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# BLOW-UP OF MULTI-COMPONENTIAL SOLUTIONS IN HEAT EQUATIONS WITH EXPONENTIAL BOUNDARY FLUX

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ABSTRACT. This paper deals with heat equations coupled via exponential boundary flux, where the solution is made up of $n$ components. Under certain monotone assumptions, necessary and sufficient conditions are obtained for simultaneous blow-up of at least two components for each initial datum. As for two components blowing up simultaneously, it is interesting that the representations of blow-up rates are quite different with respect to the different blow-up mechanisms and positions between the two components.

**1. Introduction and main results.** In this paper, we consider the multi-componential solutions of heat equations with coupled nonlinear boundary flux, taken of the form

(1.1)

$$\begin{cases} (u_i)_t = \Delta u_i & (x,t) \in B_R \times (0,T), \\ \frac{\partial u_i}{\partial \eta} = \exp\{p_i u_i + q_{i+1} u_{i+1}\} & (x,t) \in \partial B_R \times (0,T), \\ u_i(x,0) = u_{i,0}(x) \quad i = 1,2,\ldots,n,\, n \geq 2 & x \in B_R, \end{cases}$$

where $u_{n+1} := u_1$, $p_{n+1} := p_1$, $q_{n+1} := q_1$, $B_R = \{x \in \mathbf{R}^N \mid |x| < R\}$, constant exponents $p_i, q_i \geq 0$, $i = 1,2,\ldots,n$, $u_{i,0}(x)$, $i = 1,2,\ldots,n$ are positive, smooth and radially symmetric functions satisfying the compatibility conditions on the boundary. The existence and uniqueness of local classical solutions to (1.1) and the comparison principle are well known, see [**4**]. Let $T$ be the maximal existence time of system (1.1). The $n$-componential parabolic systems such as (1.1) derive from chem-

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ical reactions, heat transfer, population dynamics, etc., which describe the phenomena in real-life terms more precisely than the parabolic systems with only two components, see, for example, [**10, 15**]. The components $u_1, u_2, \dots, u_n$ represent, for example, concentrations of the chemical reactants, temperatures of the materials during heat propagations and the densities of biological populations during migrations. For more detailed information, the interested reader may refer to [**9, 14**].

Zhao and Zheng [**17**] considered radially symmetric solutions of the system

$$(1.2)\qquad \begin{cases} u_t = \Delta u, \quad v_t = \Delta v & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = \exp\{p_1 u + q_2 v\} & (x,t) \in \partial B_R \times (0,T), \\ \frac{\partial v}{\partial \eta} = \exp\{p_2 u + q_1 v\} & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x \in B_R. \end{cases}$$

Simultaneous blow-up rates are obtained in the regions $q_1 > p_1 \geq 0$ and $q_2 > p_2 \geq 0$ as follows:

$$\exp\{u(R,t)\} \sim (T-t)^{-(q_2-p_2)/(2(q_1q_2-p_1p_2))},$$
$$\exp\{v(R,t)\} \sim (T-t)^{-(q_1-p_1)/(2(q_1q_2-p_1p_2))},$$

where $f \sim g$ denotes that some positive constants $c$ and $C$ exist such that $cf \leq g \leq Cf$. It is also proved that the blow-up can only occur on the boundary of the space domain.

Non-simultaneous blow-up for parabolic systems has merited much attention, see, for example, [**1, 2, 11, 12, 13, 16, 18, 19, 20**]. Recently, Fan and Du [**3**] considered the parabolic system

$$(1.3)\qquad \begin{cases} u_t = u_{xx}, \quad v_t = v_{xx} & (x,t) \in (0,1) \times (0,T), \\ -u_x(0,t) = \exp\{p_1 u(0,t) + q_2 v(0,t)\} & t \in (0,T), \\ -v_x(0,t) = \exp\{q_1 u(0,t) + p_2 v(0,t)\} & t \in (0,T), \\ u_x(1,t) = v_x(1,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x \in [0,1], \end{cases}$$

where exponents $p_i, q_i \geq 0$, $i = 1, 2$, $p_1 + p_2 > 0$, $q_1 + q_2 > 0$, under the conditions

$$u_0(x), v_0(x) \geq \delta_1 > 0, \qquad u_0'(x), v_0'(x) \leq 0, \qquad u_0''(x), v_0''(x) \geq \delta_2 > 0,$$

$x \in [0, 1]$, for some constants $\delta_1$ and $\delta_2$. Fan and Du obtained **[3]** that:

(i) there exists initial data such that non-simultaneous blow-up occurs if and only if $q_1 < p_1$ or $q_2 < p_2$. Hence, $u$ and $v$ blow up simultaneously if $q_1 \geq p_1$ and $q_2 \geq p_2$;

(ii) if $q_1 < p_1$ and $q_2 < p_2$, both simultaneous and non-simultaneous blow-up may occur for suitable initial data;

(iii) let $p_1 > 0$ and $p_2 > 0$. If $q_1 < p_1$ and $q_2 \geq p_2$ (or $q_1 \geq p_1$ and $q_2 < p_2$), then $u$ (or $v$) blows up alone for each initial datum. The blow-up phenomenon in the exponent region $q_1 < p_1$ and $p_2 = 0$ or $q_2 < p_2$ and $p_1 = 0$ is unsettled in **[3]**. We obtained in **[8]** that only non-simultaneous blow-up occurs in that region, which completes the classifications of simultaneous versus non-simultaneous blow-up for (1.3). Moreover, we proved the blow-up set is made up of only a single point, and the solutions always blow up completely. It is interesting that, even if non-simultaneous blow-up occurs, thermal avalanche also occurs.

For $n$-componential parabolic systems, Pedersen and Lin **[10]** and Wang **[15]**, respectively, discussed the heat equations coupled via nonlinear boundary flux

$$\frac{\partial u_i}{\partial \eta} = u_{i+1}^{q_{i+1}}, \quad i = 1, 2, \ldots, n.$$

It may be verified that any blow-up must be simultaneous. If $q_1 q_2 \cdots q_n > 1$, simultaneous blow-up rates of solutions are obtained.

In **[7]**, non-simultaneous blow-up phenomena were studied for heat equations with coupled nonlinear boundary flux

$$\frac{\partial u_i}{\partial \eta} = {u_i}^{p_i} {u_{i+1}}^{q_{i+1}}, \quad (x, t) \in B_R \times (0, T).$$

Three types of non-simultaneous versus simultaneous blow-up phenomena were discussed.

Inspired by **[7, 8]**, we consider the non-simultaneous versus simultaneous blow-up of system (1.1) which enlarges from two components in **[17]** to $n$ components. In the present paper, the non-simultaneous blow-up for $n$ components means that at least one component of the $n$s still remains bounded up to blow-up time. The initial data positions of and relationship among the components play important roles in the blow-up mechanism of the $n$ components, respectively, see [**7**]. Even in the same exponent region, different initial data can lead to different blow-up phenomena and blow-up rates, see [**8**]. Moreover, the blow-up set and classifications are important to the existence of avalanching phenomena after blow-up time of the solutions, see [**8**]. What occurs to the corresponding blow-up phenomena of system (1.1) coupled via exponential nonlinearities? To our knowledge, such problems for (1.1) have not previously been considered and are worthy of study.

Rossi [**13**] showed that any positive solutions of (1.1) blow up if and only if

$$\max\left\{p_i,\ i=1,2,\dots,n,\ \prod_{j=1}^{n} q_j\right\} > 0,$$

where the blow-up means that

$$\sum_{i=1}^{n} \|u_i\|_\infty \longrightarrow +\infty \quad \text{as } t \to T.$$

In the sequel, only blow-up solutions of (1.1) will be considered.

The main results of the present paper, in general, are as follows. Let $\xi_i := \xi_{i+n}$ if the subscript $i \le 0$. Denote a set

$$\begin{aligned}(1.4)\quad \mathbb{V}_0 = \Big\{&(u_{1,0}, u_{2,0}, \dots, u_{n,0}) \mid u_{i,0} \ge \zeta > 0,\ (u_{i,0})_r \ge 0,\\ &(u_{i,0})_{rr} + \frac{N-1}{r}(u_{i,0})_r \ge 0,\ r \in [0,R),\\ &\frac{\partial u_{i,0}(R)}{\partial \eta} = \exp\{p_i u_{i,0}(R) + q_{i+1} u_{i+1,0}(R)\},\ 1 \le i \le n.\Big\}\end{aligned}$$

The conditions in $\mathbb{V}_0$ are reasonable, guaranteeing the monotonicity and compatibility conditions on the boundary for solutions. Such conditions may also be found in [**2, 5, 17**], etc. Clearly,

$$U_i(t) = u_i(R,t) = \max\{u_i(y,s),\ (y,s) \in [0,R] \times [0,t]\}, \quad 1 \le i \le n.$$

**Theorem 1.1.** *At least two components of the ns blow up simultaneously for each initial datum in* $\mathbb{V}_0$ *if and only if* $p_i \le q_i$, $i = 1, 2, \dots, n$.

**Corollary 1.2.** *There exist suitable initial data such that* $u_i$, $i \in \{1, 2, \ldots, n\}$, *blows up while the other* $(n-1)$*s remain bounded if and only if* $q_i < p_i$. *Moreover,* $\exp\{U_i(t)\} \sim (T-t)^{-1/(2p_i)}$.

