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LOWER SEMI-CONTINUITY OF ENTROPY IN A FAMILY OF K3 SURFACE AUTOMORPHISMS

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ABSTRACT. We compute topological entropies for a large family of automorphisms of K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Similarly to a result by Xie **[17]**, we find that the entropies vary in a lower semi-continuous manner as the Picard ranks of the K3 surfaces vary.

1. Introduction. We compute entropies in a family of automorphisms of complex K3 surfaces in

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = \{(x = [x_0 : x_1],\ y = [y_0 : y_1],\ z = [z_0 : z_1])\}.$$

The set of all effective divisors on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of tri-degree $(2, 2, 2)$ is parametrized by $\mathbb{P}^{26}$, and every non-singular prime divisor in this set is a K3 surface; therefore, a general effective divisor of tri-degree $(2, 2, 2)$ is a K3 surface. Throughout this paper, $Q = Q(x_0, x_1, y_0, y_1, z_0, z_1)$ is a tri-homogeneous polynomial of tri-degree $(2, 2, 2)$, and S is a K3 surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the form $\{Q = 0\}$.

We write

$$Q(x_0, x_1, y_0, y_1, z_0, z_1) = \sum_{j\in\{0,1,2\}} x_0^j x_1^{2-j} Q_{x,j}(y_0, y_1, z_0, z_1)$$

(such that each non-trivial $Q_{x,j} = Q_{x,j}(y_0, y_1, z_0, z_1)$ is bi-homogeneous of bi-degree $(2, 2)$), and for irreducible Q, we define a birational invo-

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lution τ_x on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$\tau_x(x, y, z) = ([x_0 Q_{x,2} + x_1 Q_{x,1} : -x_1 Q_{x,2}], y, z).$$

For $(x, y, z) \in S$ in the domain of τ_x,

$$\tau_x(x, y, z) = ([x_1 Q_{x,0} : x_0 Q_{x,2}], y, z) \in S;$$

since S is its own unique minimal model, it follows that τ_x defines an automorphism of S. We define τ_y and τ_z similarly; thus, $\mathrm{Aut}(S)$ contains the subgroup generated by $\{\tau_x, \tau_y, \tau_z\}$.

Silverman and Mazur **[12]** first suggested compositions of the involutions just described as interesting examples of infinite-order automorphisms of K3 surfaces. Wang **[16]** and Baragar **[1]** used automorphisms in this subgroup to study rational points on S (when S is defined over a number field). Cantat **[6]** and McMullen **[13]** highlighted $f := \tau_z \circ \tau_y \circ \tau_x$ on various choices of S as examples of K3 surface automorphisms with positive topological entropy. Cantat observed that results by Gromov **[9]**, Yomdin **[18]** and Friedland **[8]** imply that the entropy of f is the logarithm of the spectral radius $\lambda(f)$ of

$$f^* : \mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S).$$

Wang, Cantat and McMullen showed how to compute f^* in the very general case where S has Picard rank $\rho(S) = 3$. Baragar **[2]** showed how to compute f^* in a special family where $\rho(S) = 4$, and thereby showed that $\lambda(f)$ is not constant among all $K3$ surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here, we compute f^* for a much larger set of choices of S, with $\rho(S)$ ranging from 3–11.

For all $p \in \mathbb{P}^1$, we let $E_{x=p}$, respectively, $E_{y=p}$ and $E_{z=p}$, denote the restriction to S of the prime divisor $\{x = p\}$, respectively, $\{y = p\}$ and $\{z = p\}$, on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; we call each $E_{x=p}$, respectively, $E_{y=p}$ and $E_{z=p}$, a fiber of S over the x-axis, respectively, y- and z-axes. Each fiber is an effective divisor of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and hence, is an elliptic curve if it is a non-singular prime divisor; thus, a general fiber is an elliptic curve.

For all $p = (p_1, p_2) \in \mathbb{P}^1 \times \mathbb{P}^1$, we define, in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,

$$C_{x,p} := \{y = p_1\} \cap \{z = p_2\}$$
$$C_{y,p} := \{x = p_2\} \cap \{z = p_1\}$$

and

$$C_{z,p} := \{x = p_1\} \cap \{y = p_2\};$$

we call each $C_{x,p}$, respectively, $C_{y,p}$ and $C_{z,p}$, a curve parallel to the x-axis, respectively, y- and z-axes. It may occur that S contains a curve parallel to an axis. If, for example, $C_{x,p} \subseteq S$, then neither $E_{y=p_1}$ nor $E_{z=p_2}$ is a prime divisor.

For a divisor D on S, we let $[D]$ denote the class of D in $\mathrm{Pic}(S)$. We let $(\_ \cdot \_)$ denote the intersection form on both $\mathrm{Pic}(S)$ and $\mathrm{Div}(S)$. In light of the fact that the fibers of S over a fixed axis are all linearly equivalent, we let E_x, E_y and E_z in $\mathrm{Pic}(S)$ denote the classes of the fibers over, respectively, the x-, y- and z-axes. We let $\mathcal{B}_x(S)$, $\mathcal{B}_y(S)$ and $\mathcal{B}_z(S)$ denote the sets of all classes of curves parallel to, respectively, the x-, y- and z-axes which are contained in S, and we set

$$\mathcal{B}(S) := \{E_x, E_y, E_z\} \cup \mathcal{B}_x(S) \cup \mathcal{B}_y(S) \cup \mathcal{B}_z(S).$$

Since K_S is trivial, the adjunction formula gives $(E_\omega \cdot E_\omega) = 0$ for each E_ω and $(C \cdot C) = -2$ for each curve $C \subseteq S$ parallel to an axis; it follows that the number of distinct classes in $\mathcal{B}(S)$ is three plus the number of distinct curves parallel to axes in S.

Definition 1.1. For an ordered triple (k, l, m) of non-negative integers, we say that S is *pure of type* (k, l, m) if the following conditions hold:

(a) $|\mathcal{B}_x(S)| = k$, $|\mathcal{B}_y(S)| = l$ and $|\mathcal{B}_z(S)| = m$;
(b) $\mathcal{B}(S)$ is a basis for $\mathrm{Pic}(S)$; and
(c) $(\mathcal{L} \cdot \mathcal{L}') = 0$ whenever $\mathcal{L}$ and $\mathcal{L}'$ are distinct classes in

$$\mathcal{B}_x(S) \cup \mathcal{B}_y(S) \cup \mathcal{B}_z(S).$$

We let $\mathcal{U}_{k,l,m} \subseteq \mathbb{P}^{26}$ denote the set of all K3 surfaces which are pure of type (k, l, m). If (k', l', m') is a reordering of (k, l, m), then $\mathcal{U}_{k',l',m'} \cong \mathcal{U}_{k,l,m}$. If $S \in \mathcal{U}_{k,l,m}$, then the conditions in Definition 1.1 provide sufficient information for the computation of f^*. However, it is a significant step to show actual existence of pure K3 surfaces of various types. For distinct ordered triples (k, l, m) and (k', l', m'), we write

$$(k, l, m) < (k', l', m')$$

if $k \le k'$, $l \le l'$ and $m \le m'$. We set

$$\mathcal{N}'' := \{(6,0,0),(5,1,1),(4,2,2),(3,3,3)\},$$

let $\mathcal{N}'$ denote the set of all permutations of ordered triples in $\mathcal{N}''$, and let $\mathcal{N}$ denote the set of all ordered triples (k,l,m) of non-negative integers satisfying $(k,l,m) \le \nu$ for some $\nu \in \mathcal{N}'$.

Theorem 1.2. *For $(k,l,m) \in \mathcal{N} - \{(3,3,3)\}$, the dimension of the space of isomorphism classes of* K3 *surfaces contained in $\mathcal{U}_{k,l,m}$ is $17-k-l-m$. If $(k',l',m') \in \mathcal{N}$ satisfies $(k,l,m) < (k',l',m')$, then $\mathcal{U}_{k',l',m'}$ is contained in the closure of $\mathcal{U}_{k,l,m}$. For $(k,l,m) \notin \mathcal{N}$, $\mathcal{U}_{k,l,m} = \emptyset$.*

We prove Theorem 1.2 in Section 2. The proof relies on the surjectivity of the period map for K3 surfaces to show the existence of $S \in \mathcal{U}_{k,l,m}$, and thus, does not yield any explicit equations defining pure K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Baragar and van Luijk **[3]** have given explicit equations for some pure K3 surfaces of type $(0,0,0)$, and Barager **[2]** has given explicit equations for some pure K3 surfaces of type $(1,0,0)$. Little else in the form of concrete examples has appeared in the literature, and it is typically quite challenging to show that a particular polynomial Q defines a pure K3 surface. We do not know whether $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ contains pure K3 surfaces of type $(3,3,3)$.

