphi(s),\lambda)\right)ds, \qquad x(T) = Qx(0)$$

$$\implies (e^{-lT}\mathrm{id} - Q)x(0) = -e^{-lT}\int_0^T e^{ls}F(s,\varphi(s),\lambda)\,ds$$

$$\implies x(0) = -(e^{-lT}\mathrm{id} - Q)^{-1}e^{-lT}\int_0^T e^{ls}F(s,\varphi(s),\lambda)\,ds,$$

which implies

$$(2.5)\quad x(t) = e^{-lt}\left(-(e^{-lT}\mathrm{id} - Q)^{-1}e^{-lT}\int_0^T e^{ls}F(s,\varphi(s),\lambda)\,ds\right.$$
$$\left. + \int_0^t e^{ls}F(s,\varphi(s),\lambda)\,ds\right).$$

Consequently, $x(t)$ is unique. We define a map

$$P : C \longrightarrow C$$

by

$$P_\lambda(\varphi)(t) = x_\varphi(t) \quad \text{for all } \varphi \in C,$$

where $x_\varphi(t)$ is the unique Q-rotating-periodic solution of (2.4) with the form (2.5). By using a standard argument, P_λ is completely continuous. Set

$$D = \{\varphi \in C : \alpha(t) < \varphi(t) < \beta(t) \text{ for all } t\}.$$

By assumption (i), D is open and bounded in C.

Consider the homotopy

$$H(\varphi,\lambda) = \varphi - P_\lambda(\varphi) \quad \text{on } D \times [0,1].$$

If, for $\lambda = 1$, P_1 has a fixed point x_* on $\overline{D}$, then $x_*(t)$ is a Q-rotating-periodic solution of (1.1) with

$$\alpha(t) \le x_*(t) \le \beta(t) \quad \text{for all } t,$$

which proves the theorem. Hence, we assume that

$$0 \notin (\mathrm{id} - P_1)(\partial D).$$

Thus, it follows from the complete continuity of $H(\varphi, \lambda)$ that there exist $\triangle, \sigma_1 > 0$ such that

$$\alpha(t) + \sigma_1 \leq x(t) \leq \beta(t) - \sigma_1 \quad \text{for all } t$$

for every possible Q-rotating-periodic solution $x(t)$ lying in $\overline{D}$ when $\lambda \in [\triangle, 1]$.

Now, we are able to prove that

$$0 \notin H(\partial D \times [0, \triangle]).$$

If this is false, then there exist solutions $x_k(t)$ of (2.3) with $x_k \in \overline{D}$, $t_k \in \mathbb{R}^1$, $\varepsilon_k \searrow 0$ and $i \in \{1, \ldots, n\}$ such that

$$(x_k)_i(t_k) - \beta_i(t_k) = -\varepsilon_k, \qquad (x_k)'_i(t_k) - \beta'_i(t_k) \longrightarrow 0,$$

or

$$(x_k)_i(t_k) - \alpha_i(t_k) = \varepsilon_k, \qquad (x_k)'_i(t_k) - \alpha'_i(t_k) \longrightarrow 0.$$

We only discuss the former since the latter is similar. Since β is an upper solution of (1.1) and f is of Kamke type, by Definition 1.3 (i) and (ii), we have

$$
\begin{aligned}
(x_k)'_i(t_k) &= -(1-\lambda)l((x_k)_i(t_k) - a_i(t_k)) + \lambda f_i(t_k, x_k(t_k)) \\
&= -(1-\lambda)l_0((x_k)_i(t_k) - a_i(t_k)) + \lambda f_i(t_k, x_k(t_k)) \\
&\quad - (1-\lambda)(l-l_0)((x_k)_i(t_k) - a_i(t_k)) \\
&= -(1-\lambda)l_0(\beta_i(t_k) - a_i(t_k)) + \lambda f_i(t_k, [\beta_i(t_k)]) \\
&\quad + ((1-\lambda)l_0(\beta_i(t_k) - x_i(t_k)) \\
&\qquad + \lambda(f_i(t_k, x_k(t_k)) - f_i(t_k, [\beta_i(t_k)])) \\
&\quad - (1-\lambda)(l-l_0)((x_k)_i(t_k) - a_i(t_k)) \\
&\leq (1-\lambda)\beta'_i(t_k) + \lambda\beta'_i(t_k) + (1-\lambda)l_0\varepsilon_k \\
&\quad - (1-\lambda)(l-l_0)((x_k)_i(t_k) - a_i(t_k)) \\
&\leq \beta'_i(t_k) + O(\varepsilon_k) - (1-\lambda)(l-l_0)(\beta_i(t_k) - a_i(t_k)) \\
&\leq -(1-\triangle)(l-l_0)\sigma + \beta'_i(x_k) + O(\varepsilon_k).
\end{aligned}
$$