**Theorem 1.3.**

• *Assume that* $n \geq 3$. *Let* $i \in \{1, 2, \ldots, n\}$ *and* $k \in \{1, 2, \ldots, n-1\}$. *If* $q_i < p_i$ *and* $q_{i-k} < p_{i-k}$, *then suitable initial data exist such that* $u_{i-k}$ *and* $u_i$ *blow up simultaneously while the others remain bounded up to the blow-up time* $T$. *Moreover,*

$$(\exp\{U_{i-k}(t)\}, \exp\{U_i(t)\}) \sim \begin{cases} ((T-t)^{-(p_i-q_i)/(2p_ip_{i-1})}, (T-t)^{-1/(2p_i)}) \\ \qquad \text{for } k=1; \\ ((T-t)^{-1/(2p_{i-k})}, (T-t)^{-1/(2p_i)}) \\ \qquad \text{for } k \in \{2, 3, \ldots, n-2\}; \\ ((T-t)^{-1/(2p_{i-k})}, (T-t)^{-(p_{i-k}-q_{i-k})/(2p_{i-k}p_i)}) \\ \qquad \text{for } k = n-1. \end{cases}$$

• *Assume that* $n = 2$. *If* $q_1 < p_1$ *and* $q_2 < p_2$, *then there exist suitable initial data such that* $u_1$ *and* $u_2$ *blow up simultaneously at time* $T$. *Moreover,*

$$(\exp\{U_1(t)\}, \exp\{U_2(t)\}) \sim \left((T-t)^{-(q_2-p_2)/(2(q_2q_1-p_1p_2))}, (T-t)^{-(q_1-p_1)/(2(q_2q_1-p_1p_2))}\right).$$

**Remark 1.4.** For $n = 2$, Theorem 1.3 (ii) gives a new exponent region $\{q_1 < p_1,\ q_2 < p_2\}$ for the system (1.2) considered in [**17**], where simultaneous blow-up may occur, with the blow-up rates of the same form that

$$(\exp\{U_1(t)\}, \exp\{U_2(t)\}) \sim ((T-t)^{-(q_2-p_2)/(2(q_2q_1-p_1p_2))},\ (T-t)^{-(q_1-p_1)/(2(q_2q_1-p_1p_2))}).\square$$

**Remark 1.5.** It may be seen that, with the parameter region $\{q_i < p_i,\ q_{i-k} < p_{i-k}\}$, there exist suitable initial data such that $u_i$ (or $u_{i-k}$)

blows up alone by Corollary 1.2, and there also exist other particular initial data such that $u_{i-k}$ and $u_i$ blow up simultaneously. □

Similarly to [**5,** Theorem 4.8], we have the blow-up set estimates, provided that the upper bounds of blow-up rates are obtained:

**Theorem 1.6.** *If $u_i$ blows up with $U_i(t) \le C(T-t)^{-\alpha}$ for any $i \in \{1,2,\dots,n\}$ and constant $\alpha > 0$, then the blow-up can only occur on the boundary.*

In the next two sections, Theorems 1.1 and 1.3, respectively, will be proved.

**2. Proof of Theorem 1.1.** In order to prove Theorem 1.1, it suffices to prove Corollary 1.2. We introduce a lemma for some estimate of $u_i$.

**Lemma 2.1.** *Let $T$ be the blow-up time of system* (1.1). *If $p_i > 0$, then*

$$(u_i)_t(R,t) \ge \varepsilon \mathrm{e}^{2p_i u_i(R,t)} \mathrm{e}^{2q_{i+1} u_{i+1}(R,t)} \tag{2.1}$$

*for the initial data satisfying $(u_{i,0})_{rr} + ((N-1)/r)(u_{i,0})_r \ge \varepsilon[(u_{i,0})_r]^2$. Hence, there exists some constant $C_0 > 0$ such that*

$$\mathrm{e}^{U_i(t)} \le C_0(T-t)^{-1/(2p_i)},$$

*where $C_0$ depends only upon $\varepsilon$ and $p_i$.*

*Proof.* It can be seen by the comparison principle that $(u_i)_t \ge 0$ due to

$$(u_{i,0})_{rr} + \frac{N-1}{r}(u_{i,0})_r \ge 0.$$

Define $J_i(r,t) = (u_i)_t - \varepsilon[(u_i)_r]^2$. For small $\varepsilon > 0$, it may be verified that

$$(J_i)_t - (J_i)_{rr} - \frac{N-1}{r}(J_i)_r \ge 2\varepsilon \frac{N-1}{r^2}\, u_r^2 \ge 0,$$
$$(r,t) \in (0,R) \times (0,T);$$

$$(J_i)_r(R) \ge \mathrm{e}^{p_i u_i(R,t)} \mathrm{e}^{q_{i+1} u_{i+1}(R,t)}$$

$$\times[(p_i - 2\varepsilon)(u_i)_t(R,t) + q_{i+1}(u_{i+1})_t(R,t)] \geq 0, \quad t \in (0,T);$$

$$J_i(r,0) = (u_{i,0})_{rr} + \frac{N-1}{r}(u_{i,0})_r - \varepsilon[(u_{i,0})_r]^2 \geq 0, \quad r \in (0,R).$$

By the comparison principle, (2.1) holds. Then, $e^{U_i(t)} \leq C_0(T-t)^{-1/(2p_i)}$ is obtained. □

*Proof of Corollary* 1.2. Without loss of generality, we prove only the case for $i = n$. We first prove the sufficient condition. Let $G(x,y,t,\tau)$ be Green's function of the heat equation in $B_R$, satisfying

$$\left.\frac{\partial G}{\partial \eta}\right|_{\partial B_R} = 0 \quad \text{and} \quad \int_{\partial B_R} G(x,y,t,\tau)\, dS_y \leq \overline{C}(t-\tau)^{-1/2},$$

where the constant $\overline{C} > 0$ depends only upon $B_R$, see **[6, 9]**.

Fix $u_{1,0}(R), u_{2,0}(R), \ldots, u_{n-1,0}(R)$, and then take $M_m > 2u_{m,0}(R)$, $m = 1,2,\ldots,n-1$. Choose $(u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in \mathbb{V}_0$ such that $T$ is small and satisfies

$$2u_{m,0}(R) + 2\overline{C}\mathrm{e}^{q_{m+1}M_{m+1}}\mathrm{e}^{p_m M_m}T^{1/2} < M_m, \quad m = 1,2,\ldots,n-2,$$

$$2u_{n-1,0}(R) + \frac{2p_n}{p_n - q_n}\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}T^{(p_n-q_n)/(2p_n)} < M_{n-1}.$$

Consider the auxiliary problem

$$\begin{cases} (\overline{u}_{n-1})_t = \Delta\overline{u}_{n-1} & (x,t) \in B_R \times (0,T), \\ \frac{\partial \overline{u}_{n-1}}{\partial \eta} = \mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}(T-t)^{-q_n/(2p_n)} & (x,t) \in \partial B_R \times (0,T), \\ \overline{u}_{n-1}(x,0) = \overline{u}_{n-1,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric initial data $\overline{u}_{n-1,0}$ satisfies that

$$\left.\frac{\partial \overline{u}_{n-1,0}}{\partial \eta}\right|_{\partial B_R} = \mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}T^{-q_n/(2p_n)},$$

$\overline{u}_{n-1,0}(R) = 2u_{n-1,0}(R)$; $\Delta\overline{u}_{n-1,0} \geq 0$, $\overline{u}_{n-1,0} \geq u_{n-1,0}$ in $B_R$. For $q_n < p_n$, we have

$$\overline{u}_{n-1} \leq 2u_{n-1,0}(R) + \frac{2p_n}{p_n - q_n}\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}T^{(p_n-q_n)/(2p_n)} \leq M_{n-1}.$$

Thus, $\overline{u}_{n-1}$ satisfies

$$\begin{cases}(\overline{u}_{n-1})_t = \Delta \overline{u}_{n-1} & (x,t) \in B_R \times (0,T),\\ \frac{\partial \overline{u}_{n-1}}{\partial \eta} \geq C_0^{q_n}(T-t)^{-q_n/(2p_n)} \mathrm{e}^{p_{n-1}\bar{u}_{n-1}} & (x,t) \in \partial B_R \times (0,T),\\ \overline{u}_{n-1}(x,0) = \overline{u}_{n-1,0}(x) & x \in B_R.\end{cases}$$

From Lemma 2.1, $\mathrm{e}^{U_n(t)} \leq C_0(T-t)^{-1/(2p_n)}$. Then, $u_{n-1}$ satisfies (2.2)

$$\begin{cases}(u_{n-1})_t = \Delta u_{n-1} & (x,t) \in B_R \times (0,T),\\ \frac{\partial u_{n-1}}{\partial \eta} \leq C_0^{q_n}(T-t)^{-q_n/(2p_n)} \mathrm{e}^{p_{n-1}u_{n-1}} & (x,t) \in \partial B_R \times (0,T),\\ u_{n-1}(x,0) = u_{n-1,0}(x) & x \in B_R.\end{cases}$$