Theorem 1.2 shows that we can compute and compare entropies among many different types of K3 surface automorphisms as well by focusing only on automorphisms of pure K3 surface automorphisms.

Theorem 1.3. *As S varies among all pure* K3 *surfaces, $\lambda(f)$ depends only upon the type of S. Writing $\lambda(f) = \lambda(k,l,m)$ as a function of the type of S, we have*

$$\lambda(k,l,m) > \lambda(k',l',m')$$

whenever $(k,l,m) < (k',l',m')$.

We prove Theorem 1.3 in Section 3 by computing $\lambda(f)$ for every pure K3 surface. We note that $\lambda(f)$ actually depends only upon the unordered triple (k,l,m), that is,

$$\lambda(k',l',m') = \lambda(k,l,m),$$

if (k', l', m') is a reordering of (k, l, m). However, the computation of f^* does depend upon the order of (k, l, m). We compute $\lambda(3, 3, 3) = 1$, which suggests that f has some very special behavior on pure K3 surfaces of type $(3, 3, 3)$ if any exist (and, thus, perhaps suggests the nonexistence of such K3 surfaces).

Theorems 1.2 and 1.3 show that $\lambda(f)$ is a strictly lower semi-continuous (lsc) function of the parameters in the union of all of the spaces $\mathcal{U}_{k,l,m}$. Thus, the set of all pure K3 surfaces provides an example that demonstrates the following result of Xie.

Theorem 1.4 ([**17**, Theorem 4.3])**.** *Suppose that W is a quasi-projective variety,*

$$\mathcal{S} \longrightarrow W$$

is a family of projective surfaces and

$$F : \mathcal{S} \dashrightarrow \mathcal{S}$$

is a birational map that restricts to an automorphism of each fiber over W. For $s \in W$, let $h(s)$ denote the entropy of the restriction of F to the fiber over s. Then, h is an lsc *function on W.*

Remark 1.5. The hypothesis on the restrictions of F to fibers in Theorem 1.4 should not be taken to imply that F is in fact biregular, but rather that any such restriction extends biregularly to the whole fiber; in [**17**], Theorem 1.4 is stated in terms of first dynamical degrees rather than entropies, which allows for restrictions of F that are birational but not biregular.

Theorem 1.4 applies to $\lambda(f)$ in the following way:

$$\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

admits a birational self-map that restricts to f on every fiber

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

of the projection to $\mathbb{P}^{26}$, where f is well-defined; this involution preserves the variety in $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, defined by

$$Q(x_0, x_1, y_0, y_1, z_0, z_1) = 0,$$

and hence, realizes most quasi-subvarieties of $\mathbb{P}^{26}$ as parameter spaces for families of K3 surface automorphisms of the type treated in Theorem 1.4. In Section 4, we describe the indeterminacy locus of the birational self-map on $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Although pure K3 surfaces are very general among all K3 surfaces $S \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, they certainly do not account for all S. The procedure in this paper could be adapted to the computation of $\lambda(f)$ among all S satisfying (a) and (b), but not necessarily (c), in Definition 1.1, since $\mathrm{Pic}(S)$ and f^* can still be sufficiently well understood for such an S. The challenge then would be to determine which arrangements of curves parallel to axes actually occur on such an S. However, as first observed by Rowe **[15]**, a K3 surface S can fail even to satisfy (b), in which case it is impossible to compute $\lambda(f)$ in the manner used here without some means of determining $\mathrm{Pic}(S)$. The K3 surface $\widetilde{S} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ below is an example which fails to satisfy (b).

2. Finding pure K3 surfaces. Every prime divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the zero locus of an irreducible tri-homogeneous polynomial (and every such zero locus is a prime divisor). The classes of $\{x_0 = 0\}$, $\{y_0 = 0\}$ and $\{z_0 = 0\}$ generate $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. It is a well-known fact, e.g., **[12, 13, 16]**, that every smooth prime divisor S of tri-degree $(2, 2, 2)$ is a K3 surface; this may be verified by using the Lefschetz hyperplane theorem (applied to S as a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) to show that $h^1(S) = 0$ and using the adjunction formula (applied to S as a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) to show that K_S is trivial.

Lemma 2.1. *Let S' be a smooth prime divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of tri-degree (a, b, c). If $abc > 0$ and $(a, b, c) \neq (2, 2, 2)$, then S' is neither a* K3 *surface, nor a copy of $\mathbb{P}^2$, nor a Hirzebruch surface. If $abc = 0$, then S' is a product with one of the coordinate copies of $\mathbb{P}^1$ as a factor.*

Proof. First, suppose that $abc > 0$ and $(a, b, c) \neq (2, 2, 2)$. The effective divisors

$$D_1 := \{x_0 = 0\}|_{S'}, \quad D_2 := \{y_0 = 0\}|_{S'} \quad \text{and} \quad D_3 := \{z_0 = 0\}|_{S'}$$

all satisfy $(D_j \cdot D_j) = 0$ and $(D_j \cdot D_{j' \neq j}) > 0$. Thus, $\{[D_1], [D_2], [D_3]\}$ is a linearly independent set in $\mathrm{Pic}(S')$. By the adjunction formula,

$$K_{S'} = (a - 2)[D_1] + (b - 2)[D_2] + (c - 2)[D_3],$$

which is not trivial. Therefore, S' is not a K3 surface. Also, $\rho(S') \geq 3$ implies that S' is neither a copy of $\mathbb{P}^2$ nor a Hirzebruch surface.

If $abc = 0$, the claim is evident from the form of the polynomial defining S'. □

A lattice of rank $r \in \mathbb{N}$ is a group $L \cong \mathbb{Z}^r$ equipped with a bilinear form $(\_ \cdot \_)_L$, which is integral, symmetric and non-degenerate. Given a basis for L, there is a unique integer matrix M such that

$$(\vec{g}_1 \cdot \vec{g}_2)_L = \vec{g}_1{}^t M \vec{g}_2 \quad \text{for all } \vec{g}_1, \vec{g}_2 \in L.$$

Since M is symmetric with $\det(M) \neq 0$, its eigenvalues are all non-zero real numbers. The signature of L is (p, q), where p and q denote the number (counting multiplicity) of, respectively, positive and negative eigenvalues of M. If T is a projective K3 surface, it is a well-known consequence of the Hodge index theorem, e.g., **[4]**, that the intersection form changes $\operatorname{Pic}(T) \cong \operatorname{NS}(T)$ into a lattice of signature $(1, \rho(T) - 1)$.

For every K3 surface

$$S \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

the intersection form on $\langle E_x, E_y, E_z \rangle \leq \operatorname{Pic}(S)$ is given by

$$M_{0,0,0} := \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

For every ordered triple (k, l, m) of non-negative integers, the conditions in Definition 1.1 indicate how to write a matrix $M_{k,l,m}$ that gives the intersection form on $\operatorname{Pic}(S)$ in the basis $\mathcal{B}(S)$ whenever S is pure of type (k, l, m), for example,

$$M_{2,0,1} = \begin{pmatrix} 0 & 2 & 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

Lemma 2.2. *For any ordered triple (k, l, m) of non-negative integers,*

$$\det(M_{k,l,m}) = -(-2)^{k+l+m-3}(128 - 16(k + l + m) + klm).$$

Proof. The formula given follows by computation from the general formula

$$\det((a_{i,j})_{1\leq i,j\leq n}) = \sum \operatorname{sgn}(\xi) \prod_{i=1}^{n} a_{i,\xi(i)},$$

where the sum is taken over all permutations ξ of $\{1,\dots,n\}$. □

For (k,l,m) such that $\det(M_{k,l,m}) \neq 0$, which includes all $(k,l,m) \in \mathcal{N}$, let $L_{k,l,m}$ denote the lattice given by $M_{k,l,m}$. If (k',l',m') is a reordering of (k,l,m), then $L_{k',l',m'}$ is isometric to $L_{k,l,m}$.

For any K3 surface T, the Riemann-Roch theorem and the adjunction formula imply the following useful facts about the intersection form on $\operatorname{Pic}(T)$, e.g., [**4, 11, 7**]:

- if $\mathcal{L} \in \operatorname{Pic}(T)$ satisfies $(\mathcal{L}\cdot\mathcal{L}) \geq -2$, then either $\mathcal{L}$ or $-\mathcal{L}$ is effective;
- if $\mathcal{L} \in \operatorname{Pic}(T)$ is effective, then $h^0(\mathcal{L}) \geq 2 + (\mathcal{L}\cdot\mathcal{L})/2$;
- if $D \in \operatorname{Div}(T)$ is reduced, effective and connected, then $h^0([D]) = 2 + (D\cdot D)/2$;
- if D is a prime divisor on T, then $h^0([D]) \geq -2$.