(2.6)

Consequently, for k large enough,

$$0 \longleftarrow (x_*)'_{i_k}(t_k) - \beta'_{i_k}(t_k) < -\frac{1}{2}(1-\triangle)(l-l_0)\sigma,$$

a contradiction. According to homotopy invariance, we have

$$\deg(H(\cdot,1),D,0) = \deg(H(\cdot,0),D,0).$$

Note that P_0 is a constant map. Thereby, if $P_0(0) \in D$, then

$$\deg(H(\cdot,0),D,0) = 1.$$

In fact, let $x(t)$ satisfy $P_0(x) = x$, i.e., $x(t)$ is a Q-rotating-periodic solution of the equation

$$x' = -l(x - a(t)).$$

Then, we claim that $x \in D$. If this is false, there exist $t_k \in \mathbb{R}^1$, $\varepsilon_k \searrow 0$ and $i \in \{1,\ldots,n\}$ such that

$$x_i(t_k) - \beta_i(t_k) \longrightarrow \delta - \varepsilon_k, \qquad x_i'(t_k) - \beta_i'(t_k) \longrightarrow 0,$$

or

$$x_i(t_k) - \alpha_i(t_k) \longrightarrow -\delta + \varepsilon_k, \qquad x_i'(t_k) - \alpha_i'(t_k) \longrightarrow 0,$$

where

$$\delta = \sup_{\mathbb{R}^1}\{x_i(t) - \beta_i(t)\} \geq 0,$$

or

$$\delta = \sup_{\mathbb{R}^1}\{\alpha_i(t) - x_i(t)\} \geq 0.$$

We assume, without loss of generality, that the former holds. Then, it follows from assumption (i) and the definitions of $\{t_k\}$ and δ that, for k large enough,

$$
\begin{aligned}
x_i'(t_k) &= -l(x_i(t_k) - a_i(t_k)) \\
&= -l_0(\beta_i(t_k) - a_i(t_k)) + l_0(\beta_i(t_k) - x_i(t_k)) \\
&\quad - (l - l_0)(x_i(t_k) - a_i(t_k)) \\
&< \beta_i'(t_k) + l_0(-\delta + \varepsilon_k) - (l - l_0)(\beta_i(t_k) + \delta - \varepsilon_k - a_i(t_k)) \\
&\leq \beta_i'(t_k) + l(-\delta + \varepsilon_k) - (l - l_0)(\beta_i(t_k) - a_i(t_k)) \\
&\leq \beta_i'(t_k) + l(-\delta + \varepsilon_k) - (l - l_0)\sigma \\
&< \beta_i'(t_k) + l_0(-\delta + \varepsilon_k) - (l - l_0)\sigma \\
&\Longrightarrow x_i'(t_k) - \beta_i'(t_k) \leq l_0\varepsilon_k - (l - l_0)\sigma < -\tfrac{1}{2}(l - l_0)\sigma,
\end{aligned}
$$

a contradiction. Thus,

$$\deg(H(\cdot, 1), D, 0) = \deg(H(\cdot, 0), D, 0) = 1.$$

Hence, P_1 has a fixed point $x_* \in D$. Then, $x_*(t)$ is the desired Q-rotating-periodic solution. The proof of Theorem 1.5 is complete. $\square$