By the comparison principle, $u_{n-1} \leq \overline{u}_{n-1} \leq M_{n-1}$ on $\overline{B}_R \times [0,T)$. We introduce the problem

$$\begin{cases}(\overline{u}_{n-2})_t = \Delta \overline{u}_{n-2} & (x,t) \in B_R \times (0,+\infty),\\ \frac{\partial \overline{u}_{n-2}}{\partial \eta} = \mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}M_{n-2}} & (x,t) \in \partial B_R \times (0,+\infty),\\ \overline{u}_{n-2}(x,0) = \overline{u}_{n-2,0}(x) & x \in B_R,\end{cases} \tag{2.3}$$

where radial $\overline{u}_{n-2,0}(x)$ satisfies

$$\frac{\partial \overline{u}_{n-2,0}}{\partial \eta} = \mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}M_{n-2}}, \quad \overline{u}_{n-2,0} = 2u_{n-2,0} \quad \text{on } \partial B_R;$$

$$\Delta \overline{u}_{n-2,0} \geq 0,\ \overline{u}_{n-2,0} \geq u_{n-2,0} \quad \text{in } B_R.$$

Considering system (2.3) in $[0,T)$, we have

$$\overline{u}_{n-2} \leq 2u_{n-2,0}(R) + 2\overline{C}\mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}M_{n-2}} T^{1/2} \leq M_{n-2}.$$

Then, $\overline{u}_{n-2}$ satisfies $(\partial \overline{u}_{n-2})/\partial \eta \geq \mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}\overline{u}_{n-2}}$, $(x,t) \in \partial B_R \times (0,T)$. Since $u_{n-1} \leq M_{n-1}$, $u_{n-2}$ satisfies $(\partial u_{n-2})/\partial \eta \leq \mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}u_{n-2}}$ for $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_{n-2} \leq \overline{u}_{n-2} \leq M_{n-2}$ on $\overline{B}_R \times [0,T)$. The boundedness of $u_{n-3}, u_{n-4}, \ldots, u_1$ can be similarly proved. Thus, only $u_n$ blows up.

Now, we prove the necessary condition. Assume $u_1 \leq C$. Then, $u_n$ satisfies that

$$\begin{cases}(u_n)_t = \Delta u_n & (x,t) \in B_R \times (0,T),\\ \frac{\partial u_n}{\partial \eta} \leq \mathrm{e}^{q_1 C} \mathrm{e}^{p_n u_n} & (x,t) \in \partial B_R \times (0,T),\\ u_n(x,0) = u_{n,0}(x) & x \in B_R.\end{cases} \tag{2.4}$$

By Green's identity, we have $U_n(t) \le U_n(z) + 2\overline{C} \mathrm{e}^{q_1 C} \mathrm{e}^{p_n U_n(t)} (T-z)^{1/2}$, $z < t < T$. Take $z$ such that $U_n(z) + C' = U_n(t)$ for some $C' > 0$. Then $\mathrm{e}^{U_n(z)} \ge c(T-z)^{-1/(2p_n)}$, $z \in (0,T)$. Also, by Green's identity,

$$\frac{1}{2} U_{n-1}(t) \ge c \int_0^t (T-\tau)^{-q_n/(2p_n)} (t-\tau)^{-1/2} d\tau.$$

The boundedness of $u_{n-1}$ requires that $q_n < p_n$. □

**3. Proof of Theorem 1.3.** In this section, we discuss the existence of merely two components blowing up simultaneously. Without loss of generality, we prove only the case for $i = n$. We divide Theorem 1.3 for $n \ge 3$ into three subcases: $k = 1$; $k \in \{2, 3, \dots, n-2\}$; $k = n-1$. Firstly, we deal with the subcase $k = 1$.

**Proposition 3.1.** *If $q_n < p_n$ and $q_{n-1} < p_{n-1}$, then suitable initial data exist such that $u_{n-1}$ and $u_n$ blow up simultaneously at some time $T$ while the others remain bounded up to $T$. Moreover,*

$$(\mathrm{e}^{U_{n-1}(t)}, \mathrm{e}^{U_n(t)}) \sim ((T-t)^{-p_n - q_n/(2p_n p_{n-1})}, (T-t)^{-1/(2p_n)}).$$

Next, we introduce a subset of $\mathbb{V}_0$ as follows:

$$\begin{aligned} \mathbb{V}_1 = \Big\{ &(u_{1,0}(r), u_{2,0}(r), \dots, u_{n,0}(r)) \mid u_{m,0}(r) \\ &= N_m + \frac{R}{2}\sqrt{M_m^2 + 4} - \frac{R}{2} M_m \\ &\quad - \sqrt{R^2 - \Big(\tfrac{1}{2} M_m \sqrt{M_m^2+4} - \tfrac{1}{2} M_m^2\Big) r^2}, \quad r \in [0,R], \\ &\text{with } M_m = \mathrm{e}^{p_m u_{m,0}(R)} \mathrm{e}^{q_{m+1} u_{m+1,0}(R)}, \\ &N_m = u_{m,0}(R), \ m = 1, 2, \dots, n, \quad \text{where } u_{1,0}(R) = \frac{R}{\lambda_1}, \\ &u_{l,0}(R) = \frac{R}{\prod_{j=1}^{l-1} (1-\lambda_j)\lambda_l}, \quad l = 2, 3, \dots, n-1, \\ &u_{n,0}(R) = \frac{R}{\prod_{j=1}^{n-1}(1-\lambda_j)}, \quad \lambda_1, \lambda_2, \dots, \lambda_{n-1} \in (0,1), \\ &(u_{n,0})_{rr} + \frac{N-1}{r}(u_{n,0})_r \ge \varepsilon[(u_{n,0})_r]^2, \end{aligned}$$

$$(u_{n-1,0})_{rr}+\frac{N-1}{r}(u_{n-1,0})_r\geq\varepsilon[(u_{n-1,0})_r]^2,\ r\in[0,R]\Big\}.$$

We use the following four lemmata to prove it.

**Lemma 3.2.** *If $q_n < p_n$ and $q_{n-1} < p_{n-1}$, then there exists some $\overline{\lambda}_{n-2}\in(1/2,1)$ such that, for any initial data satisfying $u_{j,0}(R)=2^jR$, $j=1,2,\dots,n-3$, and $u_{n-2,0}(R)=(2^{n-3}R)/(\overline{\lambda}_{n-2})$ in $\mathbb{V}_1$, non-simultaneous blow-up must occur with $u_1,u_2,\dots,u_{n-2}$ remaining bounded.*

*Proof.* Take $M_j>2^{j+1}R$, $j=1,2,\dots,n-2$. Consider the following auxiliary problem

$$\text{(3.1)}\qquad\begin{cases}(\underline{u}_{n-1})_t=\Delta\underline{u}_{n-1} & (x,t)\in B_R\times(0,\underline{T}_{n-1}),\\ \frac{\partial\underline{u}_{n-1}}{\partial\eta}=\mathrm{e}^{q_n(2^{n-2}R-R)}\mathrm{e}^{p_{n-1}\underline{u}_{n-1}} & (x,t)\in\partial B_R\times(0,\underline{T}_{n-1}),\\ \underline{u}_{n-1}(x,0)=\underline{u}_{n-1,0}(x) & x\in B_R,\end{cases}$$

where radial symmetric initial data $\underline{u}_{n-1,0}(x)$ satisfy the compatibility condition and

$$\frac{2^{n-3}R}{1-\lambda_{n-2}}-2R\leq\underline{u}_{n-1,0}(x)\leq\frac{2^{n-3}R}{1-\lambda_{n-2}}-R$$

with $\lambda_{n-2}$ to be determined. For problem (3.1), some $\overline{\lambda}_{n-2}\in(1/2,1)$ must exist such that, if $\lambda_{n-2}=\overline{\lambda}_{n-2}$, then $\underline{T}_{n-1}$ satisfies

$$M_j\geq 2^{j+1}R+2\overline{C}\mathrm{e}^{q_{j+1}M_{j+1}}\mathrm{e}^{p_jM_j}\underline{T}_{n-1}^{1/2},\quad j=1,2,\dots,n-3,$$
$$M_{n-2}\geq 2^{n-1}R+\frac{2p_{n-1}}{p_{n-1}-q_{n-1}}\overline{C}\mathrm{e}^{p_{n-2}M_{n-2}}C_0^{q_{n-1}}\underline{T}_{n-1}^{p_{n-1}-q_{n-1}/(2p_{n-1})}.$$

For any $(u_{1,0},u_{2,0},\dots,u_{n,0})\in\mathbb{V}_1$ satisfying $u_{j,0}(R)=2^jR$, $j=1,2,\dots,n-3$, and $u_{n-2,0}(R)=(2^{n-3}R)/(\overline{\lambda}_{n-2})$, we have

$$u_{n-1,0}(R)=\frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}}\geq\frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}}\quad\text{for any }\lambda_{n-1}\in(0,1).$$

Then,

$$\frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}}-2R\leq\underline{u}_{n-1,0}(x)\leq\frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}}-R$$

$$\leq u_{n-1,0}(x) \leq \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}}.$$

For $(u_n)_t \geq 0$, $u_n(x,t) \geq u_{n,0}(x) \geq 2^{n-2}R - R$. By the comparison principle, $\underline{u}_{n-1} \leq u_{n-1}$ and $T \leq \underline{T}_{n-1}$. Hence,

$$M_j \geq 2^{j+1}R + 2\overline{C}\mathrm{e}^{q_{j+1}M_{j+1}}\mathrm{e}^{p_jM_j}T^{1/2}, \qquad j = 1,2,\ldots,n-3,$$
$$M_{n-2} \geq 2^{n-1}R + \frac{2p_{n-1}}{p_{n-1}-q_{n-1}}\,\overline{C}\mathrm{e}^{p_{n-2}M_{n-2}}C_0^{q_{n-1}}T^{p_{n-1}-q_{n-1}/(2p_{n-1})}.$$

Consider the second auxiliary problem

$$\text{(3.2)}\qquad \begin{cases} (\overline{u}_{n-2})_t = \Delta\overline{u}_{n-2} & (x,t)\in B_R\times(0,T),\\ \frac{\partial\overline{u}_{n-2}}{\partial\eta} = \mathrm{e}^{p_{n-2}M_{n-2}}C_0^{q_{n-1}} & \\ \quad\cdot(T-t)^{-q_{n-1}/(2p_{n-1})} & (x,t)\in\partial B_R\times(0,T),\\ \overline{u}_{n-2}(x,0) = \overline{u}_{n-2,0}(x) & x\in B_R, \end{cases}$$

where the radial $\overline{u}_{n-2,0}(x)$ satisfies

$$\frac{\partial\overline{u}_{n-2,0}(x)}{\partial\eta} = \mathrm{e}^{p_{n-2}M_{n-2}}C_0^{q_{n-1}}T^{-q_{n-1}/(2p_{n-1})},$$

$\overline{u}_{n-2,0}(x) = 2^{n-1}R$ for $x \in \partial B_R$, $\Delta\overline{u}_{n-2,0}(x) \geq 0$ and $\overline{u}_{n-2,0}(x) \geq u_{n-2,0}(x)$ for $x \in B_R$.