2.1. Global sections in pure Picard lattices. Fix an ordered triple (k,l,m) of non-negative integers, and suppose that T is a K3 surface such that $\operatorname{Pic}(T)$ is isometric to $L_{k,l,m}$. (It is then implicit here that $\det(M_{k,l,m} \neq 0)$.) Since $L_{k,l,m}$ contains elements with positive self-intersection, it follows from Grauert's criterion, e.g., [**4**], that T is projective. Let

$$\mathcal{B} = \{B_1, B_2, B_3, B_{x,1}, \dots, B_{x,k}, B_{y,1}, \dots, B_{y,l}, B_{z,1}, \dots, B_{z,m}\}$$

be a basis for $\operatorname{Pic}(T)$ in which $M_{k,l,m}$ gives the intersection form, and suppose further that each B_j is nef. For $(k,l,m) \in \mathcal{N}$, we will show that there is an embedding $T \subseteq \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ as a pure K3 surface of type (k,l,m).

If, for some B_j, there were $\mathcal{L} \in \langle B_j\rangle^{\perp} \leq \operatorname{Pic}(T)$ satisfying $(\mathcal{L}\cdot\mathcal{L}) = 0$ and $\mathcal{L} \notin \langle B_j\rangle$, then $\langle B_j, \mathcal{L}\rangle$ would be a totally isotropic sublattice of $\operatorname{Pic}(T)$ of rank 2; however, it is a well-known fact, e.g., [**13**], that the signature of $\operatorname{Pic}(T)$ implies that $\operatorname{Pic}(T)$ cannot contain a totally isotropic sublattice of rank $r > 1$. It follows that each $\langle B_j\rangle^{\perp}$ is negative definite away from $\langle B_j\rangle$. It may be verified that every $\mathcal{L} \in$

$\langle B_1, B_2, B_3, \rangle^{\perp}$ satisfies $(\mathcal{L}\cdot\mathcal{L}) \equiv 0 \bmod 4$, thus that, in particular, $\langle B_1, B_2, B_3\rangle^{\perp}$ cannot contain the class of any prime divisor on T.

Lemma 2.3. *Every element of $\mathcal{B}$ is the class of a prime divisor on T.*

Proof. Since $(B_{x,1}\cdot B_{x,1}) = -2$ and $(B_1\cdot B_{x,1}) = 1$, assuming $k > 0$, $B_{x,1}$ must be effective. Write

$$B_{x,1} = [D_1] + \cdots + [D_n],$$

where each D_j is a prime divisor (however, the prime divisors may not be pairwise a priori distinct). Since $B_{x,1} \in \langle B_2, B_3\rangle^{\perp}$, $(B_{x,1}\cdot B_1) = 1$ and no D_j can have its class in $\langle B_1, B_2, B_3\rangle^{\perp}$, the only possibility is $n = 1$ so that $B_{x,1}$ is the class of a prime divisor. It similarly follows that each $B_{\omega,j}$ is the class of a prime divisor $D_{\omega,j}$.

We now show that B_1 is the class of a prime divisor. It similarly follows that B_2 and B_3 are classes of prime divisors. Each B_j is effective with $h^0(B_j) \geq 2$ since it is nef and satisfies $(B_j\cdot B_j) = 0$.

First, suppose $l = m = 0$. In this case, $(\mathcal{L}\cdot\mathcal{L}) \equiv 0 \bmod 4$ whenever $\mathcal{L} \in \langle B_1\rangle^{\perp}$. Also, $(B_2\cdot\mathcal{L})$ and $(B_3\cdot\mathcal{L})$ are even for every $\mathcal{L} \in \mathrm{Pic}(T)$. It then follows from the intersection numbers given by $M_{0,0,0}$ that B_1 cannot be written as a sum of more than one class of a prime divisor.

Now, suppose $l > 0$; the case $m > 0$ similarly follows. Since $B_1' := B_1 - B_{y,1}$ satisfies $(B_1'\cdot B_1') = -2$ and $(B_1'\cdot B_2) = 1$, it is effective. Write

$$B_1' = [D_1] + \cdots + [D_n],$$

where each D_j is a prime divisor (but the prime divisors may not be pairwise a priori distinct). The intersection numbers of B_1' with B_1, B_2 and B_3 force $n \leq 3$ and $(D_j\cdot D_j) = -2$ for each D_j. Moreover, there is a unique D_j satisfying $([D_j]\cdot B_2) > 0$. Take D_1 to be this divisor such that $([D_1]\cdot B_2) = 1$ and $([D_j]\cdot B_3) > 0$ for $j > 1$.

If $n = 1$, then $(D_1\cdot D_{y,1}) = 2$. If $n = 2$, then $(B_1'\cdot B_1') = -2$ implies $(D_1\cdot D_2) = 1$. If $D_1 \in \mathcal{B}$, then

$$([D_1]\cdot B_1') \in \{0, 2\}$$

gives a contradiction. Thus, since $B_{y,1} + [D_1]$ and $B_{y,1} + [D_2]$ are both in $\langle B_1\rangle^\perp$, $(B_{y,1}\cdot B_1')=2$ implies

$$(D_1\cdot D_{y,1})=(D_2\cdot D_{y,1})=1.$$

If $n=3$, then

$$([D_2]\cdot B_3)=([D_3]\cdot B_3)=1$$

and

$$([D_1]\cdot B_3)=0.$$

If $D_1=D_{y,1}$, then $(B_1'\cdot B_{y,1})=2$ and $(B_1'\cdot B_1')=-2$ force $D_3=D_2$. If, conversely, $D_3=D_2$, then $(B_1'\cdot B_1')=-2$ implies $(D_1\cdot D_2)=2$ such that $\langle B_1,[D_1+D_2]\rangle$ is totally isotropic. Since $(B_1\cdot B_2)=2$ and $([D_1+D_2]\cdot B_2)=1$, it follows that

$$B_1=2[D_1]+2[D_2]$$

and

$$D_1=D_{y,1}.$$

Therefore, $D_1=D_{y,1}$ if and only if $D_2=D_3$; however, then $(1/2)B_1\in\operatorname{Pic}(T)$ is a contradiction in this case. Thus, $[D_2+D_3]\in\langle B_1,B_2\rangle^\perp$ and $[D_2-D_3]\in\langle B_1,B_2,B_3\rangle^\perp$ imply $(D_2\cdot D_3)=0$. Also, by similar reasoning, $(D_1\cdot D_{y,1})=0$. Since $(1/2)B_1\notin\operatorname{Pic}(T)$ and $\operatorname{Pic}(T)$ cannot contain a totally isotropic sublattice of rank 2, none of $(D_1\cdot D_2)$, $(D_1\cdot D_3)$, $(D_{y,1}\cdot D_2)$ nor $(D_{y,1}\cdot D_3)$ can equal 2. Therefore, $([D_1]\cdot B_1')=0$ and $(B_{y,1}\cdot B_1')=2$ imply

$$(D_1\cdot D_2)=(D_1\cdot D_3)=(D_{y,1}\cdot D_2)=(D_{y,1}\cdot D_3)=1.$$

In all three cases for n, B_1 is realized as the class of a reduced, effective and connected divisor E with the property that every effective divisor E' satisfying $E'<E$ has $h^0(E')=1$. Fix $\{s,s'\}\subseteq H^0(B_1)$ such that s vanishes on all of E and s' does not. If s' vanishes on some nontrivial effective divisor E' satisfying $E'<E$, then $h^0(E-E')=1$ contradicts the fact that s'/s is not constant. Thus, B_1 has no fixed component, and **[11,** Proposition 1] shows that B_1 is the class of an elliptic curve. □

Proposition 2.4. *There is an embedding*

$$T \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

If $(k, l, m) \in \mathcal{N}$, *then* T *is pure of type* (k, l, m).