Proof of Corollary 1.7. It suffices to modify (2.6) in the proof of Theorem 1.5 as follows:

$$
\begin{aligned}
(x_k)_i'(t_k) &= -(1-\lambda)l((x_k)_i(t_k) - a_i(t_k)) + \lambda f_i(t_k, x_k(t_k)) \\
&= -(1-\lambda)l_0(\beta_i(t_k) - a_i(t_k)) + \lambda f_i(t_k, [\beta_i(t_k)]) \\
&\quad + ((1-\lambda)l_0(\beta_i(t_k) - x_i(t_k)) + \lambda(f_i(t_k, x_k(t_k)) - f_i(t_k, [\beta_i(t_k)]))) \\
&\quad - (1-\lambda)(l - l_0)((x_k)_i(t_k) - a_i(t_k)) \\
&\le -(1-\triangle)(l - l_0)\sigma - (1 - \lambda)l_0(\beta_i(t_k) - a_i(t_k)) \\
&\quad + \lambda f_i(t_k, [\beta_i(x_k)]) + O(\varepsilon_k) \\
&\le -(1-\triangle)(l - l_0)\sigma + \beta_i'(x_k) + O(\varepsilon_k). \hspace{2cm} \square
\end{aligned}
$$

3. Examples.

Example 3.1. Consider the equation

$$(3.1) \hspace{4cm} x' + 2x = e^{-t}.$$

Set $f(t, x) = -2x + e^{-t}$. The general solution of (3.1) is

$$x(t) = e^{-2t}c + e^{-t} \quad (c \text{ is any constant}).$$

Obviously, for a given $\tau > 0$,

$$f(t + \tau, x) = e^{-\tau}f(t, e^{\tau}x),$$

and any solution $x(t)$ satisfies

$$
\begin{aligned}
|(e^{-\tau})^{-m}x(t + m\tau)| &= |e^{m\tau}e^{-2(t+m\tau)}c + e^{m\tau}e^{-(t+m\tau)}| \\
&\le e^{-(2t+m\tau)}|c| + 1.
\end{aligned}
$$

Hence, by Theorem 2.2, (3.1) has an $e^{-\tau}$-periodic solution. This solution is only $x(t) = e^{-t}$ and different from the usual periodic solutions.

Example 3.2. Consider the system

(3.2) $\qquad x' = -A(t) \operatorname{diag}(|x_i|^{2\sigma})x + e(t) = f(t, x),$

where $\sigma \geq 0$, $A : \mathbb{R}^1 \times \mathbb{R}^{n \times n}$ and $e : \mathbb{R}^1 \to \mathbb{R}^n$ are continuous and

$$A(t + T) = A(t), \qquad e(t + T) = -e(t);$$

$$A(t) = (a_{ij}(t)), \qquad -a_{ij}(t) \geq 0, \quad i \neq j, \quad \sum_{j=1}^{n} a_{ij}(t) > 0.$$

It is easily seen that $f(t, x)$ is of Kamke type for $x \in \mathbb{R}^n$, and, for $Q = -\operatorname{id}$,

$$f(t + T, x) = Qf(t, Q^{-1}x).$$

Set

$$\alpha(t) = -\lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \beta(t) = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

for all t, where $\lambda > 0$ is a constant such that

$$\sum_{j=1}^{n} a_{ii}(t)\lambda^{2\sigma+1} > \max_{\mathbb{R}^1} |e_i(t)|, \quad i = 1, \ldots, n.$$

Note that

$$f(t, \alpha) = -A(t) \operatorname{diag}(\lambda^{2\sigma})(-\lambda) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + e(t)$$

$$= \lambda^{2\sigma+1} \begin{pmatrix} \sum_{j=1}^{n} a_{1j}(t) \\ \vdots \\ \sum_{j=1}^{n} a_{nj}(t) \end{pmatrix} + e(t) > 0 = \alpha'.$$

Similarly, $\beta' > f(t, \beta)$. Thus,

$$\alpha' < f(t, \alpha), \qquad \beta' > f(t, \beta),$$

and by Theorem 1.5, (3.2) has an anti-periodic solution.