By Green's identity and $q_{n-1} < p_{n-1}$, we have

$$\overline{u}_{n-2} \leq 2^{n-1}R + \frac{2p_{n-1}}{p_{n-1}-q_{n-1}}\,\overline{C}\mathrm{e}^{p_{n-2}M_{n-2}}C_0^{q_{n-1}}T^{p_{n-1}-q_{n-1}/(2p_{n-1})} \leq M_{n-2}.$$

Thus, $\overline{u}_{n-2}$ satisfies

$$\frac{\partial\overline{u}_{n-2}}{\partial\eta} \geq C_0^{q_{n-1}}(T-t)^{-q_{n-1}/(2p_{n-1})}\mathrm{e}^{p_{n-2}\bar{u}_{n-2}}, \quad (x,t)\in\partial B_R\times(0,T).$$

From Lemma 2.1 and $p_{n-1} > 0$, we have $\mathrm{e}^{u_{n-1}} \leq C_0(T-t)^{-1/(2p_{n-1})}$, and hence,

$$\frac{\partial u_{n-2}}{\partial\eta} \leq C_0^{q_{n-1}}(T-t)^{-(q_{n-1})/(2p_{n-1})}\mathrm{e}^{p_{n-2}u_{n-2}}, \quad (x,t)\in\partial B_R\times(0,T).$$

Then, by the comparison principle, $u_{n-2} \leq \overline{u}_{n-2} \leq M_{n-2}$.

Now, we introduce the third auxiliary problem

$$(3.3)\qquad \begin{cases} (\overline{u}_{n-3})_t = \Delta \overline{u}_{n-3} & (x,t) \in B_R \times (0,+\infty), \\ \frac{\partial \overline{u}_{n-3}}{\partial \eta} = \mathrm{e}^{q_{n-2}M_{n-2}} \mathrm{e}^{p_{n-3}M_{n-3}} & (x,t) \in \partial B_R \times (0,+\infty), \\ \overline{u}_{n-3}(x,0) = \overline{u}_{n-3,0}(x) & x \in B_R, \end{cases}$$

where the radial $\overline{u}_{n-3,0}(x)$ satisfies

$$\frac{\partial \overline{u}_{n-3,0}(x)}{\partial \eta} = \mathrm{e}^{q_{n-2}M_{n-2}} \mathrm{e}^{p_{n-3}M_{n-3}},$$

$\overline{u}_{n-3,0}(x) = 2^{n-2}R$ for $x \in \partial B_R$, $\Delta \overline{u}_{n-3,0}(x) \geq 0$ and $\overline{u}_{n-3,0}(x) \geq u_{n-3,0}(x)$ for $x \in B_R$. Considering problem (3.3) in $(0,T)$, we have

$$\overline{u}_{n-3} \leq 2^{n-2}R + 2\overline{C}\mathrm{e}^{q_{n-1}M_{n-1}} \mathrm{e}^{p_{n-2}M_{n-2}} T^{1/2} \leq M_{n-3}.$$

Thus, $\overline{u}_{n-3}$ satisfies

$$\frac{\partial \overline{u}_{n-3}}{\partial \eta} \geq \mathrm{e}^{q_{n-2}M_{n-2}} \mathrm{e}^{p_{n-3}\overline{u}_{n-3}} \quad \text{for } (x,t) \in \partial B_R \times (0,T).$$

For $u_{n-2} \leq M_{n-2}$, $u_{n-3}$ satisfies

$$\frac{\partial u_{n-3}}{\partial \eta} \leq \mathrm{e}^{q_{n-2}M_{n-2}} \mathrm{e}^{p_{n-3}u_{n-3}} \quad \text{for } (x,t) \in \partial B_R \times (0,T).$$

By the comparison principle, $u_{n-3} \leq \overline{u}_{n-3} \leq M_{n-3}$. The boundedness of $u_{n-4}, u_{n-5}, \ldots, u_1$ can be similarly proved. □

**Lemma 3.3.** *If $q_n < p_n$ and $q_{n-1} < p_{n-1}$, then, for the fixed $\overline{\lambda}_{n-2} \in (1/2,1)$ in Lemma 3.2, there exists some $\lambda'_{n-1} \in (0,1/2)$ such that non-simultaneous blow-up occurs with $u_{n-1}$ blowing up and the other components remaining bounded, where $u_{j,0}(R) = 2^j R$, $j = 1,2, \ldots, n-3$,*

$$u_{n-2,0}(R) = \frac{2^{n-3}R}{\overline{\lambda}_{n-2}}, \qquad u_{n-1,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda'_{n-1}}$$

*and*

$$u_{n,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda'_{n-1})} \quad \text{in } \mathbb{V}_1.$$

*Proof.* Take $M_n > (2^{n-1}R)/(1-\overline{\lambda}_{n-2})$. We then introduce the following auxiliary problem

$$
\begin{cases}
(\overline{u}_n)_t = \Delta \overline{u}_n & (x,t) \in B_R \times (0,+\infty),\\
\frac{\partial \overline{u}_n}{\partial \eta} = \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n} & (x,t) \in \partial B_R \times (0,+\infty),\\
\overline{u}_n(x,0) = \overline{u}_{n,0}(x) & x \in B_R,
\end{cases}
\tag{3.4}
$$

where the radial symmetric $\overline{u}_{n,0}(x)$ satisfies

$$
\frac{\partial \overline{u}_{n,0}(x)}{\partial \eta} = \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n}, \qquad \overline{u}_{n,0}(x) = \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}}
$$

for $x \in \partial B_R$ and $\Delta \overline{u}_{n,0}(x) \geq 0$, $\overline{u}_{n,0}(x) \geq u_{n,0}(x)$ for $x \in B_R$.

Consider problem (3.1); however, here, the initial datum $\underline{u}_{n-1,0}$ satisfies

$$
\frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}} - 2R \leq \underline{u}_{n-1,0}(x) \leq \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}} - R
$$

with $\lambda_{n-1}$ to be determined. There exists some $\lambda'_{n-1} \in (0,1/2)$ such that, if $\lambda_{n-1} = \lambda'_{n-1}$, then $\underline{T}_{n-1}$ satisfies

$$
M_n \geq \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + 2\overline{C}\mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n}\underline{T}_{n-1}^{1/2}.
$$

Similarly to Lemma 3.2, $\underline{u}_{n-1} \leq u_{n-1}$ and $T \leq \underline{T}_{n-1}$. Hence, $M_n \geq (2^{n-1}R)/(1-\overline{\lambda}_{n-2}) + 2\overline{C}\mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n}T^{1/2}$. Consider (3.4) in $[0,T)$. By Green's identity,

$$
\overline{u}_n \leq \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + 2\overline{C}\mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n}T^{1/2} \leq M_n.
$$

Then, $\overline{u}_n$ satisfies that $\partial \overline{u}_n/\partial \eta \geq \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n \overline{u}_n}$ for $(x,t) \in \partial B_R \times (0,T)$. Since $u_1 \leq M_1$, $u_n$ satisfies $\partial u_n/\partial \eta \leq \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n u_n}$ for $(x,t) \in \partial B_R \times (0,T)$. By the comparison principle, $u_n \leq \overline{u}_n \leq M_n$. Thus, only $u_{n-1}$ blows up. □

**Lemma 3.4.** *If $q_n < p_n$ and $q_{n-1} < p_{n-1}$, then, for the fixed $\overline{\lambda}_{n-2} \in (1/2,1)$ in Lemma 3.2, there exists some $\lambda''_{n-1} \in (1/2,1)$ such that non-simultaneous blow-up occurs with $u_n$ blowing up and the other components remaining bounded, where $u_{j,0}(R) = 2^j R$, $j = 1,2,\ldots,n-3$,*

$$u_{n-2,0}(R) = \frac{2^{n-3}R}{\overline{\lambda}_{n-2}}, \qquad u_{n-1,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda''_{n-1}}$$

*and*

$$u_{n,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda''_{n-1})} \quad \textit{in } \mathbb{V}_1.$$

*Proof.* Now, we introduce the following auxiliary problem

$$\begin{cases} (\underline{u}_n)_t = \Delta \underline{u}_n & (x,t) \in B_R \times (0, \underline{T}_n), \\ \frac{\partial \underline{u}_n}{\partial \eta} = \mathrm{e}^{q_1 R}\mathrm{e}^{p_n \underline{u}_n} & (x,t) \in \partial B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,0) = \underline{u}_{n,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric $\underline{u}_{n,0}(x)$ satisfies the compatibility condition and

$$\frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda_{n-1})} - 2R \le \underline{u}_{n,0}(x) \le \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda_{n-1})} - R$$

with $\lambda_{n-1}$ to be determined.