Proof. By Lemma 2.3, each B_j satisfies both $h^0(B_j) = 2$ and $(B_j \cdot B_j) = 0$, and furthermore, has no fixed component. Thus, each B_j induces a morphism

$$\psi_j : T \longrightarrow \mathbb{P}^1.$$

Set

$$\psi := \psi_1 \times \psi_2 \times \psi_3 : T \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

and

$$A := B_1 + B_2 + B_3,$$

and let ϕ denote the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^7$. Then, $A = (\phi \circ \psi)^* \mathcal{O}(1)$. Since each B_j is nef and no prime divisor on T can have its class in $\langle B_1, B_2, B_3 \rangle^{\perp}$, Nakai's criterion, e.g., [**4**], implies that A is ample; in addition, A has no fixed component since neither does B_j. Therefore, $(\phi \circ \psi)$ does not collapse any curve on T, and [**11,** Proposition 2] shows that $(\phi \circ \psi)$ is either an embedding or a ramified double covering. Thus, $\psi(T)$ is a prime divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since each $B_j + B_{j' \neq j}$ is nef, big and effective with no fixed component, [**11,** Proposition 2] also shows that each $\psi_j \times \psi_{j'}$ is surjective. Thus, in particular, $\psi(T)$ is not a product with one of the coordinate copies of $\mathbb{P}^1$ as a factor. If $(\phi \circ \psi)$ is a ramified double covering, then the main result in [**14**] shows that $\psi(T)$ is either a copy of $\mathbb{P}^2$ or a Hirzebruch surface, which contradicts Lemma 2.1. Therefore, ψ is an embedding.

For each $B_{\omega,j'}$ and B_j with $(B_j \cdot B_{\omega,j'}) = 0$, $h^0(B_j)$ must contain a section whose zero locus is disjoint from $D_{\omega,j'}$, which means that $\psi_j(D_{\omega,j'})$ is a point. Thus, each $\psi(D_{\omega,j'})$ is a curve parallel to an axis (specifically, the axis corresponding to the B_j which satisfies $(B_j \cdot B_{\omega,j'}) = 1$), and $\psi(T)$ is of pure type (k, l, m) if it has no curves parallel to axes beyond those whose classes are contained in $\mathcal{B}$.

Now, consider the case $(k, l, m) \in \mathcal{N}$. Suppose that $\psi(T)$ contains some $C_{x,p}$ with $[C_{x,p}] \notin \mathcal{B}$. By the construction of ψ, $([C_{x,p}] \cdot B_1) = 1$ and $[C_{x,p}]$ must have a zero intersection with B_2, B_3 and every $B_{x,j}$.

If $[C_{x,p}]$ has a zero intersection with every $B_{y,j}$ and $B_{z,j}$, then the intersection form on $\langle \mathcal{B} \cup \{[C_{x,p}]\}\rangle$ is given by $M_{k+1,l,m}$; however, then Lemma 2.2 shows that

$$\mathcal{B} \cup \{[C_{x,p}]\}$$

is linearly independent, a contradiction. Writing $p = (p_1, p_2)$ and $p' = (p'_1, p'_2)$, every curve $C_{y,p'}$ satisfies

$$C_{y,p'} \cap C_{x,p} = \emptyset$$

if $p'_1 \neq p_2$ and

$$|C_{y,p'} \cap C_{x,p}| = 1$$

with multiplicity 1 if $p'_1 = p_2$. Since $E_{z=p_2}$ has bi-degree $(2, 2)$, there are at most two $D_{y,j}$ on T such that $(C_{x,p} \cdot D_{y,j}) = 1$. If $(C_{x,p} \cdot D_{y,j'}) = 1$ for some $D_{y,j'}$, then $(C_{x,p} \cdot D_{y,j})$ is odd, and hence, equal to 1, for every $D_{y,j}$; thus, $l \leq 2$ in this case. Similarly, $m \leq 2$ and $(C_{x,p} \cdot D_{z,j}) = 1$ for every $D_{z,j}$ if there is some $D_{z,j'}$ such that $(C_{x,p} \cdot D_{z,j'}) = 1$. Now, we may compute $\det(M) \neq 0$ for each matrix M that gives a possible intersection form on $\langle \mathcal{B} \cup \{[C_{x,p}]\}\rangle$, a contradiction. It would similarly be a contradiction if T contained some curve $C_{y,p}$ or $C_{z,p}$ whose class was not in $\mathcal{B}$. □

Remark 2.5. Proposition 2.4 shows that $n = 1$ is the only case that can actually occur in the latter part of the proof of Lemma 2.3 when $(k, l, m) \in \mathcal{N}$; otherwise, $\psi(D_2)$ would be a curve parallel to the z-axis such that $[D_2] \notin \mathcal{B}$ (since $(D_2 \cdot D_{y,1}) = 1$).

2.2. Nef classes in pure Picard lattices. Fix $(k, l, m) \in \mathcal{N}$, set

$$\Gamma := \{\gamma \in L_{k,l,m} \mid (\gamma \cdot \gamma)_{L_{k,l,m}} = -2\}$$

and write $\Gamma = \Gamma^+ \cup \Gamma^-$ such that

$$\Gamma^+ \cap \Gamma^- = \emptyset, \qquad \Gamma^- = \{\gamma | -\gamma \in \Gamma^+\}$$

and

$$\Gamma \cap \{\gamma + \gamma' \mid \{\gamma, \gamma'\} \subset \Gamma^+\} \subseteq \Gamma^+;$$

we will call a choice of Γ^+ satisfying these conditions "allowable." Let $\mathcal{B}$ as above be a basis for $L_{k,l,m}$ in which $M_{k,l,m}$ gives the intersection form. We will show that Γ^+ can be chosen so that each B_j satisfies $(B_j \cdot \gamma) \geq 0$ for every $\gamma \in \Gamma^+$. Thus, any effective isometry between

$L_{k,l,m}$ and the Picard lattice of a K3 surface, that is, any isometry which sends each $\gamma \in \Gamma^+$ to an effective class, will send each B_j to an nef class.

Set

$$\widetilde{Q}(x_0, x_1, y_0, y_1, z_0, z_1) \\ := (x_0^2 + x_1^1)(y_0^2 + y_1^2)(z_0^2 + z_1^2) + 3x_0x_1y_0y_1z_0z_1 - 2x_1^2y_0y_1z_0z_1,$$

and set $\widetilde{S} := \{\widetilde{Q} = 0\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It may be verified by directly testing possible factors that

$$\widetilde{Q}(0, 1, y_0, y_1, z_0, z_1) = y_0^2z_0^2 + y_0^2z_1^2 + y_1^2z_0^2 - 2y_0y_1z_0z_1 + y_1^2z_1^2$$

is irreducible over $\mathbb{C}$. Therefore, since it has no factor of tri-degree $(1, 0, 0)$, $\widetilde{Q}$ is irreducible over $\mathbb{C}$. It also follows from Lemma 2.6 that $\widetilde{Q}$ is irreducible since the existence of non-constants Q_1 and Q_2 satisfying $Q_1 \cdot Q_2 = \widetilde{Q}$ would imply $\{Q_1 = Q_2 = 0\} \neq \emptyset$.

Lemma 2.6. *The set*

$$\mathrm{Sing}(\widetilde{Q}) := \left\{ \widetilde{Q} = \frac{\partial \widetilde{Q}}{\partial x_0} = \frac{\partial \widetilde{Q}}{\partial x_1} = \frac{\partial \widetilde{Q}}{\partial y_0} = \frac{\partial \widetilde{Q}}{\partial y_1} = \frac{\partial \widetilde{Q}}{\partial z_0} = \frac{\partial \widetilde{Q}}{\partial z_1} = 0 \right\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

is empty.

Proof. Suppose that $([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \in \mathrm{Sing}(\widetilde{Q})$. If $y_0y_1z_0z_1 = 0$, then

$$(x_0^2 + x_1^2)(y_0^2 + y_1^2) = (x_0^2 + x_1^2)(z_0^2 + z_1^2) = (y_0^2 + y_1^2)(z_0^2 + z_1^2) = 0$$

implies that exactly one of $y_0y_1 = 0$ or $z_0z_1 = 0$ is true such that, in addition, $x_0^2 + x_1^2 = 0$ and $x_0x_1 \neq 0$; however, then $3x_0 - 2x_1 = 0$ gives a contradiction.