Example 3.3. Consider system (3.2) with

$$A(t) = \text{diag}(A_p(t), A_a(t)),$$
$$e(t) = (e_p(t), e_a(t))^\top,$$
$$Q = \text{diag}(\text{id}|_{n_1}, -\text{id}|_{n_2}),$$

where $\text{id}|_{n_i}$ denotes the identical matrix of order $n_i \times n_i$,

$$e_p(t+T) = e_p(t) \in \mathbb{R}^{n_1}, \qquad e_a(t+T) = -e_a(t) \in \mathbb{R}^{n_2},$$

$$A(t+T) = A(t), \qquad -a_{ij}(t) \geq 0, \quad i \neq j, \quad \sum_{j=1}^{n} a_{ij}(t) > 0.$$

By similar arguments as Example 3.2, (3.2) has Q-rotating-periodic solutions. Now, a Q-rotating-periodic solution possesses former n_1 periodic components and the latter n_2 anti-periodic components.

Example 3.4. Consider the system

$$(3.3) \qquad x' = -|x|^2 x + \begin{pmatrix} \sin \omega_1 t \\ \cos \omega_1 t \\ \vdots \\ \sin \omega_m t \\ \cos \omega_m t \end{pmatrix} \equiv f(t, x), \quad 2m = n,$$

where $\omega_1, \ldots, \omega_m$ are positive constants. Clearly, when $\omega = (\omega_1, \ldots, \omega_m)$ satisfies

$$k_1 \omega_1 + k_2 \omega_2 + \cdots + k_m \omega_m \neq 0 \quad \text{for all } k \in \mathbb{Z}^m \backslash \{0\},$$

$e(t) = (\sin \omega_1 t, \cos \omega_1 t, \ldots, \sin \omega_m t, \cos \omega_m t)^\top$ is a quasi-periodic function with the frequency ω. We fix $T > 0$ and choose

$$Q = \text{diag}\left(\begin{pmatrix} \cos \omega_1 T & \sin \omega_1 T \\ -\sin \omega_1 T & \cos \omega_1 T \end{pmatrix}, \ldots, \begin{pmatrix} \cos \omega_m T & \sin \omega_m T \\ -\sin \omega_m T & \cos \omega_m T \end{pmatrix} \right).$$

Then,

$$e(t+T) = Qe(t) \quad \text{for all } t \implies f(t+T, x) = Qf(t, Q^{-1}x).$$

Set

$$\alpha(t) = -\lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \beta(t) \equiv -\alpha(t) \quad \text{for all } t,$$

where $\lambda > 1$. Consider the system

$$(3.4) \qquad x' = -\text{diag}(|x_i|^2)x + \begin{pmatrix} \sin \omega_1 t \\ \cos \omega_1 t \\ \vdots \\ \sin \omega_m t \\ \cos \omega_m t \end{pmatrix} \equiv g(t, x).$$

Then, $g(t, x)$ is of Kamke type on $\Omega(t)$, and

$$\alpha' = 0 < f(t, \alpha), \qquad \beta' = 0 > f(t, \beta).$$

Hence, Corollary 1.7 (i), (ii) are satisfied. Note that

$$\begin{aligned} f_i(t, [\alpha_i(t)]) &= -|[\alpha_i(t)]|^2 \alpha_i(t) + \sin \omega_j t \quad (\text{or } \cos \omega_j t) \\ &= \lambda |[\alpha_i(t)]|^2 + \sin \omega_j t \quad (\text{or } \cos \omega_j t) \\ &\geq \lambda |\alpha_i(t)|^2 + \sin \omega_j t \quad (\text{or } \cos \omega_j t) = g_i(t, [\alpha_i(t)]), \\ f_i(t, [\beta_i(t)]) &= -|[\beta_i(t)]|^2 \beta_i(t) + \sin \omega_j t \quad (\text{or } \cos \omega_j t) \\ &\leq -|\beta_i(t)|^2 \beta_i(t) + \sin \omega_j t \quad (\text{or } \cos \omega_j t) = g_i(t, [\beta_i(t)]). \end{aligned}$$

Thus, Corollary 1.7 (iii) also holds. Corollary 1.7 implies the existence of quasi-periodic solution for (3.3) with the frequency ω.

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