Take $M_{n-1} > (2^{n-1}R)/(1-\overline{\lambda}_{n-2})$. There exists some $\lambda''_{n-1} \in (1/2, 1)$ such that, if $\lambda_{n-1} = \lambda''_{n-1}$, then

$$M_{n-1} \ge \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + \frac{2p_n}{p_n - q_n}\,\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}} C_0^{q_n} \underline{T}_n^{\;p_n - q_n/(2p_n)}.$$

Take the initial data $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ in $\mathbb{V}_1$ such that $\lambda_j = 1/2$, $j = 1, 2, \ldots, n-3$, $\lambda_{n-2} = \overline{\lambda}_{n-2}$, $\lambda_{n-1} = \lambda''_{n-1}$. For

$$\underline{u}_{n,0}(x) \le \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda''_{n-1})} - R \le u_{n,0}(x)$$

and $u_1(x,t) \ge u_{1,0}(x) \ge R$, $u_n$ satisfies

$$\frac{\partial u_n}{\partial \eta} \ge \mathrm{e}^{q_1 R}\mathrm{e}^{p_n u_n} \quad \text{on } \partial B_R \times (0,T),$$

and hence, $\underline{u}_n \le u_n$ and $T \le \underline{T}_n$. Thus,

$$M_{n-1} \ge \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + \frac{2p_n}{p_n - q_n}\,\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}} C_0^{q_n} T^{p_n - q_n/(2p_n)}.$$

Consider the next auxiliary problem

$$\begin{cases}(\overline{u}_{n-1})_t = \Delta \overline{u}_{n-1} & (x,t) \in B_R \times (0,T),\\ \frac{\partial \overline{u}_{n-1}}{\partial \eta} = \mathrm{e}^{p_{n-1}M_{n-1}} C_0^{q_n}(T-t)^{-p_n/(2p_n)} & (x,t) \in \partial B_R \times (0,T),\\ \overline{u}_{n-1}(x,0) = \overline{u}_{n-1,0}(x) & x \in B_R,\end{cases}$$

where the radial symmetric $\overline{u}_{n-1,0}(x)$ satisfies the compatibility condition and $\overline{u}_{n-1,0}(x) = (2^{n-1}R)/(1-\overline{\lambda}_{n-2})$, $x \in \partial B_R$, $\Delta \overline{u}_{n-1,0}(x) \geq 0$ and $\overline{u}_{n-1,0}(x) \geq u_{n-1,0}(x)$, $x \in B_R$.

For $q_n < p_n$, by Green's identity,

$$\overline{u}_{n-1} \leq \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + \frac{2p_n}{p_n - q_n} \overline{C}\mathrm{e}^{p_{n-1}M_{n-1}} C_0^{q_n} T^{p_n - q_n/(2p_n)} \leq M_{n-1}.$$

Thus, $\overline{u}_{n-1}$ satisfies

$$\frac{\partial \overline{u}_{n-1}}{\partial \eta} \geq C_0^{q_n}(T-t)^{-q_n/(2p_n)} \mathrm{e}^{p_{n-1}\overline{u}_{n-1}}, \quad (x,t) \in \partial B_R \times (0,T).$$

For $p_n > 0$, $\mathrm{e}^{U_n(t)} \leq C_0(T-t)^{-1/(2p_n)}$. Hence, $u_{n-1}$ satisfies

$$\frac{\partial u_{n-1}}{\partial \eta} \leq C_0^{q_n}(T-t)^{-q_n/(2p_n)} \mathrm{e}^{p_{n-1}u_{n-1}}, \quad (x,t) \in \partial B_R \times (0,T).$$

By the comparison principle, $u_{n-1} \leq \overline{u}_{n-1} \leq M_{n-1}$. Then, only $u_n$ blows up. □

**Lemma 3.5.**

(i) *The set of initial data in $\mathbb{V}_1$ such that $u_n$ blows up while the others remain bounded is open in $L^\infty$-topology.*

(ii) *The set of initial data in $\mathbb{V}_1$ such that $u_{n-1}$ blows up while the others remain bounded is open in $L^\infty$-topology.*

*Proof.* Without loss of generality, we prove only case (i). Let $(u_1, u_2, \ldots, u_n)$ be a solution of (1.1) with initial data $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ in $\mathbb{V}_1$ such that $u_n$ blows up at $t = T$ while the other components remain bounded, say $0 < 2\xi \leq u_1, u_2, \ldots, u_{n-1} \leq M$. It suffices to find an $L^\infty$-neighborhood of $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ in $\mathbb{V}_1$ such that any solution $(\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n)$ of (1.1) coming from this neighborhood maintains the property that $\widehat{u}_n$ blows up in finite time while the others remain bounded. By Corollary 1.2, $q_n < p_n$. Take $S_j > 2M + 2\xi$, $j = 1, 2, \ldots, n-1$. Let $(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)$ be the solution of:

(3.5)
$$\begin{cases} (\widetilde{u}_j)_t = \Delta \widetilde{u}_j & (x,t) \in B_R \times (0, T_0), \\ \frac{\partial \widetilde{u}_j}{\partial \eta} = \mathrm{e}^{p_j \widetilde{u}_j} \mathrm{e}^{q_{j+1} \widetilde{u}_{j+1}} & (x,t) \in \partial B_R \times (0, T_0), \\ \widetilde{u}_j(x,0) = \widetilde{u}_{j,0}(x), \ j = 1, 2, \ldots, n, \ n \geq 2 & x \in B_R, \\ \widetilde{u}_{n+1} := \widetilde{u}_1, \ p_{n+1} := p_1, \ q_{n+1} := q_1, \end{cases}$$

where the radial symmetric $(\widetilde{u}_{1,0}, \widetilde{u}_{2,0}, \ldots, \widetilde{u}_{n,0}) \in \mathbb{V}_0$ is to be determined. Denote

$$\mathcal{N}(u_{1,0}, u_{2,0}, \ldots, u_{n,0}) = \big\{ (\widetilde{u}_{1,0}, \widetilde{u}_{2,0}, \ldots, \widetilde{u}_{n,0}) \in \mathbb{V}_0 \mid \\ \|\widetilde{u}_{j,0}(x) - u_j(x, T - \varepsilon_0)\|_\infty < \xi, \ 1 \leq j \leq n \big\}.$$

Since $(u_1, u_2, \ldots, u_n)$ blows up at finite time $T$ with fixed $\xi$, some $\varepsilon_0 > 0$ exists such that $T_0$ satisfies

$$S_j \geq 2M + 2\xi + 2\overline{C} \mathrm{e}^{p_j S_j} \mathrm{e}^{q_{j+1} S_{j+1}} T_0^{1/2}, \qquad j = 1, 2, \ldots, n-2,$$
$$S_{n-1} \geq 2M + 2\xi + \frac{2p_n}{p_n - q_n} \overline{C} \mathrm{e}^{p_{n-1} S_{n-1}} C_0^{q_n} T_0^{p_n - q_n/(2p_n)},$$

provided that $(\widetilde{u}_{1,0}, \widetilde{u}_{2,0}, \ldots, \widetilde{u}_{n,0}) \in \mathcal{N}(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$.

Consider the auxiliary problem

$$\begin{cases} (\overline{u}_{n-1})_t = \Delta \overline{u}_{n-1} & (x,t) \in B_R \times (0, T_0), \\ \frac{\partial \overline{u}_{n-1}}{\partial \eta} = \mathrm{e}^{p_{n-1} S_{n-1}} C_0^{q_n} (T_0 - t)^{-q_n/(2p_n)} & (x,t) \in \partial B_R \times (0, T_0), \\ \overline{u}_{n-1}(x,0) = \overline{u}_{n-1,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric $\overline{u}_{n-1,0}(x)$ satisfies the compatibility condition, $\overline{u}_{n-1,0}(x) = 2\widetilde{u}_{n-1,0}(x)$, $x \in \partial B_R$, $\Delta \overline{u}_{n-1,0}(x) \geq 0$ and $\overline{u}_{n-1,0}(x) \geq \widetilde{u}_{n-1,0}(x)$, $x \in B_R$. By Green's identity,

$$\overline{u}_{n-1} \leq 2M + 2\xi + \frac{2p_n}{p_n - q_n} \overline{C} \mathrm{e}^{p_{n-1} S_{n-1}} C_0^{q_n} T_0^{p_n - q_n/(2p_n)} \leq S_{n-1}.$$

Then,

$$\frac{\partial \overline{u}_{n-1}}{\partial \eta} \geq C_0^{q_n} (T_0 - t)^{-q_n/(2p_n)} \mathrm{e}^{p_{n-1} \bar{u}_{n-1}}, \quad (x,t) \in \partial B_R \times (0, T_0).$$

For $p_n > 0$, $\mathrm{e}^{\tilde{u}_n} \le C_0(T_0 - t)^{-1/(2p_n)}$. Thus, $\widetilde{u}_{n-1}$ satisfies

$$\frac{\partial \widetilde{u}_{n-1}}{\partial \eta} \le C_0^{q_n}(T_0 - t)^{-q_n/(2p_n)}\mathrm{e}^{p_{n-1}\tilde{u}_{n-1}}, \quad (x,t) \in \partial B_R \times (0, T_0).$$

By the comparison principle, $\widetilde{u}_{n-1} \le \overline{u}_{n-1} \le S_{n-1}$.