From $y_0y_1z_0z_1 \neq 0$, it follows that $(y_0^2 + y_1^2)(z_0^2 + z_1^2) \neq 0$. In addition, if $x_0^2 + x_1^2 = 0$, then $3x_0 - 2x_1 = 0$ again gives a contradiction. Thus,

$$y_0^2 - y_1^2 = z_0^2 - z_1^2 = 0$$

implies

$$8x_0 \pm 3x_1 = 3x_0 + (8 \mp 4)x_1 = 0,$$

a contradiction which leaves open no further possibilities. □

Lemma 2.6 shows that $\widetilde{S}$ is a K3 surface; it is a variant of a K3 surface studied in **[13, 15]**. The set of all curves parallel to axes contained in $\widetilde{S}$ is

$$\begin{aligned}\{C_1, \dots, C_{24}\} := \{&C_{z,(i,0)}, C_{z,(i,\infty)}, C_{y,(0,i)}, C_{y,(\infty,i)}, C_{z,(2/3,i)},\\ &C_{z,(\infty,i)}, C_{x,(i,0)}, C_{x,(i,\infty)}, C_{y,(i,2/3)}, C_{y,(i,\infty)},\\ &C_{x,(0,i)}, C_{x,(\infty,i)}, C_{z,(-i,0)}, C_{z,(-i,\infty)}, C_{y,(0,-i)},\\ &C_{y,(\infty,-i)}, C_{z,(2/3,-i)}, C_{z,(\infty,-i)}, C_{x,(-i,0)}, C_{x,(-i,\infty)},\\ &C_{y,(-i,2/3)}, C_{y,(-i,\infty)}, C_{x,(0,-i)}, C_{x,(\infty,-i)}\}.\end{aligned}$$

Clearly, $\widetilde{S}$ is not pure. For example,

$$[C_{24}] = 2E_y + 2E_z - 2E_x - [C_7] - [C_8] - [C_{11}] - [C_{12}] - [C_{19}] - [C_{20}] - [C_{23}]$$

and

$$\begin{aligned}[C_{22}] &= [C_{11}] + [C_{12}] - [C_{21}] + 2E_x - 2E_y - E_z + [C_7] + [C_8] + [C_{19}] + [C_{20}]\\ &= -[C_{21}] + 2E_x - 2E_y - [C_9] - [C_{10}] + [C_7] + [C_8] + [C_{19}] + [C_{20}].\end{aligned}$$

Set

$$\Gamma^+(\widetilde{S}) := \{\mathcal{L} \in \mathrm{Pic}(\widetilde{S}) \mid (\mathcal{L} \cdot \mathcal{L}) = -2 \text{ and } \mathcal{L} \text{ is effective}\}.$$

Thus,

$$\Gamma^+(\widetilde{S}) \cap \{\mathcal{L} + \mathcal{L}' \mid \{\mathcal{L}, \mathcal{L}'\} \subseteq \Gamma^+(\widetilde{S})\} \subseteq \Gamma^+(\widetilde{S}),$$

and every $\mathcal{L} \in \mathrm{Pic}(\widetilde{S})$ satisfying $(\mathcal{L} \cdot \mathcal{L}) = -2$ also satisfies $|\{\mathcal{L}, -\mathcal{L}\} \cap \Gamma^+(\widetilde{S})| = 1$.

Proposition 2.7. *There is a lattice embedding $L_{k,l,m} \leq \mathrm{Pic}(\widetilde{S})$ such that*

$$\{B_1, B_2, B_3\} = \{E_x, E_y, E_z\}.$$

Thus, setting

$$\Gamma^+ := \Gamma \cap \Gamma^+(S)$$

is an allowable choice that yields $(B_j \cdot \gamma) \geq 0$ for each B_j and every $\gamma \in \Gamma^+$.

Proof. Since $(k, l, m) \in \mathcal{N}$, at least one of the lattice embeddings $L_{k,l,m} \leq L_{6,0,0}$, $L_{k,l,m} \leq L_{5,1,1}$, $L_{k,l,m} \leq L_{4,2,2}$ or $L_{k,l,m} \leq L_{3,3,3}$

exists; $L_{6,0,0}$ is isometric to

$$\langle E_x, E_y, E_z, [C_7], [C_8], [C_{11}], [C_{12}], [C_{19}], [C_{20}]\rangle,$$

$L_{5,1,1}$ is isometric to

$$\langle E_x, E_y, E_z, [C_2], [C_7], [C_8], [C_{11}], [C_{19}], [C_{20}], [C_{21}]\rangle,$$

$L_{4,2,2}$ is isometric to

$$\langle E_x, E_y, E_z, [C_1], [C_2], [C_7], [C_8], [C_9], [C_{10}], [C_{19}], [C_{20}]\rangle,$$

and $L_{3,3,3}$ is isometric to

$$\langle E_x, E_y, E_z, [C_1], [C_2], [C_7], [C_8], [C_9], [C_{10}], [C_{13}], [C_{19}], [C_{21}]\rangle.$$

Since E_x, E_y and E_z are all nef, each B_j satisfies $(B_j \cdot \gamma) \geq 0$ for every $\gamma \in \Gamma^+$. □

2.3. Primitive embeddings of pure Picard lattices. Let L_2 be the lattice of rank 2 given by the matrix

$$M_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

let L_8 be the lattice of rank 8 given by the matrix

$$M_8 := \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix},$$

and set

$$L_{K3} := (L_2)^{\oplus 3} \oplus (L_8)^{\oplus 2};$$

thus, L_{K3} has rank 22, is even in the sense that every element of L_{K3} has even self-intersection and is unimodular in the sense that

$$M_{K3} := (M_2)^{\oplus 3} \oplus (M_8)^{\oplus 2}$$

is invertible over $\mathbb{Z}$. For any complex K3 surface T, it is a well-known fact, e.g., [**4, 11, 13**], that the cup product changes $H^2(T, \mathbb{Z})$ into a lattice isometric to L_{K3}. A lattice embedding $L \leq L'$ is said to be primitive if $(L^{\perp})^{\perp} = L$ (where the orthogonal lattices are taken in L') or, equivalently, if $(L \otimes \mathbb{Q}) \cap L' = L$. For example, by the Lefschetz theorem on (1,1) classes, e.g., [**4**],

$$\mathrm{Pic}(T) \leq H^2(T, \mathbb{Z})$$

is a primitive lattice embedding for every complex K3 surface T.

For $(k,l,m) \in \mathcal{N}$, we have established that $L_{k,l,m}$ can be assigned an nef cone which contains every B_j, and furthermore, that any effective isometry between $L_{k,l,m}$ and the Picard lattice of a K3 surface then forces the K3 surface to be pure of type (k,l,m). In order to prove the existence of pure K3 surfaces of type (k,l,m), it remains only to show that $L_{k,l,m}$ embeds primitively in L_{K3}.

Proposition 2.8. *If $(k,l,m) \neq (3,3,3)$, then there is a primitive lattice embedding $L_{k,l,m} \leq L_{K3}$.*

Proof. Since the natural embedding of $L_{k,l,m}$ into one of $L_{6,0,0}$, $L_{5,1,1}$, $L_{4,2,2}$ or $L_{3,3,2}$ has a basis which is a subset of a basis for the larger lattice, it must be primitive. Therefore, $L_{k,l,m}$ has a primitive embedding in L_{K3} if $L_{6,0,0}$, $L_{5,1,1}$, $L_{4,2,2}$ and $L_{3,3,2}$ do.

Let $\{\beta_1, \ldots, \beta_{22}\}$ be a basis for L_{K3} in which M_{K3} gives $(\_ \cdot \_)_{L_{K3}}$. Set

$$\begin{aligned}
\mathcal{B}_{6,0,0} = \{&\beta_1 + 2\beta_2 + \beta_4 + \beta_6 + \beta_{10} + \beta_{18},\ \beta_3 + \beta_2 + \beta_6,\\
&\beta_5 + \beta_2 + \beta_4,\ \beta_7, \beta_9, \beta_{11}, \beta_{15}, \beta_{17}, \beta_{19}\},\\
\mathcal{B}_{5,1,1} = \{&\beta_1 + 2\beta_2 + \beta_4 + \beta_6 + \beta_{10} + \beta_{18}, \beta_3 + \beta_4 + \beta_2 + \beta_6 + \beta_{13},\\
&\beta_5 + \beta_6 + \beta_2 + \beta_4 + \beta_{21},\ \beta_7, \beta_9, \beta_{11}, \beta_{14}, \beta_{15}, \beta_{17}, \beta_{22}\},\\
\mathcal{B}_{4,2,2} = \{&\beta_1 + 2\beta_2 + \beta_4 + \beta_6 + \beta_{10} + \beta_{18},\ \beta_3 + \beta_4 + \beta_2 + \beta_6 + \beta_{13},\\
&\beta_5 + \beta_6 + \beta_2 + \beta_4 + \beta_{21}, \beta_7, \beta_9, \beta_{12}, \beta_{14}, \beta_{15}, \beta_{17}, \beta_{20}, \beta_{22}\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_{3,3,2} = \{&\beta_1 + \beta_2 + \beta_4 + \beta_6 + \beta_{10},\ \beta_3 + \beta_4 + \beta_2 + \beta_6 + \beta_{18},\\
&\beta_5 + 2\beta_6 + \beta_2 + \beta_4 + \beta_{13} + \beta_{21},\\
&\beta_7, \beta_9, \beta_{11}, \beta_{14}, \beta_{15}, \beta_{17}, \beta_{19}, \beta_{22}\}.
\end{aligned}$$

