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ON THE MAXIMAL INDEPENDENCE POLYNOMIAL OF CERTAIN GRAPH CONFIGURATIONS

HAN HU, TOUFIK MANSOUR AND CHUNWEI SONG

ABSTRACT. In this paper, we investigate the maximal independence polynomials of some popular graph configurations. Through careful analysis, some of the polynomials under study are shown to be Chebyshev, which helps characterize polynomial properties such as unimodality, log-concavity and real-rootedness with ease and efficiency. We partially characterize the bridge path and bridge cycle graphs of wheels and fans according to their unimodality properties and propose relevant open problems. Also, to compare with the usual independence polynomials, we provide analogous results regarding the vertebrated graph, and the firecracker graph, as studied by Wang and Zhu **[47]**.

1. Introduction. Throughout this paper, we consider only finite simple connected graphs. Let $G=(V,E)$ be such a graph with vertex set V and edge set E. For a vertex $v\in V$, let $N(v)=\{w\mid vw\in E\}$ be the collection of its neighbors, and let $N[v]=N(v)\cup\{v\}$ denote the closure of its neighborhood. The reader is referred to **[11, 48]** for graph theory terminologies not specified here.

The independence polynomial was introduced in **[14]** as a generalization of the matching polynomial:

$$I(G;x):=\sum_{k\geq 0} i_k(G)x^k,$$

where $i_k(G)$ represents the number of independent subsets of V with cardinality k, i.e.,

$$i_k(G)=|\{A\subseteq V\mid \text{ the induced subgraph } G[A] \text{ is an empty graph}\}|.$$

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The leading coefficient counts the number of maximum independent sets, those with cardinality $\alpha(G)$. Important properties of the independence polynomial, including real-rootedness, log-concavity and unimodality, have been extensively studied, see, for instance, **[2, 3, 4, 6]**–**[9, 14]**–**[16, 22, 24, 26]**–**[28, 33, 40, 42, 46, 47, 51, 52]**, etc.

In general, the polynomial

$$f(x) := \sum_{i=0}^{n} a_i x^i$$

is called *unimodal* if the sequence $a_0, a_1, \ldots, a_n$ is unimodal, i.e., there exists some peak index m (the "mode," possibly not unique, however) such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

When the coefficients $a_0, a_1, \ldots, a_n$ are all positive, $f(x)$ is said to be *logarithmic concave* if, for all $1 \leq k \leq n-1$, it holds that $a_k^2 \geq a_{k-1}a_{k+1}$, i.e., $\log a_k \geq (\log a_{k-1} + \log a_{k+1})/2$. Particularly interesting to combinatorists, unimodality problems are widely studied by mathematicians from a vast array of disciplines. It is well known that real-rootedness is stronger than log-concavity, and the latter is still stronger than unimodality **[41, 47, 50]**. Sometimes we find that the polynomial $f(x)$ has a *gap* at x^k, i.e., $a_k = 0$ when $l < k$, $r > k$, such that $a_l, a_r > 0$. Obviously, a polynomial with any gap cannot be unimodal.

In this paper, we study the maximal independence polynomial as it makes sense to focus on maximal independent sets instead of taking arbitrary independent sets that may actually be covered by those which are larger. Here, by a *maximal independent set* (MIS) of G, we require the set $A \subseteq V$ to (1) be independent, and (2) have no *strictly super* independent set W such that $A \subseteq W \subseteq V$. Accordingly, the *maximal independence polynomial* is defined by

$$I_{\max}(G; x) := \sum_{A\colon A \text{ is an MIS of } G} x^{|A|}.$$

We remark that, while maximal independent sets are also widely studied, see e.g., **[10, 12, 13, 17]**–**[21]**, **[23, 25, 29, 30, 31]**, **[34,**

35, 37, 38, 43, 44, 49], investigation of the notion of the maximal independence polynomial has been nonetheless sparse.

Apparently, $I_{\max}(G;x)$ and $I(G;x)$ have the same degree and the same leading coefficient, but in general, they are different. Partially motivated by the study of [**47**], here we are interested in investigating $I_{\max}(G;x)$ and the comparison of unimodality properties of $I_{\max}(G;x)$ and $I(G;x)$, and so on.

For usually disjoint subsets U_1, U_2 of V, define

$$I_{\max}{}^{U_1,\overline{U_2}}(G;x) := \sum_{\substack{A\colon A \text{ is an MIS of } G\\ U_1\subseteq A\subseteq V;\ U_2\subseteq \bar{A}\subseteq V}} x^{|A|},$$

which will be useful shortly. In standard notation, the overlined $\overline{U}$ denotes the complement of U in V; here, in this paper, we use it differently. If U_1, U_2, or both, have cardinality 1, we may use the shorthand notation $I_{\max}{}^{u_1,\overline{U_2}}(G;x)$, $I_{\max}{}^{U_1,\overline{u_2}}(G;x)$, $I_{\max}{}^{u_1,\overline{u_2}}(G;x)$ for $I_{\max}{}^{\{u_1\},\overline{U_2}}(G;x)$, $I_{\max}{}^{U_1,\overline{\{u_2\}}}(G;x)$ and $I_{\max}{}^{\{u_1\},\overline{\{u_2\}}}(G;x)$, respectively. We allow one of U_1, U_2 to be empty; thus, the restriction is one-fold only, in which case, the $\emptyset$ is often omitted from the notation:

$$I_{\max}{}^{U_1}(G;x) = I_{\max}{}^{U_1,\bar{\emptyset}}(G;x), \qquad I_{\max}{}^{\overline{U_2}}(G;x) = I_{\max}{}^{\emptyset,\overline{U_2}}(G;x).$$

Clearly, $I_{\max}{}^{U_1}(G;x) = 0$ if $E(G[U_1]) \neq \emptyset$.

Throughout, we use the convention that, for $a, b \in \mathbb{Z}$, the binomial coefficient $\binom{a}{b}$ is taken to be 0 whenever $a < b$ or $b < 0$.

2. General results.

Let $v \in V(G)$. For convenience, we introduce the following notation which reflects a subtle treatment of the situation. Define $I^{\tilde{v}}_{\max}(G;x)$ to be $I^{\overline{N(v)}}_{\max}(G-v;x)$. Note that $I^{\overline{N(v)}}_{\max}(G-v;x)$ is different from a somewhat similar $I^{\overline{N[v]}}_{\max}(G;x)$, which is the zero polynomial since $G[N[v]]$ is not an empty graph (recall the assumption that G is connected from the beginning).

The next lemma establishes a recurrence relation of the maximal independence polynomial.

Lemma 2.1. *Let $G = (V, E)$ be any graph with an arbitrarily fixed vertex $v \in V$. Then,*

$$I_{\max}(G;x) = \sum_{\substack{\emptyset \neq A \subseteq N(v) \\ B = N(v) - A}} I_{\max}^{A,\bar{B}}(G - v; x) + x I_{\max}^{\bar{v}}(G; x),$$

where $G - v$ is the graph obtained by deleting the vertex v from G.

Proof. Straightforward. (Similar to the standard proof of the "usual" recursive formula for the standard independence polynomial.) □

In subsequent sections, Lemma 2.1 will be used to derive maximal independence polynomials for various classes of graphs. Here, we investigate the n-path, i.e., the path on n vertices, and the n-cycle.

Let P_n be a path graph with n vertices, say, labeled by $1, 2, \ldots, n$. Then, $I_{\max}(P_0; x) = 1$, $I_{\max}(P_1; x) = x$, $I_{\max}(P_2; x) = 2x$ and, for $n \geq 3$,

$$\begin{aligned} I_{\max}(P_n; x) &= I_{\max}^{n}(P_n; x) + I_{\max}^{\bar{n}}(P_n; x) \\ &= x I_{\max}(P_{n-2}; x) + x I_{\max}(P_{n-3}; x). \end{aligned} \tag{2.1}$$

Proposition 2.2. *The maximal independence polynomial of path P_n is given by*

$$I_{\max}(P_n; x) = \sum_{j \geq 0} \binom{j+1}{n+1-2j} x^j,$$

where, as usual, $\binom{a}{b}$ is assumed to be 0 whenever $a < b$ or $b < 0$ for $a, b \in \mathbb{Z}$. Moreover, the polynomial $I_{\max}(P_n; x)$ is log-concave, and therefore, unimodal.

Proof. If we define $f(t) = \sum_{n \geq 0} I_{\max}(P_n; x) t^n$, then (2.1) gives

$$f(t) - 2xt^2 - xt - 1 = xt^2(f(t) - 1) + xt^3 f(t),$$

which implies

$$f(t) = \frac{1+xt+xt^2}{1-xt^2(1+t)} = (1+xt(1+t))\sum_{j\geq 0} x^j t^{2j}(1+t)^j$$
$$= \sum_{j\geq 0} x^j t^{2j}(1+t)^j + \sum_{j\geq 0} x^{j+1} t^{2j+1}(1+t)^{j+1}.$$

Therefore, the coefficient of t^n in the generating function $f(t)$ is given by

$$I_{\max}(P_n;x) = \sum_{j\geq 0}\binom{j}{n-2j}x^j + \sum_{j\geq 0}\binom{j+1}{n-1-2j}x^{j+1}$$
$$= \sum_{j\geq 0}\binom{j}{n-2j}x^j + \sum_{j\geq 0}\binom{j}{n+1-2j}x^j$$
$$= \sum_{j\geq 0}\binom{j+1}{n+1-2j}x^j,$$

where the last equality is based on Pascal's well-known identity

$$\binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}.$$

Hence, we have obtained the maximal independence polynomial for P_n. Clearly, $\deg I_{\max}(P_n;x) = \lfloor (n+1)/2 \rfloor$, and the lowest nonzero term of $I_{\max}(P_n;x)$ has degree $\lceil n/3 \rceil$.

Straightforward computation shows that log-concavity holds, and unimodality follows. □

Proposition 2.3. *For all* $n \geq 3$,

$$I_{\max}(C_n;x) = \sum_{j\geq 1}\frac{n}{j}\binom{j}{n-2j}x^j.$$

Moreover, the polynomial $I_{\max}(C_n;x)$ *is log-concave, and therefore, unimodal.*

Proof. Let C_n be the cycle graph on the vertices $1, 2, \ldots, n$. Then, Lemma 2.1 for $G = C_n$ gives

$$\begin{aligned}
I_{\max}(C_n;x) &= I_{\max}^{n}(C_n;x) + I_{\max}^{\bar{n}}(C_n;x)\\
&= xI_{\max}(P_{n-3};x) + I_{\max}^{1,\overline{\{n-1,n\}}}(C_n;x)\\
&\quad + I_{\max}^{n-1,\overline{\{1,n\}}}(C_n;x) + I_{\max}^{\{1,n-1\},\bar{n}}(C_n;x)\\
&= xI_{\max}(P_{n-3};x) + I_{\max}^{\{1,n-2\},\overline{\{2,n-3,n-1,n\}}}(C_n;x)\\
&\quad + I_{\max}{}^{\{2,n-1\},\overline{\{1,3,n-2,n\}}}(C_n;x) + x^2 I_{\max}(P_{n-5};x)\\
&= xI_{\max}(P_{n-3};x) + 2x^2 I_{\max}(P_{n-6};x) + x^2 I_{\max}(P_{n-5};x),
\end{aligned}$$

for all $n \geq 7$.