Next, consider the auxiliary problem

$$\begin{cases} (\overline{u}_{n-2})_t = \Delta \overline{u}_{n-2} & (x,t) \in B_R \times (0,+\infty), \\ \frac{\partial \overline{u}_{n-2}}{\partial \eta} = \mathrm{e}^{q_{n-1}S_{n-1}}\mathrm{e}^{p_{n-2}S_{n-2}} & (x,t) \in \partial B_R \times (0,+\infty), \\ \overline{u}_{n-2}(x,0) = \overline{u}_{n-2,0}(x) & x \in B_R, \end{cases}$$

where $\overline{u}_{n-2,0}(x)$ satisfies the compatibility condition and $\overline{u}_{n-2,0} = 2\widetilde{u}_{n-2,0}$ on $\partial B_R$, $\Delta \overline{u}_{n-2,0} \ge 0$, $\overline{u}_{n-2,0} \ge \widetilde{u}_{n-2,0}$ in $B_R$. From Green's identity, $\overline{u}_{n-2} \le S_{n-2}$ in $B_R \times (0, T_0)$. Thus,

$$\frac{\partial \overline{u}_{n-2}}{\partial \eta} \ge \mathrm{e}^{q_{n-1}S_{n-1}}\mathrm{e}^{p_{n-2}\bar{u}_{n-2}}, \quad (x,t) \in \partial B_R \times (0, T_0).$$

For $\widetilde{u}_{n-1} \le S_{n-1}$, $(\partial \widetilde{u}_{n-2})/\partial \eta \le \mathrm{e}^{q_{n-1}S_{n-1}}\mathrm{e}^{p_{n-2}\tilde{u}_{n-2}}$, $(x,t) \in \partial B_R \times (0, T_0)$. Thus, $\widetilde{u}_{n-2} \le \overline{u}_{n-2} \le S_{n-2}$, $(x,t) \in B_R \times (0, T_0)$. The boundedness of $\widetilde{u}_i$, $i = n-3, n-4, \ldots, 1$, can be similarly proved. Thus, $\widetilde{u}_n$ is the blow-up component.

According to the continuity on initial data for bounded solutions, there must exist a neighborhood $N(\subset \mathbb{V}_1)$ of $(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ such that every solution $(\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_n)$ starting from the neighborhood will enter $\mathcal{N}(u_{1,0}, u_{2,0}, \ldots, u_{n,0})$ at time $T - \varepsilon_0$. This upholds the property that $\widehat{u}_n$ blows up while the other components remain bounded. □

*Proof of Proposition* 3.1. Lemma 3.2 states that some $\overline{\lambda}_{n-2} \in (1/2, 1)$ exists such that any initial datum in $\mathbb{V}_1$ satisfying $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = 1/2$, $\overline{\lambda}_{n-2} \in (1/2, 1)$ develops the nonsimultaneous blow-up solution with $u_j$, $j = 1, 2, \ldots, n-2$, remaining bounded. We know from Lemma 3.3 that there exists some $\lambda'_{n-1} \in (0, 1/2)$ such that the solution of (1.1) with the initial datum in $\mathbb{V}_1$ satisfying $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = 1/2$, $\lambda_{n-2} = \overline{\lambda}_{n-2}$ and $\lambda_{n-1} = \lambda'_{n-1}$ blows up non-simultaneously, where $u_{n-1}$ blows up and the others are bounded. Lemma 3.4 guarantees that some $\lambda''_{n-1} \in (1/2, 1)$ exists such that $u_n$ blows up alone with the initial datum in $\mathbb{V}_1$, where $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = 1/2$, $\lambda_{n-2} = \overline{\lambda}_{n-2}$ and $\lambda_{n-1} = \lambda''_{n-1}$. In addition, the sets of the initial data in $\mathbb{V}_1$ such that $u_n$ blows up alone and $u_{n-1}$ blows up alone are all open by Lemma 3.5. Note that $\mathbb{V}_1$ is connected. Thus, there must exist initial data (suitable $\overline{\lambda}_{n-1} \in (\lambda'_{n-1}, \lambda''_{n-1})$) such that $u_n$ and $u_{n-1}$ blow up simultaneously while the others remain bounded.

Due to the boundedness of $u_1$ and by Green's identity, we have

$$U_n(t) \leq U_n(z) + C\mathrm{e}^{p_n U_n(t)}(T-z)^{1/2}.$$

For the blow-up property of $u_n$, take $z < t < T$ such that $C' = U_n(t) - U_n(z) > 0$. Thus, $\mathrm{e}^{U_n(z)} \geq c(T-z)^{-1/(2p_n)}$. Similarly to the method of Lemma 2.1, $\mathrm{e}^{U_n(t)} \leq C(T-t)^{-1/(2p_n)}$ and $\mathrm{e}^{U_{n-1}(t)} \leq C(T-t)^{-(p_n-q_n)/(2p_n p_{n-1})}$ may be obtained. Combining the upper estimate of $U_n$ with Green's identity to $u_{n-1}$, we have

$$U_{n-1}(t) \leq U_{n-1}(z) + C\mathrm{e}^{p_{n-1}U_{n-1}(t)}(T-z)^{p_n - q_n/(2p_n)}.$$

Similarly to the discussion for lower estimates of $u_n$, we have

$$\mathrm{e}^{U_{n-1}(t)} \geq c(T-t)^{-(p_n-q_n)/(2p_n p_{n-1})}. \qquad \square$$

Secondly, we discuss the subcase $k \in \{2, 3, \ldots, n-2\}$. It may be verified that it must be $n \geq 4$, if the blow-up rate

$$(\mathrm{e}^{U_{n-k}(t)}, \mathrm{e}^{U_n(t)}) \sim ((T-t)^{1/(2p_{n-k})}, (T-t)^{-1/(2p_n)})$$

occurs.

**Proposition 3.6.** *If $q_n < p_n$ and $q_{n-k} < p_{n-k}$, then suitable initial data exist such that $u_{n-k}$ and $u_n$ blow up simultaneously at some time $T$ while the others still remain bounded. Moreover,*

$$(\mathrm{e}^{U_{n-k}(t)}, \mathrm{e}^{U_n(t)}) \sim ((T-t)^{-1/(2p_{n-k})}, (T-t)^{-1/(2p_n)}).$$

Without loss of generality, we prove only the case for $k = 2$ by the following five lemmata. Define another subset of $\mathbb{V}_0$ as follows:

$$\begin{aligned}\mathbb{V}_2 = \Big\{&(u_{1,0}(r), u_{2,0}(r), \ldots, u_{n,0}(r)) \mid u_{m,0}(r)\\ &= N_m + \frac{R}{2}\sqrt{M_m^2+4} - \frac{R}{2}M_m\\ &\quad - \sqrt{R^2 - \Big(\tfrac{1}{2}M_m\sqrt{M_m^2+4} - \tfrac{1}{2}M_m^2\Big)r^2},\ r \in [0, R],\end{aligned}$$

$$\text{with } M_m = \mathrm{e}^{p_m u_{m,0}(R)}\mathrm{e}^{q_{m+1}u_{m+1,0}(R)},$$
$$N_m = u_{m,0}(R)\ (m = 1,2,\dots,n), \quad \text{where } u_{1,0}(R) = \frac{R}{\lambda_1},$$
$$u_{l,0}(R) = \frac{R}{\prod_{j=1}^{l-1}(1-\lambda_j)\lambda_l}\ (l = 2,3,\dots,n-3),$$
$$u_{n-1,0}(R) = \frac{R}{\prod_{j=1}^{n-3}(1-\lambda_j)\lambda_{n-2}},$$
$$u_{n-2,0}(R) = \frac{R}{\prod_{j=1}^{n-2}(1-\lambda_j)\lambda_{n-1}},$$
$$u_{n,0}(R) = \frac{R}{\prod_{j=1}^{n-1}(1-\lambda_j)},\ \lambda_1,\lambda_2,\dots,\lambda_{n-1} \in (0,1)$$
$$(u_{n,0})_{rr} + \frac{N-1}{r}(u_{n,0})_r \geq \varepsilon[(u_{n,0})_r]^2,$$
$$(u_{n-2,0})_{rr} + \frac{N-1}{r}(u_{n-2,0})_r \geq \varepsilon[(u_{n-2,0})_r]^2,\ r \in [0,R]\Big\}.$$

**Lemma 3.7.** *If $q_n < p_n$ and $q_{n-2} < p_{n-2}$, then some $\overline{\lambda}_{n-2} \in (1/2,1)$ exists such that non-simultaneous blow-up occurs with $u_1, u_2, \dots, u_{n-3}$, $u_{n-1}$ remaining bounded for the initial datum satisfying $u_{j,0}(R) = 2^j R$, $j = 1,2,\dots,n-3$, and $u_{n-1,0}(R) = (2^{n-3}R)/\overline{\lambda}_{n-2}$ in $\mathbb{V}_2$.*

*Proof.* Take $M_j > 2^{j+1}R$, $j = 1,2,\dots,n-3$, $M_{n-1} > 2^{n-1}R$. Consider the auxiliary problem

$$(3.6)\quad \begin{cases} (\underline{u}_{n-2})_t = \Delta \underline{u}_{n-2} & (x,t) \in B_R \times (0,\underline{T}_{n-2}),\\ \frac{\partial \underline{u}_{n-2}}{\partial \eta} = \mathrm{e}^{q_{n-1}(2^{n-3}R-R)}\mathrm{e}^{p_{n-2}\underline{u}_{n-2}} & (x,t) \in \partial B_R \times (0,\underline{T}_{n-2}),\\ \underline{u}_{n-2}(x,0) = \underline{u}_{n-2,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric $\underline{u}_{n-2,0}(x)$ satisfies the compatibility condition and

$$\frac{2^{n-3}R}{1-\lambda_{n-2}} - 2R \leq \underline{u}_{n-2,0}(x) \leq \frac{2^{n-3}R}{1-\lambda_{n-2}} - R,$$

with $\lambda_{n-2}$ to be determined.