Since the matrices which send $\{\beta_1, \beta_3, \beta_5\}$ to the first three entries of $\mathcal{B}_{6,0,0}$, $\mathcal{B}_{5,1,1}$, $\mathcal{B}_{4,2,2}$ and $\mathcal{B}_{3,3,2}$ and fix the remaining β_j are all invertible over $\mathbb{Z}$, $\mathcal{B}_{6,0,0}$, $\mathcal{B}_{5,1,1}$, $\mathcal{B}_{4,2,2}$ and $\mathcal{B}_{3,3,2}$ are all subsets of bases for L_{K3}; thus, they generate primitive embeddings of $L_{6,0,0}$, $L_{5,1,1}$, $L_{4,2,2}$ and $L_{3,3,2}$ in L_{K3}. □

2.4. Contradictions in pure Picard lattices of high rank. Fix an ordered triple (k,l,m) of non-negative integers such that $(k,l,m) \notin \mathcal{N}$. Up to reordering, one of $(k,l,m) \geq (7,0,0)$, $(k,l,m) \geq (6,1,0)$, $(k,l,m) \geq (5,2,0)$ or $(k,l,m) \geq (4,3,0)$ is true. Taking $\widetilde{S} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as above, we use the arrangement of the curves parallel to axes in $\widetilde{S}$ to show that there is no pure K3 surface whose Picard lattice is isometric to $L_{k,l,m}$.

Proposition 2.9. *There is no pure* K3 *surface of type* (k,l,m) *in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Since $L_{7,0,0}$ is isometric to

$$\langle E_x, E_y, E_z, [C_7], [C_8], [C_{11}], [C_{12}], [C_{19}], [C_{20}], [C_{23}] \rangle,$$

which contains $[C_{24}]$, $L_{6,1,0}$ is isometric to

$$\langle E_x, E_y, E_z, [C_7], [C_8], [C_{11}], [C_{12}], [C_{19}], [C_{20}], [C_{21}] \rangle,$$

which contains $[C_{22}]$, $L_{5,2,0}$ is isometric to

$$\langle E_x, E_y, E_z, [C_7], [C_8], [C_{11}], [C_{19}], [C_{20}], [C_{21}], [C_{22}] \rangle,$$

which contains $[C_{12}]$ and $L_{4,3,0}$ is isometric to

$$\langle E_x, E_y, E_z, [C_7], [C_8], [C_9], [C_{10}], [C_{19}], [C_{20}], [C_{21}] \rangle,$$

which contains $[C_{22}]$, each of these lattices contains an element γ_0 which satisfies

$$(\gamma_0 \cdot \gamma_0) = -2, \qquad (\gamma_0 \cdot E_{\omega'}) = 1$$

for some $E_{\omega'}$, $(\gamma_0 \cdot E_{\omega \neq \omega'}) = 0$ and $(\gamma_0 \cdot [C_j]) \geq 0$ for all $[C_j]$ in the given basis.

Suppose that $S \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is pure of type (k,l,m); thus, in light of the natural embedding of one of the lattices listed above in $L_{k,l,m}$, there must be a $\gamma_0 \in \operatorname{Pic}(S)$ with the properties described above and, moreover, the property that $\gamma_0 \notin \mathcal{B}(S)$. Since γ_0 is effective and is in $\langle E_{\omega_1}, E_{\omega_2}\rangle^{\perp}$ for some distinct E_{ω_1} and E_{ω_2}, it is a sum

$$\gamma_0 = [D_1] + \cdots + [D_n]$$

of (a priori, not necessarily distinct) classes of prime divisors all satisfying $(D_j \cdot D_j) = -2$ and $D_j \in \langle E_{\omega_1}, E_{\omega_2}\rangle^{\perp}$. Then, as in the proof of Proposition 2.4, each D_j must be a curve parallel to an axis, which leads to a contradiction. □

2.5. Proof of Theorem 1.2. Suppose that

$$(k, l, m) \in \mathcal{N} - \{(3, 3, 3)\}.$$

By Proposition 2.8, there is a primitive lattice embedding $L_{k,l,m} \leq L_{K3}$. Since $L^{\perp}_{k,l,m}$ has signature

$$(2, 17 - k - l - m),$$

$L^{\perp}_{k,l,m} \otimes \mathbb{R}$ contains a positive definite two-dimensional subspace V such that

$$V^{\perp} \cap L_{K3} = L_{k,l,m}.$$

Thus, the surjectivity of the period map for K3 surfaces, e.g., [**4, 7**], implies, with an application of the Leschetz theorem on (1,1) classes, that there is a K3 surface S with $\mathrm{Pic}(S)$ isometric to $L_{k,l,m}$. Moreover, the isometry between $\mathrm{Pic}(S)$ and $L_{k,l,m}$ may be taken to be effective for any allowable choice of Γ^{+}. Therefore, by Propositions 2.4 and 2.7, there is a pure K3 surface of type (k, l, m). In fact, since it has been established that at least one exists, the moduli space $\mathcal{M}(L_{k,l,m})$ of ample $L_{k,l,m}$-polarized K3 surfaces (with Γ^{+} fixed), e.g., [**4, 7**], is a quasi-projective variety of dimension $17 - k - l - m$. For every $T \in \mathcal{M}(L_{k,l,m})$, there is an effective primitive lattice embedding $L_{k,l,m} \leq \mathrm{Pic}(T)$; thus, either T is pure of type (k, l, m) or $\rho(T) > 3+k+l+m$ and $T \in \mathcal{M}(\mathrm{Pic}(T))$. Since there are only countably many possible such $\mathrm{Pic}(T)$ which are not effectively isometric to $L_{k,l,m}$ and the dimension of $\mathcal{M}(\mathrm{Pic}(T))$ for each of these is less than $17-k-l-m$, the space $\mathcal{M}_0(L_{k,l,m})$ of K3 surfaces S with $\mathrm{Pic}(S)$ effectively isometric to $L_{k,l,m}$ is very general in $\mathcal{M}(L_{k,l,m})$. By Propositions 2.4 and 2.7, $\mathcal{M}_0(L_{k,l,m})$ is the space of isomorphism classes of K3 surfaces contained in $\mathcal{U}_{k,l,m}$.

Let $\mathcal{V}_{k,l,m} \subseteq \mathbb{P}^{26}$ denote the space of all effective divisors of tri-degree $(2,2,2)$ whose supports contain some union of curves

$$C_{x,1} \cup \cdots \cup C_{x,k} \cup C_{y,1} \cup \cdots \cup C_{y,l} \cup C_{z,1} \cup \cdots \cup C_{z,m}$$

so that each $C_{\omega,j}$ is a curve parallel to the ω-axis and any two distinct $C_{\omega,j}$ and $C_{\omega',j'}$ are disjoint, and let $\mathcal{I}_{k,l,m}$ denote the incidence variety in

$$\begin{aligned}&\mathbb{P}^{26} \times (\mathbb{P}^1 \times \mathbb{P}^1)^{k+l+m} \\ &\quad = \{(Q, [\alpha_{x,1} : \beta_{x,1}], [\delta_{x,1} : \epsilon_{x,1}], \ldots, [\alpha_{z,m} : \beta_{z,m}], [\delta_{z,m} : \epsilon_{z,m}])\},\end{aligned}$$

defined by

$$\begin{aligned}Q_{\omega,0}(\alpha_{\omega,j}, \beta_{\omega,j}, \delta_{\omega,j}, \epsilon_{\omega,j}) &= Q_{\omega,1}(\alpha_{\omega,j}, \beta_{\omega,j}, \delta_{\omega,j}, \epsilon_{\omega,j}) \\ &= Q_{\omega,2}(\alpha_{\omega,j}, \beta_{\omega,j}, \delta_{\omega,j}, \epsilon_{\omega,j}) \\ &= 0\end{aligned}$$

for all ω and j. Since $\mathcal{V}_{k,l,m}$ is the image under the projection to $\mathbb{P}^{26}$ of a complement

$$\mathcal{V}'_{k,l,m} \subseteq \mathcal{I}_{k,l,m}$$

of finitely many sections from linear subspaces of $(\mathbb{P}^1 \times \mathbb{P}^1)^{k+l+m}$, it is a quasi-projective variety. For a fixed point

$$\zeta \in (\mathbb{P}^1 \times \mathbb{P}^1)^{k+l+m},$$

the equations defining $\mathcal{I}_{k,l,m}$ show that the fiber over ζ of the projection of $\mathcal{I}_{k,l,m}$ to $(\mathbb{P}^1 \times \mathbb{P}^1)^{k+l+m}$ is a linear subspace of $\mathbb{P}^{26}$ of codimension at most $3(k+l+m) \leq 24$. Since the projection of $\mathcal{V}'_{k,l,m}$ to $(\mathbb{P}^1 \times \mathbb{P}^1)^{k+l+m}$ is Zariski dense, it follows that $\mathcal{V}_{k,l,m}$ is irreducible. By the construction of $\mathcal{V}_{k,l,m}$, $\mathcal{U}_{k,l,m}$ is very general in $\mathcal{V}_{k,l,m}$. Thus, the closure of $\mathcal{U}_{k,l,m}$ contains $\mathcal{V}_{k',l',m'}$ for all $(k',l',m') \in \mathcal{N}$ satisfying $(k,l,m) < (k',l',m')$.