Hence, by Proposition 2.2, we obtain

$$\begin{aligned}
I_{\max}(C_n;x) &= \sum_{j\geq 0}\binom{j+1}{n-2-2j}x^{j+1} + 2\sum_{j\geq 0}\binom{j+1}{n-5-2j}x^{j+2}\\
&\quad + \sum_{j\geq 0}\binom{j+1}{n-4-2j}x^{j+2},
\end{aligned}$$

which, by Pascal's identity, implies that, for $n \geq 7$,

$$\begin{aligned}
I_{\max}(C_n;x) &= \sum_{j\geq 0}\binom{j+1}{n-2-2j}x^{j+1} + \sum_{j\geq 0}\binom{j+1}{n-5-2j}x^{j+2}\\
&\quad + \sum_{j\geq 0}\binom{j+2}{n-4-2j}x^{j+2}\\
&= 2\sum_{j\geq 0}\binom{j+1}{n-2-2j}x^{j+1} + \sum_{j\geq 0}\binom{j}{n-3-2j}x^{j+1}\\
&= \sum_{j\geq 0}\frac{n}{j}\binom{j}{n-2j}x^{j}.
\end{aligned}$$

Direct calculation shows that $I_{\max}(C_2;x) = 2x$, $I_{\max}(C_3;x) = 3x$, $I_{\max}(C_4;x) = 2x^2$, $I_{\max}(C_5;x) = 5x^2$ and $I_{\max}(C_6;x) = 3x^2 + 2x^3$.

Therefore, the first statement of Proposition 2.3 holds for all n, based on which the log-concavity, and hence unimodality, is clear. $\square$

Definition 2.4. Begin with $G = G(V, E)$, where $V = \{v_1, v_2, \ldots, v_m\}$. Now, G is repeated n times as $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $G^{(j)}$ having vertices

$v_i^{(j)}$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Then, we obtain two graphs $G \times P_n$ and $G \times C_n$. $G \times P_n$ is the graph obtained by connecting vertices $v_i^{(j)}$ and $v_i^{(j+1)}$ with an edge, for all $1 \leq i \leq m$, $1 \leq j \leq n-1$, and $G \times C_n$ is based on $G \times P_n$, obtained by further connecting vertices $v_i^{(1)}$ and $v_i^{(n)}$, $1 \leq i \leq m$. Moreover, for permutation σ on the letters $\{1, 2, \ldots, m\}$, $G \times C_n^{\sigma}$ is the graph obtained from $G \times P_n$ by further connecting the vertices $v_i^{(1)}$ and $v_{\sigma(i)}^{(n)}$ with an edge, for all $1 \leq i \leq m$.

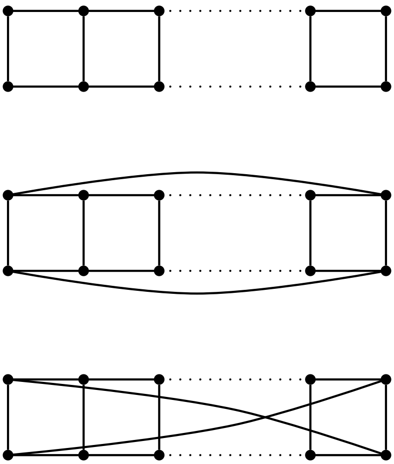

FIGURE 1. Graph $P_2 \times P_n, P_2 \times C_n$ and $P_2 \times C_n^{(12)}$.

Proposition 2.5. *The maximal independence polynomials of graphs* $P_2 \times P_n, P_2 \times C_n, P_2 \times C_n^{(12)}$ *are, respectively,*

$$I_{\max}(P_2 \times P_n; x) = 2 \sum_{d} \binom{d-1}{n-d} x^d,$$

$$I_{\max}(P_2 \times C_n; x) = 2 \sum_{2 \mid d} \frac{n}{d} \binom{d}{n-d} x^d,$$

$$I_{\max}(P_2 \times C_n^{(12)}; x) = 2 \sum_{2 \mid d+1} \frac{n}{d} \binom{d}{n-d} x^d.$$

Note that, while Proposition 2.5 may also be proven by generating functions, we give an illustration below of a purely combinatorial proof.

First, let $G = P_2$.

Proof. (*Combinatorial enumeration*). We investigate the manner in which vertices in $G = P_2 \times P_n$ or $G = P_2 \times C_n$ are "selected" in an MIS and determine $\#\{|A| = d : A \text{ is an MIS of } G\}$ for fixed d. We address $v_1^{(1)}, v_1^{(2)}, \ldots, v_1^{(n)}$ by "top" vertices and $v_2^{(1)}, v_2^{(2)}, \ldots, v_2^{(n)}$ by "bottom" vertices. Fix an MIS of size d. Observe that, if a top vertex $v_1^{(j)}$ is selected in MIS, then none of $v_1^{(j-1)}, v_1^{(j+1)}, v_2^{(j)}$ can exist in MIS; therefore, either $v_2^{(j-1)}$ is selected in MIS or $v_2^{(j-1)}$ is not selected in MIS. However, $v_2^{(j-2)}$ must be in MIS. Two conclusions may obtained.

(1) In an MIS, the "top" and "bottom" vertices occur alternately from left to right.

(2) Regardless of the subscript i, the index j of vertices $v_i^{(j)}$ in an MIS forms a strictly increasing sequence $1 \le j_1 < j_2 < \cdots < j_d \le n$ with the property $j_{t+1} - j_t \in \{1, 2\}$, for $1 \le t \le d - 1$.

For the case of $G = P_2 \times P_n$, it is obvious that one of $\{v_1^{(1)}, v_2^{(1)}\}$ must be selected in MIS and one of $\{v_1^{(n)}, v_2^{(n)}\}$ must also be selected. Note that the sequence $1 = j_1 < j_2 < \cdots < j_d = n$ with property $j_{t+1} - j_t \in \{1, 2\}$ has $\binom{d-1}{n-d}$ ways to be constructed. In addition, there are two ways to obtain either $\{v_1^{(1)}\}$ or $\{v_2^{(1)}\}$, which determines the oscillating pattern of the "top" and "bottom" sequence of d vertices. Hence, the coefficient of x^d in $I_{\max}(P_2 \times P_n; x)$ is $2\binom{d-1}{n-d}$.

For the case of $G = P_2 \times C_n$, the up-down phenomenon is the same; however, possibly none of $\{v_1^{(1)}, v_2^{(1)}\}$ would be selected in an MIS. For symmetry, the probability that one of $\{v_1^{(1)}, v_2^{(1)}\}$ is selected in MIS is d/n. Denote this probability by P_1. In order to illustrate, suppose that $v_1^{(1)}$ is in MIS. It is easy to see that either $v_2^{(n-1)}$ or $v_2^{(n)}$ must be selected, and moreover, it follows that d must be even. Next, there is a bijection between the MISs of $P_2 \times C_n$ with $v_1^{(1)}$ selected and the MISs of $P_2 \times P_{n+1}$ with both $v_1^{(1)}$ and $v_1^{(n+1)}$ selected. As in the previous case, there are

$$\binom{d+1}{(n+1)-(d+1)} = \binom{d}{n-d}$$

methods of constructing the subscript sequence $1 = j_1 < j_2 < \cdots < j_d < j_{d+1} = n+1$ with property $j_{t+1} - j_t \in \{1, 2\}$ (while the oscillating pattern is already fixed). Hence, the coefficient of x^d in $I_{\max}(P_2 \times C_n; x)$ is

$$\frac{2}{P_1}\binom{d}{n-d} = \frac{2n}{d}\binom{d}{n-d},$$

where d must be even.

For the case of $G = P_2 \times C_n^{(12)}$, the proof is similar to the case of $G = P_2 \times C_n$, except that the presence of $v_1^{(1)}$ requires either $v_1^{(n-1)}$ or $v_1^{(n)}$ to be selected, and the bijection is between the MISs of $P_2 \times C_n$ with $v_1^{(1)}$ selected and the MISs of $P_2 \times P_{n+1}$ with both $v_1^{(1)}$ and $v_2^{(n+1)}$ selected. In this case, d must be odd. □

Maximal independence polynomials $I_{\max}(P_2 \times C_n; x)$ and $I_{\max}(P_2 \times C_n^{(12)}; x)$ are nonunimodal due to their many gaps. Maximal independence polynomial $I_{\max}(P_2 \times P_n; x)$ is monotone and log-concave by checking the coefficients:

$$\binom{d-1}{n-d}^2 \geq \binom{d-2}{n-d+1}\binom{d}{n-d-1}$$

(with the condition $d - 1 \geq n - d$).

Now we focus on the following graphs: $G = P_3 \times P_n$, $G = P_3 \times C_n$, $G = C_3 \times P_n$ and $G = C_3 \times C_n$.

Proposition 2.6. *The maximal independence polynomials of graphs $C_3 \times P_n$ and $C_3 \times C_n$ are:*

$$I_{\max}(C_3 \times P_n; x) = 3 \times 2^{n-1} x^n,$$
$$I_{\max}(C_3 \times C_n; x) = (2^n + 2 \times (-1)^n) x^n, \quad n \geq 3.$$

Proof. For the cases of $G = C_3 \times P_n$ and $G = C_3 \times C_n$, it is obvious that no two vertices of each C_3 $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$ may be selected in MIS at the same time. In addition, it is impossible that none of $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$ are selected, for otherwise, either at least two of $\{v_1^{(j-1)}, v_2^{(j-1)}, v_3^{(j-1)}\}$ or at least two of $\{v_1^{(j+1)}, v_2^{(j+1)}, v_3^{(j+1)}\}$ must be included in order to cover $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$, which is problematic. Hence, exactly one of $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$ should be selected for any MIS.