For problem (3.6), some $\lambda_{n-2} = \overline{\lambda}_{n-2} \in (1/2, 1)$ must exist such that $\underline{T}_{n-2}$ satisfies

$$\begin{aligned} M_j &\geq 2^{j+1}R + 2\overline{C}\mathrm{e}^{q_{j+1}M_{j+1}}\mathrm{e}^{p_jM_j}\underline{T}_{n-2}^{1/2}, \qquad j = 1, 2, \ldots, n-4, \\ M_{n-3} &\geq 2^{n-2}R + \frac{2p_{n-2}}{p_{n-2}-q_{n-2}}\,\overline{C}\mathrm{e}^{p_{n-3}M_{n-3}}C_0^{q_{n-2}}\underline{T}_{n-2}^{(p_{n-2}-q_{n-2})/(2p_{n-2})}, \\ M_{n-1} &\geq 2^{n-1}R + \frac{2p_n}{p_n-q_n}\,\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}\underline{T}_{n-2}^{(p_n-q_n)/2p_n}. \end{aligned}$$

For any $(u_{1,0}, u_{2,0}, \ldots, u_{n,0}) \in \mathbb{V}_2$ satisfying $u_{j,0}(R) = 2^jR$, $j = 1, 2, \ldots, n-3$ and $u_{n-1,0}(R) = (2^{n-3}R)/\overline{\lambda}_{n-2}$, we have

$$u_{n-2,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}} \geq \frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}}$$

for any $\lambda_{n-1} \in (0, 1)$. Then,

$$\begin{aligned} \frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}} - 2R \leq \underline{u}_{n-2,0}(x) &\leq \frac{2^{n-3}R}{1-\overline{\lambda}_{n-2}} - R \\ &\leq u_{n-2,0}(x) \leq \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}}. \end{aligned}$$

For

$$(u_{n-1})_t \geq 0, \qquad u_{n-1}(x,t) \geq u_{n-1,0}(x) \geq 2^{n-3}R - R,$$

$u_{n-2}$ satisfies

$$\frac{\partial u_{n-2}}{\partial \eta} \geq \mathrm{e}^{q_{n-1}(2^{n-3}R-R)}\mathrm{e}^{p_{n-2}u_{n-2}} \quad \text{on } \partial B_R \times (0, T).$$

By the comparison principle, $\underline{u}_{n-2} \leq u_{n-2}$ and $T \leq \underline{T}_{n-2}$. Hence,

$$\begin{aligned} M_j &\geq 2^{j+1}R + 2\overline{C}\mathrm{e}^{q_{j+1}M_{j+1}}\mathrm{e}^{p_jM_j}T^{1/2}, \qquad j = 1, 2, \ldots, n-4, \\ M_{n-3} &\geq 2^{n-2}R + \frac{2p_{n-2}}{p_{n-2}-q_{n-2}}\,\overline{C}\mathrm{e}^{p_{n-3}M_{n-3}}C_0^{q_{n-2}}T^{(p_{n-2}-q_{n-2})/(2p_{n-2})}, \\ M_{n-1} &\geq 2^{n-1}R + \frac{2p_n}{p_n-q_n}\,\overline{C}\mathrm{e}^{p_{n-1}M_{n-1}}C_0^{q_n}T^{p_n-q_n/(2p_n)}. \end{aligned}$$

Consider the second auxiliary problem

$$\begin{cases} (\overline{u}_{n-3})_t = \Delta \overline{u}_{n-3} & (x,t) \in B_R \times (0,T), \\ \frac{\partial \overline{u}_{n-3}}{\partial \eta} = \mathrm{e}^{p_{n-3}M_{n-3}} C_0^{q_{n-2}} (T-t)^{-(q_{n-2})/(2p_{n-2})} & (x,t) \in \partial B_R \times (0,T), \\ \overline{u}_{n-3}(x,0) = \overline{u}_{n-3,0}(x) & x \in B_R, \end{cases}$$

where the radial $\overline{u}_{n-3,0}(x)$ satisfies

$$\frac{\partial \overline{u}_{n-3,0}}{\partial \eta} = \mathrm{e}^{p_{n-3}M_{n-3}} C_0^{q_{n-2}} T^{-(q_{n-2})/(2p_{n-2})},$$

$\overline{u}_{n-3,0}(x) = 2^{n-2}R$ for $x \in \partial B_R$, $\Delta \overline{u}_{n-3,0}(x) \geq 0$ and $\overline{u}_{n-3,0}(x) \geq u_{n-3,0}(x)$ for $x \in B_R$.

From Green's identity and $q_{n-2} < p_{n-2}$,

$$\overline{u}_{n-3} \leq 2^{n-2}R + \frac{2p_{n-2}}{p_{n-2}-q_{n-2}} \overline{C} \mathrm{e}^{p_{n-3}M_{n-3}} C_0^{q_{n-2}} T^{(p_{n-2}-q_{n-2})/(2p_{n-2})} \leq M_{n-3}.$$

Thus, $\overline{u}_{n-3}$ satisfies

$$\frac{\partial \overline{u}_{n-3}}{\partial \eta} \geq C_0^{q_{n-2}} (T-t)^{-q_{n-2}/(2p_{n-2})} \mathrm{e}^{p_{n-3}\overline{u}_{n-3}}, \quad (x,t) \in \partial B_R \times (0,T).$$

By Lemma 2.1 and $p_{n-2} > 0$, $\mathrm{e}^{u_{n-2}} \leq C_0 (T-t)^{-1/(2p_{n-2})}$,

$$\frac{\partial u_{n-3}}{\partial \eta} \leq C_0^{q_{n-2}} (T-t)^{-(q_{n-2})/(2p_{n-2})} \mathrm{e}^{p_{n-3}u_{n-3}}, \quad (x,t) \in \partial B_R \times (0,T).$$

Then, by the comparison principle, $u_{n-3} \leq \overline{u}_{n-3} \leq M_{n-3}$.

Similarly to the proof for $u_{n-3}$, we have $u_{n-1} \leq M_{n-1}$.

In order to obtain the boundedness of $u_{n-4}$, we introduce the third auxiliary problem:

$$\begin{cases} (\overline{u}_{n-4})_t = \Delta \overline{u}_{n-4} & (x,t) \in B_R \times (0,+\infty), \\ \frac{\partial \overline{u}_{n-4}}{\partial \eta} = \mathrm{e}^{q_{n-3}M_{n-3}} \mathrm{e}^{p_{n-4}M_{n-4}} & (x,t) \in \partial B_R \times (0,+\infty), \\ \overline{u}_{n-4}(x,0) = \overline{u}_{n-4,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric $\overline{u}_{n-4,0}(x)$ satisfies

$$\frac{\partial \overline{u}_{n-4,0}}{\partial \eta} = \mathrm{e}^{q_{n-3}M_{n-3}} \mathrm{e}^{p_{n-4}M_{n-4}}, \quad \overline{u}_{n-4,0}(x) = 2^{n-3}R \text{ for } x \in \partial B_R,$$

$\Delta\overline{u}_{n-4,0}(x) \geq 0$ and $\overline{u}_{n-4,0}(x) \geq u_{n-4,0}(x)$ for $x \in B_R$. From Green's identity, we have

$$\overline{u}_{n-4} \leq 2^{n-3}R + 2\overline{C}\mathrm{e}^{q_{n-3}M_{n-3}}\mathrm{e}^{p_{n-4}M_{n-4}}T^{1/2} \leq M_{n-4}.$$

Thus, $\overline{u}_{n-4}$ satisfies

$$\frac{\partial \overline{u}_{n-4}}{\partial \eta} \geq \mathrm{e}^{q_{n-3}M_{n-3}}\mathrm{e}^{p_{n-4}\overline{u}_{n-4}} \quad \text{for } (x,t) \in \partial B_R \times (0,T).$$

For $u_{n-3} \leq M_{n-3}$, $u_{n-4}$ satisfies

$$\frac{\partial u_{n-4}}{\partial \eta} \leq \mathrm{e}^{q_{n-3}M_{n-3}}\mathrm{e}^{p_{n-4}u_{n-4}}, \quad (x,t) \in \partial B_R \times (0,T).$$

By the comparison principle, $u_{n-4} \leq \overline{u}_{n-4} \leq M_{n-4}$. We can obtain $u_j \leq M_j$, $j = n-5, n-6, \dots, 1$, similarly. $\square$