The claim for $(k,l,m) \notin \mathcal{N}$ is given by Proposition 2.9. □

3. Computing entropies on pure K3 surfaces. Fix $S \in \mathcal{U}_{k,l,m}$ for some $(k,l,m) \in \mathcal{N}$. It is a well-known fact, e.g., **[5]**, that every birational self-map on S extends to an automorphism of S. Therefore, in particular, each τ_ω, and hence, also f, defines an automorphism of S.

3.1. Cohomological actions of involutions. We compute the action of τ_x^* on $\mathrm{Pic}(S)$; the actions of τ_y^* and τ_z^* are similar. Write

$$\mathcal{B}_x(S) = \{C_{x,p_1}, \dots, C_{x,p_k}\}.$$

Proposition 3.1. *Each $[C_{x,p_j}]$ is fixed by τ_x^*, as are E_y and E_z. For each $[C_{y,p}] \in \mathcal{B}(S)$,*

$$\tau_x^*[C_{y,p}] = E_z - [C_{y,p}].$$

For each $[C_{z,p}] \in \mathcal{B}(S)$,

$$\tau_x^*[C_{z,p}] = E_y - [C_{z,p}].$$

Finally,

$$\tau_x^* E_x = -E_x + 2E_y + 2E_z - [C_{x,p_1}] - \cdots - [C_{x,p_k}].$$

Proof. Since $\tau_x = \tau_x^{-1}$ preserves every elliptic curve, which is a fiber over either the y- or the z-axis, τ_X^* must fix E_y and E_z. For $E_{\omega=\alpha}$ containing a curve C parallel to an axis, $E_{\omega=\alpha} - C$ is an effective divisor of bi-degree $(1,2)$ or $(2,1)$. It follows from Remark 2.5 that, in fact, $E_{\omega=\alpha} - C$ is a prime divisor which is not parallel to any axis. For each C_{x,p_j}, write $p_j = (\alpha, \delta)$; since τ_x preserves both $E_{y=\alpha}$ and $E_{z=\delta}$, it must fix C_{x,p_j}. For $[C_{y,p}] \in \mathcal{B}(S)$, write $p = (\alpha, \delta)$; since τ_x preserves $E_{z=\alpha}$ and does not preserve $C_{y,p}$, it must take $C_{y,p}$ to $E_{z=\alpha} - C_{y,p}$. It follows similarly that τ_x^* takes $C_{z,p}$ to $E_{y=\delta} - C_{z,p}$ for $[C_{z,p}] \in \mathcal{B}(S)$.

With the action of τ_x^* established for all elements of $\mathcal{B}(S)$ except E_x, the conditions that τ_x is an involution and τ_x^* preserves the intersection form given by $M_{k,l,m}$ force the formula given for $\tau_x^* E_x$ to hold. □

Proposition 3.1 shows that the action of f^* in the basis $\mathcal{B}(S)$ is constant on $\mathcal{U}_{k,l,m}$ and provides the necessary information for computation of $\lambda(f)$.

Lemma 3.2. *If (k', l', m') is a reordering of (k, l, m), then $\lambda(f)$ is constant on*

$$\mathcal{U}_{k,l,m} \cup \mathcal{U}_{k',l',m'}.$$

Proof. Fix $S' \in \mathcal{U}_{k',l',m'}$. Some

$$g \in \mathcal{G} := \{\tau_z \circ \tau_y \circ \tau_x, \tau_z \circ \tau_x \circ \tau_y, \tau_y \circ \tau_x \circ \tau_z, \tau_y \circ \tau_z \circ \tau_x, \tau_x \circ \tau_z \circ \tau_y, \tau_x \circ \tau_y \circ \tau_z\}$$

has the property that the action of g^* on $\mathrm{Pic}(S')$ is essentially identical to the action of f^* on $\mathrm{Pic}(S)$. Since every element of $\mathcal{G}$ is conjugate, by some element in $\langle \tau_x, \tau_y, \tau_z \rangle$, to either f or f^{-1}, the spectral radius of f^* on $\mathrm{Pic}(S')$ is the same as that of f^* on $\mathrm{Pic}(S)$. □

3.2. Proof of Theorem 1.3. By Proposition 3.1 and Lemma 3.2, the action of f^* on $\mathrm{Pic}(S)$ depends only upon the unordered type of S. Table 1 provides the spectral radius, computed in `Mathematica`, of f^* for all types of S, and it may be verified that $\lambda(k, l, m)$ exhibits the claimed behavior. □

4. Indeterminacy loci in families of pure K3 surface automorphisms. Letting $\mathbb{P}^{26}$ parametrize all polynomials Q that are tri-homogeneous of tri-degree $(2, 2, 2)$,

$$\mathcal{S} := \{(Q, x, y, z) \mid Q(x_0, x_1, y_0, y_1, z_0, z_1) = 0\}$$

is a subvariety of $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose general fibers over $\mathbb{P}^{26}$ are K3 surfaces. Define a birational involution F_x on $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$F_x : (Q, x, y, z) \longmapsto (Q, [x_0 Q_{x,2} + x_1 Q_{x,1} : -x_1 Q_{x,2}], y, z),$$

where each $Q_{x,i}$ is as above; thus, F_x preserves $\mathcal{S}$ and restricts to τ_x on each fiber of $\mathcal{S}$ over $\mathbb{P}^{26}$. We investigate the indeterminacy of F_x considered as a birational self-map on $\mathcal{S}$. We can define and understand F_y and F_z similarly, and thus, also study the indeterminacy of $F := F_z \circ F_y \circ F_x$.

Since the birational self-map on $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, given by

$$(Q, x, y, z) \longmapsto (Q, [-x_0 Q_{x,0} : x_1 Q_{x,0} + x_0 Q_{x,1}], y, z)$$

agrees with F_x everywhere on $\mathcal{S}$ where both are defined, the indeterminacy of F_x on $\mathcal{S}$ is contained in

$$\mathcal{Q} := \{(Q, x, y, z) \mid Q_{x,2} = Q_{x,1} = Q_{x,0} = 0\},$$

which is the union of all of the curves parallel to the x-axis contained in the fibers of $\mathcal{S}$ over $\mathbb{P}^{26}$.

TABLE 1. Spectral radius of f^*.

(k,l,m)	$\lambda(f)$	Min. poly. for $\lambda(f)$
$(0,0,0)$	$17.944\ldots$	$t^2-18t+1$
$(1,0,0)$	$15.937\ldots$	$t^2-16t+1$
$(2,0,0)$	$13.928\ldots$	$t^2-14t+1$
$(3,0,0)$	$11.916\ldots$	$t^2-12t+1$
$(4,0,0)$	$9.898\ldots$	$t^2-10t+1$
$(5,0,0)$	$7.872\ldots$	t^2-8t+1
$(6,0,0)$	$5.828\ldots$	t^2-6t+1
$(1,1,0)$	$14.011\ldots$	$t^4-16t^3+29t^2-16t+1$
$(2,1,0)$	$12.113\ldots$	$t^4-14t^3+24t^2-14t+1$
$(3,1,0)$	$10.261\ldots$	$t^4-12t^3+19t-12t+1$
$(4,1,0)$	$8.487\ldots$	$t^4-10t^3+14t-10t+1$
$(5,1,0)$	$6.854\ldots$	t^2-7t+1
$(2,2,0)$	$10.375\ldots$	$t^4-12t^3+18t^2-12t+1$
$(3,2,0)$	$8.758\ldots$	$t^4-10t^3+12t^2-10t^3+1$
$(4,2,0)$	$7.327\ldots$	$t^4-8t^3+6t^2-8t+1$
$(3,3,0)$	$7.471\ldots$	$t^4-8t^3+5t^2-8t+1$
$(1,1,1)$	$12.113\ldots$	$t^4-14t^3+24t^2-14t+1$
$(2,1,1)$	$10.261\ldots$	$t^4-12t^3+19t-12t+1$
$(3,1,1)$	$8.487\ldots$	$t^4-10t^3+14t-10t+1$
$(4,1,1)$	$6.854\ldots$	t^2-7t+1
$(5,1,1)$	$5.462\ldots$	$t^4-6t^3+4t^2-6t+1$
$(2,2,1)$	$8.487\ldots$	$t^4-10t^3+14t-10t+1$
$(3,2,1)$	$6.854\ldots$	t^2-7t+1
$(4,2,1)$	$5.462\ldots$	$t^4-6t^3+4t^2-6t+1$
$(3,3,1)$	$5.462\ldots$	$t^4-6t^3+4t^2-6t+1$
$(2,2,2)$	$6.678\ldots$	$t^4-8t^3+10t^2-8t+1$
$(3,2,2)$	$5.037\ldots$	$t^4-6t^3+6t^2-6t+1$
$(4,2,2)$	$3.732\ldots$	t^2-4t+1
$(3,3,2)$	$3.441\ldots$	$t^4-4t^3+3t^2-4t+1$
$(3,3,3)$	1	$t-1$