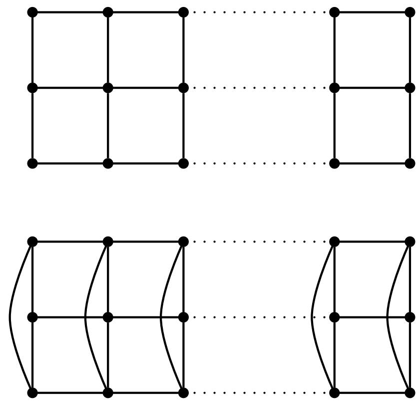

FIGURE 2. Graph $P_3 \times P_n$ and $C_3 \times P_n$.

Finally, note that 3-colorings of P_n and C_n are counted by $3 \times 2^{n-1}$ and $2^n + 2 \times (-1)^n$, respectively, [**5,** page 65], [**39,** A007283, A092297]. □

The cases of $G \times P_n$ and $G \times C_n$, here $G = P_3$, are not as simple compared with the cases of $G = P_2$ and $G = C_3$. In any MIS of $P_3 \times P_n$ (or of $P_3 \times C_n$), individual vertices of $\{V_1^{(j)}, V_2^{(j)}, V_3^{(j)}\}$ must be selected according to one of the five types.

Type 1. $\{v_1^{(j)}, v_3^{(j)}\}$ are both included.

Type 2/3/4. Exactly one vertex of $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$ is selected.

Type 5. None of $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}\}$ appears.

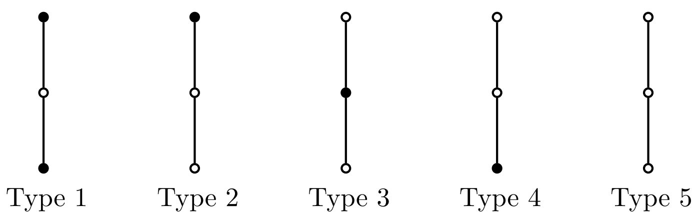

FIGURE 3. A visual graph of five types.

For example, for $P_3 \times P_n$, we may consider the generating functions,

$$g_{i,j}(x,y) = \sum_{d\geq 0}\sum_{n\geq 2} a_{d,n}^{i,j} x^d y^n,$$

where $a_{d,n}^{i,j}$ represents the number of different ways of selecting d vertices from graph $P_3 \times P_n$, such that the $(n-1)$th column has type i, the nth column has type j, and the d vertices form an MIS. Note that

$$\sum_{n\geq 2} I_{\max}(P_3 \times P_n; x) \times y^n = g_{1,3}+g_{2,3}+g_{2,4}+g_{3,1}+g_{4,2}+g_{4,3}+g_{5,1}+g_{5,3},$$

and recurrence relationships such as $g_{1,3} = xy(g_{5,1} + g_{3,1}) + x^3y^2$ aid in calculating

$$\begin{aligned}\sum_{n\geq 2} I_{\max}(P_3 \times P_n; x) y^n & (2x^2 + 2x^3)y^2 \\ & + (4x^3 - 3x^4 + x^5)y^3 + (-4x^4 + 4x^5 - 2x^6)y^4 \\ &= \frac{+(x^4 - 3x^5 - 5x^6 + x^7)y^5 + (-4x^5 + 2x^7)y^6 + (3x^6 + x^7)y^7}{(1 - xy)(1 - xy - x^3y^2 + (x^4 - 4x^3)y^3 + (2x^4 - x^3)y^4 + x^4y^5)} \\ &= (2x^2 + 2x^3)y^2 + (8x^3 + x^4 + x^5)y^3 + (10x^4 + 6x^5 + 2x^6)y^4 + \cdots .\end{aligned}$$

In fact, 12 recurrence formulas, which are linear equations on $g_{i,j}$, are used to derive the above formula. This yields $I_{\max}(P_3 \times P_2; x) = 2x^2 + 2x^3$, $I_{\max}(P_3 \times P_3; x) = 8x^3+x^4+x^5$, $I_{\max}(P_3 \times P_4; x) = 10x^4+6x^5+2x^6$, and so on. For an illustration of the MISs of $P_3 \times P_3$, see Figure 4.

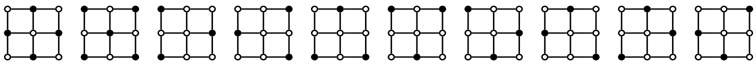

FIGURE 4. MIS in the graph $P_3 \times P_3$.

We will skip the calculation of $P_3 \times C_n$.

For the remainder of this paper we will apply our general results to several interesting graph concatenation configurations that appear in [**47**]. In Section 3, we study the type of bridge path graphs, while in Section 4, we investigate bridge cycle graphs.

3. Bridge path graph. Let $\{H_i\}_{i=1}^d$ be a sequence of finite pairwise disjoint graphs with specific $v_i \in V(H_i)$ and $V(H_i) - v_i \neq \emptyset$. The *bridge* *path graph*

$$B(H_1, H_2, \dots, H_d) \equiv B(H_1, H_2, \dots, H_d; v_1, v_2, \dots, v_d)$$

of sequence $\{H_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is obtained from the graphs $H_1, \dots, H_d$ by connecting the vertices v_i and v_{i+1} with an edge for all $i = 1, 2, \dots, d-1$. See Figure 5 for an illustration.

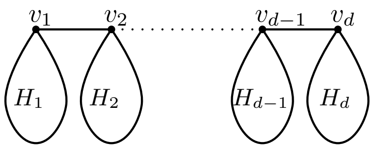

FIGURE 5. The bridge path graph.

Based upon Lemma 2.1, we have the following theorem regarding the general configuration of bridge path graphs.

Theorem 3.1. *For $i = 1, 2, \dots, d$, $d \geq 3$, let H_i be any graph of order ≥ 2. Then, the maximal independence polynomial for the bridge path graph $B_d := B(H_1, H_2, \dots, H_d; v_1, v_2, \dots, v_d)$ is recursively given by*

$$\begin{aligned}(3.1)\quad I_{\max}(B_d; x) = {} & I_{\max}^{\overline{v_d}}(H_d; x) I_{\max}(B_{d-1}; x) \\ & + I_{\max}^{v_d}(H_d; x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1}; x) + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1}; x)) \\ & \cdot I_{\max}(B_{d-2}; x) + I_{\max}^{\widetilde{v_d}}(H_d; x) I_{\max}^{v_{d-1}}(H_{d-1}; x) \\ & \cdot (I_{\max}^{\overline{v_{d-2}}}(H_{d-2}; x) + I_{\max}^{\widetilde{v_{d-2}}}(H_{d-2}; x)) I_{\max}(B_{d-3}; x),\end{aligned}$$

with $I_{\max}(B_0; x) = 1$, $I_{\max}(B_1; x) = I_{\max}(H_1; x)$ and

$$\begin{aligned}I_{\max}(B_2; x) = {} & I_{\max}^{\overline{v_1}}(H_1; x) I_{\max}^{\overline{v_2}}(H_2; x) \\ & + I_{\max}^{v_2}(H_2; x)(I_{\max}^{\overline{v_1}}(H_1; x) + I_{\max}^{\widetilde{v_1}}(H_1; x)) \\ & + I_{\max}^{v_1}(H_1; x)(I_{\max}^{\overline{v_2}}(H_2; x) + I_{\max}^{\widetilde{v_2}}(H_2; x)).\end{aligned}$$

Proof. By Lemma 2.1 and the fact that $I_{\max}(G; x) = I_{\max}^{v}(G; x) + I_{\max}^{\bar{v}}(G; x)$, we have

$$(3.2)\qquad I_{\max}(B_d; x) = I_{\max}^{v_d}(B_d; x) + I_{\max}^{\overline{v_d}}(B_d; x),$$

where

$$\begin{aligned} I_{\max}^{v_d}(B_d;x) &= I_{\max}^{v_d,\overline{v_{d-1}}}(B_d;x) \\ &= I_{\max}^{\{v_d,v_{d-2}\},\overline{v_{d-1}}}(B_d;x) + I_{\max}^{v_d,\overline{\{v_{d-1},v_{d-2}\}}}(B_d;x) \\ &= I_{\max}^{v_d}(H_d;x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1};x))I_{\max}^{v_{d-2}}(B_{d-2};x) \\ &\quad + I_{\max}^{v_d}(H_d;x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1};x))I_{\max}^{\overline{v_{d-2}}}(B_{d-2};x) \\ &= I_{\max}^{v_d}(H_d;x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1};x))I_{\max}(B_{d-2};x) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} I_{\max}^{\overline{v_d}}(B_d;x) &= I_{\max}^{v_{d-1},\overline{v_d}}(B_d;x) + I_{\max}^{\overline{\{v_{d-1},v_d\}}}(B_d;x) \\ &= (I_{\max}^{\overline{v_d}}(H_d;x) + I_{\max}^{\widetilde{v_d}}(H_d;x))I_{\max}^{v_{d-1}}(B_{d-1};x) \\ &\quad + I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}^{\overline{v_{d-1}}}(B_{d-1};x) \\ &= I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}(B_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_d}}(H_d;x)I_{\max}^{v_{d-1}}(B_{d-1};x). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned} I_{\max}^{v_d}(B_d;x) &= I_{\max}^{v_d}(H_d;x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1};x))I_{\max}(B_{d-2};x), \end{aligned} \tag{3.5}$$

$$\begin{aligned} I_{\max}^{\overline{v_d}}(B_d;x) &= I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}(B_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_d}}(H_d;x)I_{\max}^{v_{d-1}}(H_{d-1};x) \\ &\quad \cdot (I_{\max}^{\overline{v_{d-2}}}(H_{d-2};x) + I_{\max}^{\widetilde{v_{d-2}}}(H_{d-2};x))I_{\max}(B_{d-3};x). \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.2), we have

$$\begin{aligned} I_{\max}(B_d;x) &= I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}(B_{d-1};x) \\ &\quad + I_{\max}^{v_d}(H_d;x)(I_{\max}^{\overline{v_{d-1}}}(H_{d-1};x) \\ &\quad + I_{\max}^{\widetilde{v_{d-1}}}(H_{d-1};x))I_{\max}(B_{d-2};x) \\ &\quad + I_{\max}^{\widetilde{v_d}}(H_d;x)I_{\max}^{v_{d-1}}(H_{d-1};x) \\ &\quad \cdot (I_{\max}^{\overline{v_{d-2}}}(H_{d-2};x) + I_{\max}^{\widetilde{v_{d-2}}}(H_{d-2};x))I_{\max}(B_{d-3};x). \end{aligned}$$

Direct calculation gives the initial values of this recurrence relation, namely, $I_{\max}(B_0;x) = 1$, $I_{\max}(B_1;x) = I_{\max}(H_1;x)$ and

$$\begin{aligned}I_{\max}(B_2;x) = I_{\max}^{\overline{v_1}}(H_1;x)I_{\max}^{\overline{v_2}}(H_2;x)\\ + I_{\max}^{v_2}(H_2;x)(I_{\max}^{\overline{v_1}}(H_1;x) + I_{\max}^{\widetilde{v_1}}(H_1;x))\\ + I_{\max}^{v_1}(H_1;x)(I_{\max}^{\overline{v_2}}(H_2;x) + I_{\max}^{\widetilde{v_2}}(H_2;x)),\end{aligned}$$

completing the proof. □

For the remainder of this section, we consider the case where all $H_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$. Also, we define

$$\begin{aligned}\alpha &:= I_{\max}^{\bar{v}}(H;x),\\ \beta &:= I_{\max}^{v}(H;x)(I_{\max}^{\bar{v}}(H;x) + I_{\max}^{\tilde{v}}(H;x))\end{aligned}$$

and

$$\gamma := I_{\max}^{\tilde{v}}(H;x)\beta.$$

These parameters play important roles in our consequent derivations.