**Lemma 3.8.** *If $q_n < p_n$ and $q_{n-2} < p_{n-2}$, then, for the fixed $\overline{\lambda}_{n-2} \in (1/2,1)$ in Lemma 3.7, there exists $\lambda'_{n-1} \in (0,1/2)$ such that $u_{n-2}$ blows up while the other components remain bounded for the initial data satisfying $u_{j,0}(R) = 2^jR$, $j = 1,2,\dots,n-3$,*

$$u_{n-1,0}(R) = \frac{2^{n-3}R}{\overline{\lambda}_{n-2}}, \qquad u_{n-2,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda'_{n-1}}$$

*and*

$$u_{n,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda'_{n-1})} \quad \text{in } \mathbb{V}_2.$$

*Proof.* Take $M_n > (2^{n-1}R)/(1-\overline{\lambda}_{n-2})$. Consider problem (3.6) with initial data $\underline{u}_{n-2,0}$ satisfying the compatibility condition and

$$\frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}} - 2R < \underline{u}_{n-2,0}(x) \leq \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda_{n-1}} - R,$$

where $\lambda_{n-1}$ is to be determined. There exists some $\lambda'_{n-1} \in (0,1/2)$ such that, if $\lambda_{n-1} = \lambda'_{n-1}$, $\underline{T}_{n-2}$ satisfies

$$M_n \geq \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + 2\overline{C}\mathrm{e}^{q_1M_1}\mathrm{e}^{p_nM_n}\underline{T}_{n-2}^{1/2}.$$

Similarly to Lemma 3.7, $\underline{u}_{n-2} \le u_{n-2}$ and $T \le \underline{T}_{n-2}$. Hence,

$$M_n \ge \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + 2\overline{C}\mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n} T^{1/2}.$$

Considering problem (3.4) in $[0, T)$, we have

$$\overline{u}_n \le \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}} + 2\overline{C}\mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n M_n} T^{1/2} \le M_n.$$

Then, $\overline{u}_n$ satisfies

$$\frac{\partial \overline{u}_n}{\partial \eta} \ge \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n \overline{u}_n} \quad \text{for } (x,t) \in \partial B_R \times (0, T).$$

Due to $u_1 \le M_1$, $u_n$ satisfies

$$\frac{\partial u_n}{\partial \eta} \le \mathrm{e}^{q_1 M_1}\mathrm{e}^{p_n u_n} \quad \text{for } (x,t) \in \partial B_R \times (0, T).$$

By the comparison principle, $u_n \le \overline{u}_n \le M_n$. Thus, only $u_{n-2}$ blows up. □

**Lemma 3.9.** *If $q_n < p_n$ and $q_{n-2} < p_{n-2}$, then, for the fixed $\overline{\lambda}_{n-2} \in (1/2, 1)$ in Lemma 3.2, some $\lambda''_{n-1} \in (1/2, 1)$ exists such that $u_n$ blows up while the other components remain bounded for the initial data satisfying $u_{j,0}(R) = 2^j R,\ j = 1, 2, \ldots, n-3$,*

$$u_{n-1,0}(R) = \frac{2^{n-3}R}{\overline{\lambda}_{n-2}}, \qquad u_{n-2,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})\lambda''_{n-1}}$$

*and*

$$u_{n,0}(R) = \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda''_{n-1})} \quad \text{in } \mathbb{V}_2.$$

*Proof.* We now introduce the following auxiliary problem

$$\begin{cases} (\underline{u}_n)_t = \Delta \underline{u}_n & (x,t) \in B_R \times (0, \underline{T}_n), \\ \frac{\partial \underline{u}_n}{\partial \eta} = \mathrm{e}^{q_1 R}\mathrm{e}^{p_n \underline{u}_n} & (x,t) \in \partial B_R \times (0, \underline{T}_n), \\ \underline{u}_n(x,0) = \underline{u}_{n,0}(x) & x \in B_R, \end{cases}$$

where the radial symmetric $\underline{u}_{n,0}(x)$ satisfies the compatibility condition and

$$\frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda_{n-1})}-2R\le \underline{u}_{n,0}\le \frac{2^{n-3}R}{(1-\overline{\lambda}_{n-2})(1-\lambda_{n-1})}-R,$$

with $\lambda_{n-1}$ to be determined. Choose $M_{n-2}>(2^{n-1}R)/(1-\overline{\lambda}_{n-2})$. There exists some $\lambda''_{n-1}\in(1/2,1)$ such that, if $\lambda_{n-1}=\lambda''_{n-1}$, $\underline{T}_n$ satisfies

$$M_{n-2}\ge \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}}+2\overline{C}\mathrm{e}^{q_{n-1}M_{n-1}}\mathrm{e}^{p_{n-2}M_{n-2}}\underline{T}_n{}^{1/2}.$$

Take the initial data in $\mathbb{V}_2$ such that $\lambda_j=1/2$, $j=1,2,\dots,n-3$, $\lambda_{n-2}=\overline{\lambda}_{n-2}$ and $\lambda_{n-1}=\lambda''_{n-1}$. For $\underline{u}_{n,0}(x)\le u_{n,0}(x)$ and $u_1(x,t)\ge u_{1,0}(x)\ge R$, we have $\underline{u}_n\le u_n$ and $T\le \underline{T}_n$. Thus,

$$M_{n-2}\ge \frac{2^{n-1}R}{1-\overline{\lambda}_{n-2}}+2\overline{C}\mathrm{e}^{q_{n-1}M_{n-1}}\mathrm{e}^{p_{n-2}M_{n-2}}T^{1/2}.$$

Consider the next auxiliary problem:

$$\begin{cases}(\overline{u}_{n-2})_t=\Delta\overline{u}_{n-2} & (x,t)\in B_R\times(0,+\infty),\\ \frac{\partial\overline{u}_{n-2}}{\partial\eta}=\mathrm{e}^{q_{n-1}M_{n-1}}\mathrm{e}^{p_{n-2}M_{n-2}} & (x,t)\in\partial B_R\times(0,+\infty),\\ \overline{u}_{n-2}(x,0)=\overline{u}_{n-2,0}(x) & x\in B_R,\end{cases}$$

where the radial symmetric $\overline{u}_{n-2,0}(x)$ satisfies the compatibility condition and $\overline{u}_{n-2,0}(R)=(2^{n-1}R)/(1-\overline{\lambda}_{n-2})$, $\Delta\overline{u}_{n-2,0}(x)\ge 0$ and $\overline{u}_{n-2,0}(x)\ge u_{n-2,0}(x)$ for $x\in B_R$. By Green's identity, $\overline{u}_{n-2}\le M_{n-2}$. Thus, $\overline{u}_{n-2}$ satisfies

$$\frac{\partial\overline{u}_{n-2}}{\partial\eta}\ge \mathrm{e}^{q_{n-1}M_{n-1}}\mathrm{e}^{p_{n-2}\overline{u}_{n-2}},\quad (x,t)\in\partial B_R\times(0,T).$$

For $u_{n-1}\le M_{n-1}$, $u_{n-2}$ satisfies

$$\frac{\partial u_{n-2}}{\partial\eta}\le \mathrm{e}^{q_{n-1}M_{n-1}}\mathrm{e}^{p_{n-2}u_{n-2}},\quad (x,t)\in\partial B_R\times(0,T).$$

By the comparison principle, $u_{n-2}\le\overline{u}_{n-2}\le M_{n-2}$. Then, only $u_n$ blows up. □

Similarly to the proof of Lemma 3.5, we have the following lemma.

**Lemma 3.10.**

(i) *The set of initial data in* $\mathbb{V}_2$ *such that* $u_n$ *blows up while the others remain bounded is open in* $L^\infty$*-topology.*

(ii) *The set of initial data in* $\mathbb{V}_2$ *such that* $u_{n-2}$ *blows up while the others remain bounded is open in* $L^\infty$*-topology.*

**Lemma 3.11.** *If* $q_n < p_n$, $q_{n-2} < p_{n-2}$ *and* $u_{n-2}, u_n$ *blow up simultaneously while the others remain bounded up to time* $T$*, then*

$$(\mathrm{e}^{U_{n-2}(t)}, \mathrm{e}^{U_n(t)}) \sim ((T-t)^{-1/(2p_{n-2})}, (T-t)^{-1/(2p_n)}).$$

*Proof.* The proof is similar to the scale case of the system in [**17**]. Here, we omit the details. □

Until now, we have obtained Proposition 3.6. Finally, we consider the subcase $k = n-1$. Similarly to Proposition 3.1, we give the following proposition without a proof.

**Proposition 3.12.** *If* $q_n < p_n$ *and* $q_1 < p_1$*, then there exist suitable initial data such that* $u_1$ *and* $u_n$ *blow up simultaneously at some time* $T$ *while the others remain bounded up to* $T$*. Moreover,*

$$(\mathrm{e}^{U_1(t)}, \mathrm{e}^{U_n(t)}) \sim ((T-t)^{-1/(2p_1)}, (T-t)^{-(p_1-q_1)/(2p_1p_n)}). \qquad \square$$

Thereby, Theorem 1.3 for $n \geq 3$ is obtained.

*Proof of Theorem* 1.3 *for* $n = 2$. Simultaneous blow-up of $(u_1, u_2)$ can be proved similarly to the proof of Proposition 3.1. The blow-up rates may be followed by the methods used in [**17**]. □

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