4.1. Indeterminacy over spaces of pure K3 surfaces. For $j \in \mathbb{N}_0$, let $\mathcal{U}_j$ denote the union of all of the spaces $\mathcal{U}_{k,l,m}$ with $k=j$. Let π_x denote the projection from

$$\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

to

$$\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1$$

along the x-axis, given by

$$\pi_x : (Q, x, y, z) \longmapsto (Q, y, z),$$

and let π denote the natural projection from $\mathbb{P}^{26} \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^{26}$.

For $p \in \mathcal{U}_k$, the fiber of π over p intersects $\pi_x(\mathcal{Q})$ in exactly k points (since the pure K3 surface in $\mathcal{S}$ over p contains exactly k curves parallel to the x-axis). Therefore, π changes $\pi_x(\mathcal{Q})$ into a k-fold cover over a subset $\mathcal{U}'_k \subseteq \mathcal{U}_k$ which is general in $\mathcal{U}_k$. This cover extends to a k-fold cover on a general subset of each $\mathcal{U}_j$ with $j > k$, and in fact, gives part of a j-fold cover on a (possibly further restricted) general subset of each $\mathcal{U}_j$.

Proposition 4.1. *Fix $k_0 \in \mathbb{N}_0$, and consider F_x as a birational self-map on the intersection of $\mathcal{S}$ with the closure of $\mathcal{U}_{k_0} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then*:

(a) *the indeterminacy locus of F_x misses every fiber of $\pi \circ \pi_x$ over $\mathcal{U}'_{k_0}$, and*

(b) *the indeterminacy locus of F_x intersects a fiber of $\pi \circ \pi_x$ along precisely $k - k_0$ curves parallel to the x-axis whenever the fiber is over a point in $\mathcal{U}_k$ with $k > k_0$ to which the k_0-fold cover of $\mathcal{U}'_{k_0}$ by $\pi_x(\mathcal{Q})$ extends.*

Proof. For any particular K3 surface in $\mathcal{S}$ containing some $C_{x,(y',z')}$, Baragar **[2]** explicitly showed how to extend the involution τ_x to the curve: assuming $y'_1 \neq 0$ and $z'_1 \neq 0$ (and otherwise proceeding similarly with appropriate modifications), set

$$\zeta = [(z_0 z'_1 - z_1 z'_0) y_1 y'_1 : (y_0 y'_1 - y_1 y'_0) z_1 z'_1]$$

such that

$$z = [\zeta_0 z'_1 (y_0 y'_1 - y_1 y'_0) + \zeta_1 y_1 y'_1 z'_0 : \zeta_1 y_1 y'_1 z'_1].$$

Each $Q_{x,i}(y, z)$ may be written as a polynomial in y and ζ that vanishes along $\{y = y'\}$, and thus, the coordinate polynomials defining τ_x in terms of x, y and ζ can be reduced by the common factor $(y_0 y'_1 - y_1 y'_0)$; in these terms, the extension of τ_x to $C_{x,(y',z')}$ is apparent.

Now consider a neighborhood

$$N \subseteq \pi_x(\mathcal{Q})$$

over

$$\bigcup_{k \geq k_0} \mathcal{U}_k,$$

on which π is injective. Taking $(y', z') \in N$, the procedure involving ζ shows that F_x extends to all of the curves $C_{x,(y',z')}$ in $\pi_x^{-1}(N)$. Thus, F_x extends to every curve parallel to the x-axis whose image under π_x is in the k_0-fold cover $\mathcal{U}_{k_0}$ by $\pi_x(\mathcal{Q})$ or its extension to a general subset $\mathcal{V}$ of $\cup_{k>k_0}\mathcal{U}_k$.

For

$$p \in \mathcal{U}_k \cap \mathcal{V},$$

there are exactly $k - k_0$ curves parallel to the x-axis in

$$(\pi \circ \pi_x)^{-1}(\{p\})$$

that are not accounted for by the preceding construction; now, let $C = \mathcal{C}_{x,(y',z')}$ denote one such curve. For every

$$p' \in C \setminus \{([0:1], y', z'), ([1:0], y', z')\},$$

there is a neighborhood $N \subseteq \mathcal{S}$ containing p' such that $\{Q_{x,0}\}$ and $\{Q_{x,2}\}$ have an empty intersection in N over $\mathcal{U}_{k_0}$; it follows that their intersection has codimension 2 in N over $\cup_{k \geq k_0}\mathcal{U}_k$, while individually each set has codimension 1. Thus, there is a path in N meeting p' along which $Q_{x,0} = 0$ and $Q_{x,2} \neq 0$, and another such path along which the opposite holds. Then, the earlier observation that F_x restricted to $\mathcal{S}$ may be written in two distinct ways shows that it cannot extend to C. □

Remark 4.2. The fact that F has some indeterminacy over each point in $\mathcal{U}_k$ with $k > k_0$ in Proposition 4.1 is crucial to the result that the entropy of f changes at these points. It is well understood that an automorphism preserving a non-singular fibration by complex surfaces cannot exhibit a change in entropy on any fiber.

4.2. Indeterminacy over one-parameter families. Now suppose that $W \subseteq \mathbb{P}^{26}$ is a smooth, irreducible curve having infinite intersection with some $\mathcal{U}_{k_0}$, and consider F_x as a birational self-map on the three-dimensional intersection of $\mathcal{S}$ with

$$W \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Thus, F_x has no indeterminacy over $\mathcal{U}_{k_0}$ and has indeterminacy precisely along finitely many pairwise disjoint curves parallel to the x-axis over any point in $W \cap \mathcal{U}_k$ with $k > k_0$. It follows from a result of Kollár, [**10,** Theorem 11] that, locally, F_x must be a composition of flops in a neighborhood of any one of these curves.

Proposition 4.3. *Suppose that C is a curve in the indeterminacy locus of F_x over a point in $W \cap \mathcal{U}_k$ with $k > k_0$. Then, F_x is a single flop in a neighborhood of C.*

Proof. From [**10,** Definitions 2, 3], to prove that a neighborhood of C admits a single flop to itself, it is sufficient to find line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on a neighborhood of C with the following properties:

(a) $\mathcal{L}_1|_C$ and $\mathcal{L}_2|_C$ are both ample; and
(b) $\mathcal{L}_1$ is isomorphic to $-\mathcal{L}_2$ after C is removed from the neighborhood.

Assuming that (a) and (b) are satisfied, then there is exactly one flop from a neighborhood of C to itself. Therefore, F_x locally must be this flop (composed with an isomorphism).

Let $\mathcal{L}_x$ denote the line bundle on $\mathcal{S}$ whose restriction to every K3 surface in $\mathcal{S}$ is E_x, and let $\mathcal{L}_y$ and $\mathcal{L}_z$ be similar; then take these line bundles to be their restrictions to a neighborhood of C over W. Set

$$\mathcal{L}_1 = \mathcal{L}_x - 2\mathcal{L}_y - 2\mathcal{L}_z$$

and

$$\mathcal{L}_2 = F_x^* \mathcal{L}_x.$$

Then, $\mathcal{L}_1$ clearly restricts to $\mathcal{O}(1)$ on C, and it follows from Proposition 3.1 that $\mathcal{L}_2$ does the same. Proposition 3.1 shows, moreover, that $\mathcal{L}_2$ restricts to

$$-\mathcal{L}_x + 2\mathcal{L}_y + 2\mathcal{L}_z$$

on a suitable neighborhood of C with C removed. ∎

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