Corollary 3.2. *For the bridge path graph $\mathbb{B}_d = \mathbb{B}_d(H,v) = B(H,H,\ldots,H;v,v,\ldots,v)$ connecting the same component H, $|V(H)| \geq 2$, a total of $d \geq 3$ times,*

$$I_{\max}(\mathbb{B}_d;x) = f_d + I_{\max}^{v}(H;x)f_{d-1} + I_{\max}^{v}(H;x)I_{\max}^{\tilde{v}}(H;x)f_{d-2}, \tag{3.7}$$

$d \geq 2$, where

$$f_d := \sum_{i=0}^{d}\sum_{j=0}^{\min(i,d-i)} \binom{i}{j}\binom{j}{d-i-j}\alpha^{i-j}\beta^{2j+i-d}\gamma^{d-i-j},$$

with the initial polynomials $I_{\max}(\mathbb{B}_0;x) = 1$ and $I_{\max}(\mathbb{B}_1;x) = I_{\max}(H;x)$.

Proof. Let $H_i = H$ and $v_i = v$, $i = 1, 2, \ldots, d$. By (3.1), we have

$$I_{\max}(\mathbb{B}_d;x) = \alpha I_{\max}(\mathbb{B}_{d-1};x) + \beta I_{\max}(\mathbb{B}_{d-2};x) + \gamma I_{\max}(\mathbb{B}_{d-3};x) \tag{3.8}$$

with $I_{\max}(\mathbb{B}_0;x) = 1$, $I_{\max}(\mathbb{B}_1;x) = I_{\max}(H;x)$ and

$$I_{\max}(\mathbb{B}_2;x) = (I_{\max}^{\bar{v}}(H;x))^2 + 2I_{\max}^{v}(H;x)(I_{\max}^{\bar{v}}(H;x) + I_{\max}^{\tilde{v}}(H;x)).$$

Let $F(t)$ be the generating function for the sequence $I_{\max}(\mathbb{B}_d;x)$, that is,

$$F(t)=\sum_{d\geq 0} I_{\max}(\mathbb{B}_d;x)t^d.$$

Multiplying (3.8) by t^d and summing over all $d\geq 3$ with the above initial values, we obtain

$$\begin{aligned}&F(t)-I_{\max}(\mathbb{B}_2;x)t^2-I_{\max}(\mathbb{B}_1;x)t-1\\&\qquad=\alpha(F(t)-I_{\max}(\mathbb{B}_1;x)t-1)t+\beta(F(t)-1)t^2+\gamma F(t)t^3,\end{aligned}$$

which implies

$$\begin{aligned}F(t)&=\frac{1+I^v_{\max}(H;x)t+I^v_{\max}(H;x)I^{\tilde{v}}_{\max}(H;x)t^2}{1-\alpha t-\beta t^2-\gamma t^3}\\&=(1+I^v_{\max}(H;x)t+I^v_{\max}(H;x)I^{\tilde{v}}_{\max}(H;x)t^2)\\&\quad+\sum_{i\geq 0}t^i(\alpha+\beta t+\gamma t^2)^i\\&=(1+I^v_{\max}(H;x)t+I^v_{\max}(H;x)I^{\tilde{v}}_{\max}(H;x)t^2)\\&\quad+\sum_{i\geq 0}\sum_{j=0}^{i}\sum_{s=0}^{j}\binom{i}{j}\binom{j}{s}\alpha^{i-j}\beta^{j-s}\gamma^s t^{i+j+s}.\end{aligned}\tag{3.9}$$

Finally, comparing the coefficient of t^d on both sides of equation (3.9), we obtain the explicit formula for $I_{\max}(\mathbb{B}_d;x)$, as claimed. □

Corollary 3.3. *Consider the bridge path graph $\mathbb{B}_d=\mathbb{B}_d(H,v)=B(H,H,\ldots,H;v,v,\ldots,v)$ connecting the same component H, $|V(H)|\geq 2$ a total of d times. If H has the feature $I^{\tilde{v}}_{\max}(H;x)=0$, whence $\gamma=0$, then*

$$I_{\max}(\mathbb{B}_d;x)=\alpha I_{\max}(\mathbb{B}_{d-1};x)+\beta I_{\max}(\mathbb{B}_{d-2};x),\quad d\geq 2,\tag{3.10}$$

with the initial values $I_{\max}(\mathbb{B}_0;x)=1$, $I_{\max}(\mathbb{B}_1;x)=I_{\max}(H;x)$ and $\alpha:=I^{\bar{v}}_{\max}(H;x)$, $\beta:=I^v_{\max}(H;x)\alpha$. Moreover, for all $d\geq 0$,

$$I_{\max}(\mathbb{B}_d;x)=\frac{(i\sqrt{\beta})^{d+1}}{\alpha}U_{d+1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right),\tag{3.11}$$

where $i^2=-1$, and U_m is the mth Chebyshev polynomial of the second kind **[53]**.

Proof. In the case of $\gamma I_{\max} = 0$, in the above summation of (3.9), we take $s = 0$, i.e., $i + j = d$, and in (3.7), we have only two terms, or simply use (3.8) so that (3.10) is established.

Let $g_d := (I_{\max}(\mathbb{B}_d; x))/(i\sqrt{\beta})^d$. Note that the recurrence (3.10) may be written as

$$g_d = \frac{\alpha}{i\sqrt{\beta}}\, g_{d-1} - g_{d-2},$$

with $g_0 = 1$ and $g_1 = (I_{\max}(H; x))/(i\sqrt{\beta}) = \alpha/(i\sqrt{\beta}) + (I^v_{\max}(H; x))/(i\sqrt{\beta})$.

Using the fact that Chebyshev polynomials satisfy the recurrence relation $U_s(t) = 2tU_{s-1}(t) - U_{s-2}(t)$ with $U_0(t) = 1$ and $U_1(t) = 2t$, we inductively derive

$$\begin{aligned} g_d &= U_{d-1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right) g_1 - U_{d-2}\left(\frac{\alpha}{2i\sqrt{\beta}}\right) g_0 \\ &= U_d\left(\frac{\alpha}{2i\sqrt{\beta}}\right) + \frac{I^v_{\max}(H; x)}{i\sqrt{\beta}} U_{d-1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} I_{\max}(\mathbb{B}_d; x) &= (i\sqrt{\beta})^d U_d\left(\frac{\alpha}{2i\sqrt{\beta}}\right) + I^v_{\max}(H; x)(i\sqrt{\beta})^{d-1} U_{d-1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right) \\ &= \frac{(i\sqrt{\beta})^{d+1}}{\alpha}\left(\frac{\alpha}{i\sqrt{\beta}} U_d\left(\frac{\alpha}{2i\sqrt{\beta}}\right) - U_{d-1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right)\right) \\ &= \frac{(i\sqrt{\beta})^{d+1}}{\alpha} U_{d+1}\left(\frac{\alpha}{2i\sqrt{\beta}}\right), \end{aligned}$$

completing the proof of (3.11). □

Some applications of Theorem 3.1 and Corollaries 3.2 and 3.3 are presented next.

Our first application concerns the vertebrated graph. Its model widely inspires studies of mathematical biology and bioinformatics, see **[1, 32, 36, 45]**. Let H be the star graph $K_{1,m}$ with center v. Then, the bridge path graph $B(H, H, \ldots, H; v, v, \ldots, v)$ (d-times) is called the *vertebrated graph* $V_d^{(m)}$, see Figure 6.

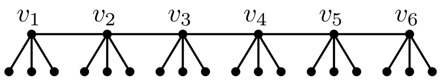

FIGURE 6. The vertebrated graph $V_6^{(3)}$.

Corollary 3.4. *For all $d \geq 0$ and $m \geq 1$,*

$$I_{\max}(V_d^{(m)}; x) = i^{d+1} \sqrt{x}^{(d-1)m+d+1} U_{d+1}\left(\frac{\sqrt{x}^{m-1}}{2i}\right) = \sum_{k=0}^{\lfloor (d+1)/2 \rfloor} \binom{d+1-k}{k} x^{dm-k(m-1)}$$

with $i^2 = -1$, and U_m is the mth Chebyshev polynomial of the second kind.

Proof. Note that

$$I_{\max}(K_{1,m}; x) = x + x^m, \qquad I_{\max}^{v}(K_{1,m}; x) = x,$$
$$I_{\max}^{\bar{v}}(K_{1,m}; x) = x^m \qquad \text{and} \qquad I_{\max}^{\tilde{v}}(K_{1,m}; x) = 0.$$

Therefore, $\alpha = x^m$, $\beta = x^{m+1}$ and $\gamma = 0$. Thus, by (3.11),

$$I_{\max}(V_d^{(m)}; x) = i^{d+1} \sqrt{x}^{(d-1)m+d+1} U_{d+1}\left(\frac{\sqrt{x}^{m-1}}{2i}\right).$$

For the second equality, we use the well-known fact that Chebyshev polynomials of the second kind satisfy

$$U_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2t)^{n-2k}. \tag{3.12}$$

Thus,

$$I_{\max}(V_d^{(m)}; x) = i^{d+1} \sqrt{x}^{(d-1)m+d+1} \sum_{k=0}^{\lfloor (d+1)/2 \rfloor} (-1)^k \cdot \binom{d+1-k}{k} \left(\frac{\sqrt{x}^{m-1}}{i}\right)^{d+1-2k}$$

$$= \sum_{k=0}^{\lfloor (d+1)/2 \rfloor} \binom{d+1-k}{k} x^{dm-k(m-1)},$$

which completes the proof. □

Example 3.5. Corollary 3.4 for $m = 1$ gives $I_{\max}(V_d^{(1)}; x) = i^{d+1} x^d U_{d+1}(1/2i) = \mathrm{Fib}_{d+2}\, x^d$, where $\mathrm{Fib}_n = \mathrm{Fib}_{n-1} + \mathrm{Fib}_{n-2}$ with $\mathrm{Fib}_0 = 0$ and $\mathrm{Fib}_1 = 1$ (the Fibonacci numbers). Thus, in this case, $I_{\max}(V_d^{(1)}; x)$ is a monomial with order d and coefficient Fib_{n+2}. It thus trivially has only real zeros $x = 0$, log-concave and unimodal. (The graph $V_d^{(1)}$ is the so-called *n-centipede graph* due to its appearance.)

Corollary 3.4 for $m = 2$ gives $I_{\max}(V_d^{(2)}; x) = i^{d+1} \sqrt{x}^{\,3d-1} U_{d+1} (\sqrt{x}/2i)$. Using the fact that

$$U_n(t) = \prod_{k=1}^{n} \left(2t - 2\cos \frac{k\pi}{n+1} \right),$$

cf., [**53**], we obtain

$$\begin{aligned} I_{\max}(V_d^{(2)}; x) &= i^{d+1} \sqrt{x}^{\,3d-1} \prod_{k=1}^{d+1} \left(\frac{\sqrt{x}}{i} - 2\cos \frac{k\pi}{d+2} \right) \\ &= x^{d-1} \prod_{k=1}^{d+1} \left(x - 2i\sqrt{x} \cos \frac{k\pi}{d+2} \right) \\ &= x^{d-(1-(-1)^d)/2} \prod_{k=1}^{\lfloor (d+1)/2 \rfloor} \left(x^2 + 4x \cos^2 \frac{k\pi}{d+2} \right). \end{aligned}$$

This shows that the polynomial $I_{\max}(V_d^{(2)}; x)$ has degree $2d$. Furthermore, $I_{\max}(V_d^{(2)}; x)$ has only real zeros and hence is log-concave and unimodal.

Also, Corollary 3.4 for $m = 3$ gives

$$\begin{aligned} I_{\max}(V_d^{(3)}; x) &= i^{d+1} x^{2d-1} U_{d+1} \left(\frac{x}{2i} \right) \\ &= x^{2d-1} \prod_{k=1}^{d+1} \left(x - 2i \cos \frac{k\pi}{d+2} \right) \end{aligned}$$

$$= x^{2d-((1-(-1)^d)/2)} \prod_{k=1}^{\lfloor (d+1)/2 \rfloor} \left(x^2 + 4\cos^2 \frac{k\pi}{d+2}\right),$$

which shows that *not all* of the zeros of the polynomial $I_{\max}(V_d^{(3)};x)$ are real. In fact, it is easy to see that the coefficients of $I_{\max}(V_d^{(3)};x)$ jump in the pattern where zero coefficients are interlaced with nonzero coefficients and hence cannot be unimodal, let alone log-concave.

Remark 3.6. In [**52**], it is conjectured that, for all $n,m \geq 0$, the (ordinary) independence polynomial $I(V_n^m)$ is unimodal, which is confirmed in [**47**] by establishing log-concavity. Also, it is verified in [**47,** Proposition 3.1] that, for the cases $m = 0,1,2$, $I(V_n^m)$ has only real zeros. Here, however, in the theory of maximal independence polynomials, the analogous unimodality does not hold for $m \geq 3$, as is clear from the second equality of Corollary 3.4, while real-rootedness holds true for $m = 0,1,2$.

Another application is the following. Keep H as the star graph, but this time let the connecting vertex v be one of the leaves rather than the center. In this case, the bridge path graph

$$\mathbb{B}_d = B(H,H,\ldots,H;v,v,\ldots,v)$$

(d-times) is the *firecracker graph* $F_d^{(m)}$, see Figure 7.

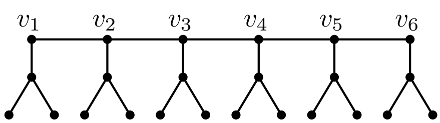

FIGURE 7. The firecracker graph $F_6^{(3)}$.

Corollary 3.7. *For all* $d \geq 0$ *and* $m \geq 2$,

$$I_{\max}(F_d^{(m)};x) = f_d + x^m f_{d-1} + x^{2m-1} f_{d-2}, \tag{3.13}$$

where

$$f_d = \sum_{i=0}^{d} \sum_{j=0}^{\min(i,d-i)} \binom{i}{j}\binom{j}{d-i-j} x^{(m-1)(d-i)+i}(x+x^{m-1})^j. \tag{3.14}$$

Proof. Note that $I_{\max}(H;x)=x+x^m$, $I^v_{\max}(H;x)=x^m$, $I^{\bar{v}}_{\max}(H;x) = x$ and $I^{\tilde{v}}_{\max}(H;x)=x^{m-1}$. According to Corollary 3.2,

$$I_{\max}(F_d^{(m)};x) = f_d + x^m f_{d-1} + x^{2m-1} f_{d-2},$$

where

$$f_d = \sum_{i=0}^{d} \sum_{j=0}^{\min(i,d-i)} \binom{i}{j}\binom{j}{d-i-j} x^{(m-1)(d-i)+i}(x+x^{m-1})^j,$$

which completes the proof. $\square$

Furthermore, note that Corollary 3.7 gives

$$I_{\max}(F_1^{(m)};x)=x+x^m,$$
$$I_{\max}(F_2^{(m)};x)=x^2+2x^{m+1}+2x^{2m-1},$$
$$I_{\max}(F_3^{(m)};x)=x^3+3x^{m+2}+4x^{2m}+x^{2m+1}+x^{3m-2}+x^{3m-1},$$
$$I_{\max}(F_4^{(m)};x)=x^4+4x^{m+3}+6x^{2m+1}+3x^{2m+2}+2x^{3m-1}+6x^{3m}+3x^{4m-2}.$$

Remark 3.8. In [**47**], it is shown that, for all d, $m \geq 0$, the independence polynomial $I(F_d^m)$ is log-concave and unimodal. Here, according to (3.13) and (3.14), the analogous properties for maximal independence polynomials hold if and only if $m = 0, 1, 2$; note that the lowest nonzero term of $I_{\max}(F_d^m)$ has degree d, the lowest has degree $d-1+m$ (from both f_d and $x^m f_{d-1}$), and their difference is > 2 when $m \geq 3$.

We complete this section with three additional examples.

Example 3.9. Let $H = K_m$ be the complete graph on m vertices. Clearly, $I_{\max}(H;x) = mx$, $I^v_{\max}(H;x) = x$, $I^{\bar{v}}_{\max}(H;x) = (m-1)x$ and $I^{\tilde{v}}_{\max}(H;x) = 0$. Thus, Corollary 3.3 gives

$$I_{\max}(\mathbb{B}_d(K_m,v);x) = x^d\, i^{d+1}\sqrt{m-1}^{\,d-1} U_{d+1}\left(\frac{\sqrt{m-1}}{2i}\right),$$

where $i^2=-1$. Hence, the polynomial $I_{\max}(\mathbb{B}_d(K_m,v);x)$ is a monomial in the form of $c_d x^d$ where $c_d := i^{d+1}\sqrt{m-1}^{\,d-1} U_{d+1}(\sqrt{m-1}/2i)$

is a constant. This, of course, is not surprising. However, now the coefficient of the monomial is known.

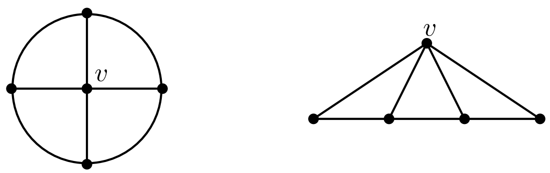

FIGURE 8. The wheel and the fan graphs both on $4+1=5$ vertices with center v.

Example 3.10. Let $H = W_m$ be the *wheel graph* on $m+1$ vertices with center v, see the left hand side of Figure 8 for an illustration. Furthermore, let the center v be connected to all vertices of the cycle graph C_m. Clearly, $I_{\max}(H;x) = x + I_{\max}(C_m;x)$, $I^{v}_{\max}(H;x) = x$, $I^{\bar{v}}_{\max}(H;x) = I_{\max}(C_m;x)$ and $I^{\tilde{v}}_{\max}(H;x) = 0$. Thus, Corollary 3.3 gives

$$I_{\max}(\mathbb{B}_d(W_m,v);x) = i^{d+1}\sqrt{x}^{\,d+1}\sqrt{I_{\max}(C_m;x)}^{\,d-1} U_{d+1}\left(\frac{\sqrt{I_{\max}(C_m;x)}}{2i\sqrt{x}}\right),$$

where $i^2 = -1$. Then, by (3.12), we have

(3.15)
$$I_{\max}(\mathbb{B}_d(W_m,v);x) = \sum_{k=0}^{\lfloor (d+1)/2 \rfloor} \binom{d+1-k}{k} (I_{\max}(C_m;x))^{d-k} x^k.$$

Using the fact that $U_n(t) = \prod_{k=1}^{n}(2t - 2\cos(k\pi/(n+1)))$, see [**53**], we obtain

(3.16)
$$\begin{aligned} I_{\max}(\mathbb{B}_d(W_m,v);x) &= \sqrt{I_{\max}(C_m;x)}^{\,d-1} \\ &\quad \times \prod_{k=1}^{d+1}\left(\sqrt{I_{\max}(C_m;x)} - 2i\sqrt{x}\cos\frac{k\pi}{d+2}\right) \\ &= \sqrt{I_{\max}(C_m;x)}^{\,d-(1-(-1)^d)/2} \end{aligned}$$

$$\times \prod_{k=1}^{\lfloor (d+1)/2 \rfloor} \left(I_{\max}(C_m;x) + 4x\cos^2 \frac{k\pi}{d+2} \right).$$

Recall that, by Proposition 2.3,

$$I_{\max}(C_m;x) = \sum_{j\geq 1} \frac{m}{j} \binom{j}{m-2j} x^j.$$

Specifically,

$$\begin{aligned} I_{\max}(C_1;x) &= x, & I_{\max}(C_2;x) &= 2x, \\ I_{\max}(C_3;x) &= 3x, & I_{\max}(C_4;x) &= 2x^2, \\ I_{\max}(C_5;x) &= 5x^2, & I_{\max}(C_6;x) &= 3x^2 + 2x^3 \end{aligned}$$

and

$$I_{\max}(C_7;x) = 7x^3.$$

By (3.16), the polynomial $I_{\max}(\mathbb{B}_d(W_m,v);x)$ contains only real zeros for $m = 1,2,3,4,5$. For $m = 6$, there are non-real zeros if $d \geq 2$, whence, $3^2 - 4 \cdot 2 \cdot 4\cos^2 \pi/(d+2) \leq 9 - 32\cos^2 \pi/4 < 0$. Nonetheless, since every factor of $I_{\max}(\mathbb{B}_d(W_6,v);x)$ is log-concave, the polynomial is log-concave and thus unimodal. If $d = 1$, all zeros of $I_{\max}(\mathbb{B}_1(W_6,v);x)$ are obviously real. For $m = 7$, since $\sqrt{I_{\max}(C_7;x)}^{\,d-(1-(-1)^d)/2}$ is a monomial, and every factor of the product

$$\prod_{k=1}^{\lfloor (d+1)/2 \rfloor} \left(I_{\max}(C_7;x) + 4x\cos^2 \frac{k\pi}{d+2} \right)$$

has a gap at x^2; in fact, $I_{\max}(\mathbb{B}_d(W_7,v);x)$ is nonunimodal.

Next, we focus on the specific values of $d = 1,2$. When $d = 1$, $I_{\max}(\mathbb{B}_d(W_m,v);x) = x + I_{\max}(C_m;x)$. As discussed, for $1 \leq m \leq 6$, all zeros of $I_{\max}(\mathbb{B}_1(W_m,v);x)$ are real; for $m \geq 7$, $I_{\max}(\mathbb{B}_1(W_m,v);x)$ is nonunimodal since, according to Proposition 2.3, we know that $x^3 \mid I_{\max}(C_m;x)$, and thus, $x + I_{\max}(C_m;x)$ has a gap at x^2.

The above conclusions hold exactly the same for $d = 2$, except that, here, $I_{\max}(\mathbb{B}_2(W_6,v);x)$ is log-concave, unimodal, but not real-rooted. Based on previous discussions, we only need to prove that $I_{\max}(\mathbb{B}_2(W_m,v);x)$ is nonunimodal for all $m \geq 8$. First, we show that the polynomial $I_{\max}(\mathbb{B}_2(W_m,v);x)$ is nonunimodal for all $m \geq$ 13. For

TABLE 1.

m	$I_{\max}(\mathbb{B}_2(W_m, v); x) = I_{\max}(C_m; x)(2x + I_{\max}(C_m; x))$
8	$16x^4 + 4x^5 + 64x^6 + 32x^7 + 4x^8$
9	$6x^4 + 18x^5 + 9x^6 + 54x^7 + 81x^8$
10	$30x^5 + 4x^6 + 225x^8 + 60x^9 + 4x^{10}$
11	$22x^5 + 22x^6 + 121x^8 + 242x^9 + 121x^{10}$
12	$6x^5 + 48x^6 + 4x^7 + 9x^8 + 144x^9 + 588x^{10} + 96x^{11} + 4x^{12}$

convenience, let $q_m(x) := I_{\max}(C_m; x)$, and let $p_m(x) := I_{\max}(\mathbb{B}_2(W_m, v); x)$, noting that $p_m(x) = 2xq_m(x) + (q_m(x))^2$. However, since the degree of $2xq_m(x)$ is $\lfloor m/2 \rfloor + 1$, the degree of the minimum nonzero term of $(q_m(x))^2$ is $2\lceil m/3 \rceil$, and the difference of these is at least 2 for all $m \geq 13$, there is a gap at $x^{\lfloor m/2 \rfloor + 2}$, thus implying $p_m(x)$ nonunimodal.

Table 1 shows that $I_{\max}(\mathbb{B}_2(W_m, v); x)$ is nonunimodal for $8 \leq m \leq 12$, to complete our discussion.

Question 3.11. *Characterize the subset $\mathcal{A}(\mathbb{B}_d(W_m, v))$ of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that, when $(m, d) \in \mathcal{A}(\mathbb{B}_d(W_m, v))$, the polynomial $I_{\max}(\mathbb{B}_d(W_m, v); x)$ is unimodal.*

In summary, we have shown above that $(m, d) \in \mathcal{A}(\mathbb{B}_d(W_m, v))$ for $1 \leq m \leq 6$ and any d, that $(m, d) \notin \mathcal{A}$ for $m = 7$ and any d, that $(m, d) \notin \mathcal{A}$ for all $m \geq 8$ and $d = 1, 2$.

Example 3.12. Let $H = F_m$ be the *fan graph* on $m + 1$ vertices with center v, see the right hand side of Figure 8 for an illustration. The vertex v is connected to all vertices of the path graph P_m. Clearly, $I_{\max}(H; x) = x + I_{\max}(P_m; x)$, $I^v_{\max}(H; x) = x$, $I^{\bar{v}}_{\max}(H; x) = I_{\max}(P_m; x)$ and $I^{\tilde{v}}_{\max}(H; x) = 0$. Thus, Corollary 3.3 yields

$$I_{\max}(\mathbb{B}_d(F_m, v); x) = i^{d+1}\sqrt{x}^{\,d+1}\sqrt{I_{\max}(P_m; x)}^{\,d-1} U_{d+1}\left(\frac{\sqrt{I_{\max}(P_m; x)}}{2i\sqrt{x}}\right),$$

where $i^2 = -1$. Therefore, by (3.12), we have

(3.17)

$$I_{\max}(\mathbb{B}_d(W_m, v); x) = \sum_{k=0}^{\lfloor (d+1)/2 \rfloor} \binom{d+1-k}{k} (I_{\max}(P_m; x))^{d-k} x^k.$$

Using the fact that $U_n(t) = \prod_{k=1}^{n}(2t - 2\cos(k\pi/(n+1)))$, see **[53]**, we obtain

(3.18)

$$\begin{aligned} I_{\max}(\mathbb{B}_d(F_m, v); x) &= \sqrt{I_{\max}(P_m; x)}^{\,d-1} \\ &\quad \times \prod_{k=1}^{d+1}\left(\sqrt{I_{\max}(P_m; x)} - 2i\sqrt{x}\cos\frac{k\pi}{d+2}\right) \\ &= \sqrt{I_{\max}(P_m; x)}^{\,d-(1-(-1)^d)/2} \\ &\quad \times \prod_{k=1}^{\lfloor (d+1)/2 \rfloor}\left(I_{\max}(P_m; x) + 4x\cos^2\frac{k\pi}{d+2}\right). \end{aligned}$$

Recall that, by Proposition 2.2,

$$I_{\max}(P_m; x) = \sum_{j\geq 0}\binom{j+1}{m+1-2j}x^j.$$

Specifically,

$$\begin{aligned} &I_{\max}(P_1; x) = x, && I_{\max}(P_2; x) = 2x, \\ &I_{\max}(P_3; x) = x + x^2, && I_{\max}(P_4; x) = 3x^2, \\ &I_{\max}(P_5; x) = 3x^2 + x^3 \quad \text{and} && I_{\max}(P_6; x) = x^2 + 4x^3 \end{aligned}$$

By (3.18), the polynomial $I_{\max}(\mathbb{B}_d(F_m, v); x)$ has only real zeros for $m = 1, 2, 3, 4$. For $m = 5$, there are non-real zeros when $d \geq 3$. Nonetheless, since every factor of $I_{\max}(\mathbb{B}_d(F_5, v); x)$ in (3.18) is log-concave, the polynomial is log-concave, and thus unimodal, for all $d \geq 1$.

When $d = 1$, by (3.17),

$$I_{\max}(\mathbb{B}_d(F_m, v); x) = x + I_{\max}(P_m; x).$$

As already discussed, for $1 \leq m \leq 5$, all zeros of $I_{\max}(\mathbb{B}_1(F_m, v); x)$ are real; $I_{\max}(\mathbb{B}_1(F_6, v); x) = x + x^2 + x^3$ has non-real zeros but is log-concave and thus unimodal; $I_{\max}(\mathbb{B}_1(F_m, v); x)$ is nonunimodal for $m \geq 7$ since $x^3 \mid I_{\max}(P_m; x)$, and thus produces a gap at x^2.

The above conclusions hold exactly the same for $d = 2$. We only need to prove that $I_{\max}(\mathbb{B}_2(F_m, v); x)$ is nonunimodal for all $m \geq 6$

and $m \neq 8$. First, as in the treatment of Example 3.10, it is not difficult to see that the polynomial $I_{\max}(\mathbb{B}_2(F_m, v); x)$ is nonunimodal for all $m \geq 16$. A special case is $m = 8$ since $I_{\max}(\mathbb{B}_2(F_m, v); x) = 8x^4 + 10x^5 + 16x^6 + 40x^7 + 25x^8$ is unimodal but not log-concave. For $6 \leq m \leq 15$ and $m \neq 8$, numerical computation yields that $I_{\max}(\mathbb{B}_2(F_m, v); x)$ is nonunimodal; thus, our discussion is complete.

4. Bridge cycle graph. Let $\{H_i\}_{i=1}^d$ be a set of finite pairwise disjoint graphs with a specifically chosen $v_i \in V(H_i)$ and $|V(H_i)| \geq 2$. The *bridge cycle graph*

$$C(H_1, H_2, \dots, H_d) \equiv C(H_1, H_2, \dots, H_d; v_1, v_2, \dots, v_d)$$

of $\{H_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is obtained from the components $H_1, \dots, H_d$ by connecting vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \dots, d-1$, as well as connecting vertices v_d and v_1 by an edge, see Figure 9.

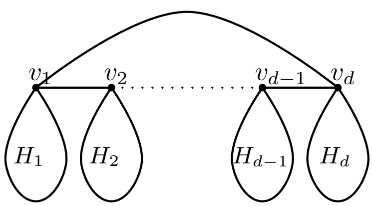

FIGURE 9. The bridge cycle graph.

Now, let us derive a recurrence relation for the sequence of maximal independence polynomials

$$\{I_{\max}(C(H_1, H_2, \dots, H_d); x)\}_{d \geq 0}.$$

For convenience, define $I'^{v}_{\max}(G; x) = I^{\bar{v}}_{\max}(G; x) + I^{\tilde{v}}_{\max}(G; x)$. As in Section 3, $B_d := B(H_1, H_2, \dots, H_d; v_1, v_2, \dots, v_d)$ represents the bridge path graph of $\{H_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$. Let $\mathbb{B}_d$ denote $B(H, H, \dots, H; v, v, \dots, v)$ when all of the components H_i are the same and $v_i = v$ for all $1 \leq i \leq d$. By definition, it is easy to see that

$$\begin{aligned}(4.1)\qquad & I_{\max}^{v_d}(C(H_1,H_2,\ldots,H_d);x)\\ &= I_{\max}^{v_d}(H_d;x)I_{\max}^{\prime v_{d-1}}(H_{d-1};x)I_{\max}^{\prime v_1}(H_1;x)\\ &\quad\cdot I_{\max}(B(H_2,H_3,\ldots,H_{d-2});x).\end{aligned}$$

Also,

$$\begin{aligned}(4.2)\qquad & I_{\max}^{\overline{v_d}}(C(H_1,H_2,\ldots,H_d);x)\\ &= I_{\max}^{\{v_1,v_{d-1}\},\overline{v_d}}(C(H_1,H_2,\ldots,H_d);x)\\ &\quad+ I_{\max}^{v_1,\overline{\{v_{d-1},v_d\}}}(C(H_1,H_2,\ldots,H_d);x)\\ &\quad+ I_{\max}^{v_{d-1},\overline{\{v_1,v_d\}}}(C(H_1,H_2,\ldots,H_d);x)\\ &\quad+ I_{\max}^{\overline{\{v_1,v_{d-1},v_d\}}}(C(H_1,H_2,\ldots,H_d);x)\\ &= I_{\max}^{\prime v_d}(H_d;x)I_{\max}^{\{v_1,v_{d-1}\}}(B_{d-1};x)\\ &\quad+ I_{\max}^{\prime v_d}(H_d;x)I_{\max}^{v_1,\overline{v_{d-1}}}(B_{d-1};x)\\ &\quad+ I_{\max}^{\prime v_d}(H_d;x)I_{\max}^{v_{d-1},\overline{v_1}}(B_{d-1};x)\\ &\quad+ I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}^{\overline{\{v_1,v_{d-1}\}}}(B_{d-1};x)\\ &= I_{\max}^{\prime v_d}(H_d;x)I_{\max}(B_{d-1};x)\\ &\quad- I_{\max}^{\widetilde{v_d}}(H_d;x)I_{\max}^{\overline{\{v_1,v_{d-1}\}}}(B_{d-1};x).\end{aligned}$$

It remains to find a recurrence for $I_{\max}^{\overline{\{v_1,v_{d-1}\}}}(B_{d-1};x)$, which can be accomplished as follows:

$$\begin{aligned}I_{\max}^{\overline{\{v_1,v_d\}}}(B_d;x) &= I_{\max}^{\overline{\{v_1,v_{d-1},v_d\}}}(B_d;x) + I_{\max}^{v_{d-1},\overline{\{v_1,v_d\}}}(B_d;x)\\ &= I_{\max}^{\overline{v_d}}(H_d;x)I_{\max}^{\overline{\{v_1,v_{d-1}\}}}(B_{d-1};x)\\ &\quad+ I_{\max}^{\prime v_d}(H_d;x)I_{\max}^{v_{d-1},\overline{v_1}}(B_{d-1};x).\end{aligned}$$

Thus, by the fact that

$$I_{\max}^{\overline{\{v_1,v_d\}}}(B_d;x) + I_{\max}^{v_d,\overline{v_1}}(B_d;x) = I_{\max}^{\overline{v_1}}(B_d;x),$$

we obtain

$$\begin{aligned}(4.3)\quad I_{\max}^{\overline{\{v_1,v_d\}}}(B_d;x) + I_{\max}^{\widetilde{v_d}}(H_d;x)I_{\max}^{\overline{\{v_1,v_{d-1}\}}}(B_{d-1};x)&\\ = I_{\max}^{\prime v_d}(H_d;x)I_{\max}^{\overline{v_1}}(B_{d-1};x).&\end{aligned}$$

Indeed, the recurrences (4.1)–(4.3) with right initial values may be used to compute $I_{\max}(C(H_1, H_2, \ldots, H_d); x)$ for any graph components $H_1, \ldots, H_d$, each of which has at least two vertices.

For nicer exhibitions, from now on, we focus on cases $H_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$. Denote the bridge cycle graph $C(H, H, \ldots, H, v, v, \ldots, v)$ (d-times) by $\mathbb{C}_d = \mathbb{C}_d(H, v)$. Thus, (4.1)–(4.3) give

$$\begin{aligned}
I^{v}_{\max}(\mathbb{C}_d; x) &= I^{v}_{\max}(H; x)(I'^{v}_{\max}(H; x))^2 I_{\max}(\mathbb{B}_{d-3}; x),\\
I^{\bar{v}}_{\max}(\mathbb{C}_d; x) &= I'^{v}_{\max}(H; x) I_{\max}(\mathbb{B}_{d-1}; x)\\
&\quad - I^{\tilde{v}}_{\max}(H; x) I^{\overline{\{v_1, v_{d-1}\}}}_{\max}(\mathbb{B}_{d-1}; x),\\
I^{\overline{\{v_1, v_d\}}}_{\max}(\mathbb{B}_d; x) &+ I^{\tilde{v}}_{\max}(H; x) I^{\overline{\{v_1, v_{d-1}\}}}_{\max}(\mathbb{B}_{d-1}; x)\\
&= I'^{v}_{\max}(H; x) I^{\bar{v}}_{\max}(\mathbb{B}_{d-1}; x).
\end{aligned}$$

This implies

$$\begin{aligned}
&I^{v}_{\max}(\mathbb{C}_d; x) + I^{\tilde{v}}_{\max}(H; x) I^{v}_{\max}(\mathbb{C}_{d-1}; x)\\
&\quad = I^{v}_{\max}(H; x)(I'^{v}_{\max}(H; x))^2 I_{\max}(\mathbb{B}_{d-3}; x)\\
&\qquad + I^{\tilde{v}}_{\max}(H; x) I^{v}_{\max}(H; x)(I'^{v}_{\max}(H; x))^2 I_{\max}(\mathbb{B}_{d-4}; x)\\
&I^{\bar{v}}_{\max}(\mathbb{C}_d; x) + I^{\tilde{v}}_{\max}(H; x) I^{\bar{v}}_{\max}(\mathbb{C}_{d-1}; x)\\
&\quad = I'^{v}_{\max}(H; x)(I_{\max}(\mathbb{B}_{d-1}; x) + I^{\tilde{v}}_{\max}(H; x) I_{\max}(\mathbb{B}_{d-2}; x))\\
&\qquad - I^{\tilde{v}}_{\max}(H; x) I'^{v}_{\max}(H; x) I^{\bar{v}}_{\max}(\mathbb{B}_{d-2}; x).
\end{aligned}$$

Adding the above two equations, we obtain

$$\begin{aligned}
&I_{\max}(\mathbb{C}_d; x) + I^{\tilde{v}}_{\max}(H; x) I_{\max}(\mathbb{C}_{d-1}; x)\\
&\quad = I'^{v}_{\max}(H; x) I_{\max}(\mathbb{B}_{d-1}; x) + I'^{v}_{\max}(H; x) I^{\tilde{v}}_{\max}(H; x) I_{\max}(\mathbb{B}_{d-2}; x)\\
&\qquad - I'^{v}_{\max}(H; x) I^{\tilde{v}}_{\max}(H; x) I^{\bar{v}}_{\max}(\mathbb{B}_{d-2}; x)\\
&\qquad + I^{v}_{\max}(H; x)(I'^{v}_{\max}(H; x))^2 I_{\max}(\mathbb{B}_{d-3}; x)\\
&\qquad + I^{\tilde{v}}_{\max}(H; x)(I'^{v}_{\max}(H; x))^2 I^{v}_{\max}(H; x) I_{\max}(\mathbb{B}_{d-4}; x).
\end{aligned}$$

Hence, replacing $I^{\overline{v_{d-2}}}_{\max}(\mathbb{B}_{d-2}; x)$ by (3.6), we obtain

Theorem 4.1. *Let H be any graph which has at least two vertices. Let $\mathbb{C}_d = \mathbb{C}_d(H, v) = C(H, H, \ldots, H; v, v, \ldots, v)$ (d-times, $d \geq 5$)*

be a bridge cycle graph. The sequence of the maximal independence polynomials $I_{\max}(\mathbb{C}_d;x)$ satisfies the following recurrence relation

$$\begin{aligned} &I_{\max}(\mathbb{C}_d;x)+I^{\tilde{v}}_{\max}(H;x)I_{\max}(\mathbb{C}_{d-1};x)\\ &\quad= I'^{v}(H;x)_{\max}I_{\max}(\mathbb{B}_{d-1};x)+I'^{v}_{\max}(H;x)I^{\tilde{v}}_{\max}(H;x)I_{\max}(\mathbb{B}_{d-2};x)\\ &\qquad+I'^{v}_{\max}(H;x)(I'^{v}_{\max}(H;x)I^{v}_{\max}(H;x)\\ &\qquad-I^{\tilde{v}}_{\max}(H;x)I^{\bar{v}}_{\max}(H;x))I_{\max}(\mathbb{B}_{d-3};x)\\ &\qquad+(I'^{v}_{\max}(H;x))^2I^{\tilde{v}}_{\max}(H;x)I^{v}_{\max}(H;x)I_{\max}(\mathbb{B}_{d-4};x)\\ &\qquad-(I'^{v}_{\max}(H;x))^2(I^{\tilde{v}}_{\max}(H;x))^2I^{v}_{\max}(H;x)I_{\max}(\mathbb{B}_{d-5};x).\end{aligned}$$

As a corollary, we have the following result.

Corollary 4.2. *Let H be any graph with least two vertices, and let v be a vertex in H with $I^{\tilde{v}}_{\max}(H;x)=0$. Let $\mathbb{C}_d=\mathbb{C}_d(H,v)=C(H,H,\dots,H;v,v,\dots,v)$ (d-times) be a bridge cycle graph, and let $\mathbb{B}_d=\mathbb{B}_d(H,v)=B(H,H,\dots,H;v,v,\dots,v)$ (d-times) be a bridge path graph. Then,*

$$\begin{aligned} I_{\max}(\mathbb{C}_d;x)=\;&I^{\bar{v}}_{\max}(H;x)I_{\max}(\mathbb{B}_{d-1};x)\\ &+I^{v}_{\max}(H;x)(I^{\bar{v}}_{\max}(H;x))^2I_{\max}(\mathbb{B}_{d-3};x),\end{aligned}$$

for all $d\geq 3$.

As a consequence of Corollaries 3.3 and 4.2 we have a formula for the polynomial $I_{\max}(\mathbb{C}_d;x)$.

Corollary 4.3. *Let H be any graph with at least two vertices, and let v be a vertex in H such that $I^{\tilde{v}}_{\max}(H;x)=0$. Let $\mathbb{C}_d=\mathbb{C}_d(H,v)=B(H,H,\dots,H;v,v,\dots,v)$ (d-times) be a bridge cycle graph. Then, for all $d\geq 3$,*

$$I_{\max}(\mathbb{C}_d;x)=(i\sqrt{\beta})^d\,U_d\left(\frac{\alpha}{2i\sqrt{\beta}}\right)-(i\sqrt{\beta})^d\,U_{d-2}\left(\frac{\alpha}{2i\sqrt{\beta}}\right) \tag{4.4a}$$

$$=\alpha^{(1-(-1)^d)/2}\prod_{k=0}^{\lfloor d/2\rfloor-1}\left(\alpha^2+4\beta\cos^2\frac{(2k+1)\pi}{2d}\right), \tag{4.4b}$$

when $\alpha:=I^{\bar{v}}_{\max}(H;x)$, $\beta:=I^{v}_{\max}(H;x)I^{\bar{v}}_{\max}(H;x)$ and $i^2=-1$.

Proof. The first equality, i.e., (4.4a), is an immediate consequence of Corollaries 3.3 and 4.2.

For the second equality, i.e., (4.4b), we observe the following. In general, the real-rooted polynomial $U_d(t)-U_{d-2}(t)$ has d distinct roots, and they are $\cos(2k+1)\pi/(2d)$, $k=0,\dots,d-1$. In order to see this, we use the trigonometric representations of the Chebyshev polynomials $U_n(t)=\sin(n+1)t/\sin t$ and solve for the zeros of (4.4a) from there. The rest is clear. $\square$

Now, we present a few examples to show applications of the above theory.

Example 4.4. Let $H=K_m$ be the complete graph on m vertices. Recall that $I_{\max}(H;x)=mx$, $I^{v}_{\max}(H;x)=x$, $I^{\bar{v}}_{\max}(H;x)=(m-1)x$ and $I^{\tilde{v}}_{\max}(H;x)=0$. Thus, $\alpha=(m-1)x$, $\beta=(m-1)x^2$ and (4.4a) gives, for all $d\geq 3$,

$$I_{\max}(\mathbb{C}_d(K_m,v);x)=(i\sqrt{(m-1)x})^d\bigg[U_d\bigg(\frac{\sqrt{m-1}}{2i}\bigg)-U_{d-2}\bigg(\frac{\sqrt{m-1}}{2i}\bigg)\bigg],$$

where $i^2=-1$. Hence, $I_{\max}(\mathbb{C}_d(K_m,v);x)$ is a monomial of the form $c_d x^d$, where c_d is the constant

$$(i\sqrt{(m-1)})^d\bigg[U_d\bigg(\frac{\sqrt{m-1}}{2i}\bigg)-U_{d-2}\bigg(\frac{\sqrt{m-1}}{2i}\bigg)\bigg].$$

Equivalently, when applying (4.4b), we obtain

$$I_{\max}(\mathbb{C}_d(K_m,v);x)=x^d(m-1)^{\lfloor (d+1)/2\rfloor}\prod_{k=0}^{\lfloor d/2\rfloor-1}\bigg(m-1+4\cos^2\frac{2k+1}{2d}\pi\bigg).$$

Example 4.5. Let $H=K_{1,m}$ be the star graph on $m+1$ vertices where v is its center. Again, $I_{\max}(H;x)=x+x^m$, $I^{v}_{\max}(H;x)=x$, $I^{\bar{v}}_{\max}(H;x)=x^m$ and $I^{\tilde{v}}_{\max}(H;x)=0$. Hence, for the *cyclic vertebrated graph*, $\alpha=x^m$, $\beta=x^{m+1}$ and (4.4b) yield that, for all $d\geq 3$,

$$\begin{aligned}I_{\max}(\mathbb{C}_d(K_{1,m},v);x)&=x^{m(1-(-1)^d)/2}\prod_{k=0}^{\lfloor d/2\rfloor-1}\bigg(x^{2m}+4x^{m+1}\cos^2\frac{(2k+1)\pi}{2d}\bigg)\\ &=\sqrt{x}^{\,(m+1)d+(m-1)(1-(-1)^d)/2}\end{aligned} \tag{4.5}$$

$$\times \prod_{k=0}^{\lfloor d/2\rfloor-1}\left(x^{m-1}+4\cos^2\frac{(2k+1)\pi}{2d}\right)$$

where $i^2=-1$. Therefore, for $m\geq 3$, the polynomial $I_{\max}(\mathbb{C}_d(K_{1,m},v);x)$ is nonunimodal due to the many obvious gaps. By (4.5), for $m=1$, $I_{\max}(\mathbb{C}_d(P_2,v);x)$ is a monomial $I_{\max}(\mathbb{C}_d(P_2,v);x)$ with the coefficient

$$\prod_{k=0}^{\lfloor d/2\rfloor-1}(1+4\cos^2(2k+1)\pi)/(2d).$$

For $m=2$, we have

$$I_{\max}(\mathbb{C}_d(K_{1,2},v);x)=\sqrt{x}^{\,3d+(1-(-1)^d)/2}\prod_{k=0}^{\lfloor d/2\rfloor-1}\left(x+4\cos^2\frac{(2k+1)\pi}{2d}\right),$$

implying that the polynomial $I_{\max}(\mathbb{C}_d(K_{1,2},v);x)$ has only real zeros and is thus log-concave and unimodal.

Example 4.6. Let $H=W_m$ be the wheel graph on $m+1$ vertices with center v, that is, the vertex v is connected to all vertices of the cycle C_m. Clearly, $I_{\max}(H;x)=x+I_{\max}(C_m;x)$, $I_{\max}^{v}(H;x)=x$, $I_{\max}^{\bar{v}}(H;x)=I_{\max}(C_m;x)$ and $I_{\max}^{\tilde{v}}(H;x)=0$. As in the previous examples, applying Corollary 4.3 yields

(4.6)

$$I_{\max}(\mathbb{C}_d(W_m,v);x)=I_{\max}(C_m;x)^{\lfloor (d+1)/2\rfloor}\times\prod_{k=0}^{\lfloor d/2\rfloor-1}\left(I_{\max}(C_m;x)+4x\cos^2\frac{(2k+1)\pi}{2d}\right).$$

Recall again, by Proposition 2.3, that $I_{\max}(C_m;x)=\sum_{j\geq 1}(m/j)\binom{j}{m-2j}x^j$. Specifically, $I_{\max}(C_1;x)=x$, $I_{\max}(C_2;x)=2x$, $I_{\max}(C_3;x)=3x$, $I_{\max}(C_4;x)=2x^2$, $I_{\max}(C_5;x)=5x^2$, $I_{\max}(C_6;x)=3x^2+2x^3$ and $I_{\max}(C_7;x)=7x^3$.

By (4.6), the polynomial $I_{\max}(\mathbb{C}_d(W_m,v);x)$ has only real zeros for $m=1,2,3,4,5$. For $m=6$, there are non-real zeros when $d\geq 2$. However, $I_{\max}(\mathbb{C}_d(W_6,v);x)$ is log-concave and unimodal for any d.

For $m = 7$, $I_{\max}(\mathbb{C}_d(W_7, v); x)$ is nonunimodal following the discussion of Example 3.10.

As for $d = 1, 2$, the polynomial $I_{\max}(\mathbb{C}_1(W_m, v); x)$ is log-concave and unimodal, the polynomial $I_{\max}(\mathbb{C}_2(W_m, v); x)$ is nonunimodal for $m \geq 8$. This is due to the fact that, when $d = 1, 2$, there is no difference between the bridge path and bridge cycle graphs.

Example 4.7. Let $H = F_m$ be the fan graph on $m + 1$ vertices with center v, that is, v is connected to all vertices of the path P_m. We know that $I_{\max}(H; x) = x + I_{\max}(P_m; x)$, $I^v_{\max}(H; x) = x$, $I^{\bar{v}}_{\max}$ $(H; x) = I_{\max}(P_m; x)$ and $I^{\tilde{v}}_{\max}(H; x) = 0$. Similar to Example 4.6, Corollary 4.3 yields

(4.7)

$$I_{\max}(\mathbb{C}_d(F_m, v); x) = I_{\max}(P_m; x)^{\lfloor (d+1)/2 \rfloor} \times \prod_{k=0}^{\lfloor d/2 \rfloor - 1} \left(I_{\max}(P_m; x) + 4x \cos^2 \frac{(2k+1)\pi}{2d} \right).$$

Again, recall that, by Proposition 2.2, $I_{\max}(P_m; x) = \sum_{j \geq 0} \binom{j+1}{m+1-2j} x^j$. Specifically, $I_{\max}(P_1; x) = x$, $I_{\max}(P_2; x) = 2x$, $I_{\max}(P_3; x) = x + x^2$, $I_{\max}(P_4; x) = 3x^2$, $I_{\max}(P_5; x) = 3x^2 + x^3$ and $I_{\max}(P_6; x) = x^2 + 4x^3$.

By (4.7), the polynomial $I_{\max}(\mathbb{C}_d(F_m, v); x)$ has only real zeros for $m = 1, 2, 3, 4$. For $m = 5$, there are non-real zeros when $d \geq 3$. Nonetheless, $I_{\max}(\mathbb{C}_d(F_5, v); x)$ is log-concave and unimodal for all $d \geq 1$.

Cases $d = 1, 2$ are treated in Example 3.12.

Question 4.8. *Questions similar to Question* 3.11 *may be formulated. For example, let $\mathcal{A}(\mathbb{C}_d(F_m, v))$ be the subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that, when $(m, d) \in \mathcal{A}(\mathbb{C}_d(F_m, v))$, the polynomial $I_{\max}(\mathbb{C}_d(F_m, v); x)$ is unimodal. By our discussion in Examples* 3.12 *and* 4.7*, we have shown that $(m, d) \in \mathcal{A}(\mathbb{C}_d(F_m, v))$ for $1 \leq m \leq 5$ and any d, that $(m, d) \notin \mathcal{A}$ for $m = 8$ and $d = 1$, that $(m, d) \notin \mathcal{A}$ for all $m \geq 6, m \neq 8$ and $d = 1, 2$. How is $\mathcal{A}(\mathbb{C}_d(F_m, v))$ completely characterized?*

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Peking University, School of Mathematical Sciences & LMAM, Beijing 100871, P.R. China
Email address: **huhan@pku.edu.cn**

University of Haifa, Department of Mathematics, 31905 Haifa, Israel
Email address: **toufik@math.haifa.ac.il**

Peking University, School of Mathematical Sciences & LMAM, Beijing 100871, P.R. China
Email address: **csong@math.pku.edu.cn**