

AMITSUR'S PROPERTY FOR SKEW POLYNOMIALS OF DERIVATION TYPE

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ABSTRACT. We investigate when radicals \mathfrak{F} satisfy Amitsur's property on skew polynomials of derivation type, namely, $\mathfrak{F}(R[x; \delta]) = (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta]$. In particular, we give a new argument that the Brown-McCoy radical has this property. We also give a new characterization of the prime radical of $R[x; \delta]$.

1. Introduction. A radical \mathfrak{F} is said to satisfy *Amitsur's property* if, for every ring R , we have

$$\mathfrak{F}(R[x]) = (\mathfrak{F}(R[x]) \cap R)[x].$$

The terminology is a consequence of Amitsur's work [1], where he showed that the Jacobson, prime and Levitzki radicals and the upper nilradical all have this property. Amitsur also gave the following two criteria (as distilled in [10, Lemma 1]) which, together, are sufficient to guarantee Amitsur's property and, in practice, are often easily verified:

- (1) \mathfrak{F} is hereditary, and,
- (2) whenever R has characteristic p , we have

$$\mathfrak{F}(R[x]) \cap R[x^p - x] \subseteq \mathfrak{F}(R[x^p - x]).$$

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These criteria were used, for instance, in [10, Theorem 3] and [12, Theorem 3.2] to show, respectively, that the Brown-McCoy and Behrens radicals have Amitsur's property. For three recent and interesting studies of Amitsur's property in connection with other natural radical properties, see [9, 13, 18]. For information on general radical theory, and Amitsur's property in particular, we recommend [7, subsection 4.9] as a good reference.

In the present paper, we are concerned with the radicals of skew polynomial rings of derivation type (which we also call differential polynomial rings).

Let R be a ring, and let δ be a derivation on R , meaning an additive map satisfying the product rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. We can then define the ring $R[x; \delta]$ consisting of left polynomials with standard addition and multiplication subject to the skewed constraint $xa = ax + \delta(a)$. Radicals of this ring were first studied in the work of Ferrero, Kishimoto and Motose [6], where it was shown that the Jacobson, prime and Wedderburn radicals again possess (an analogue of) Amitsur's property. Letting \mathfrak{J} denote the Jacobson radical, they raised the question of whether $\mathfrak{J}(R[x; \delta]) \cap R$ is a nil ideal of R , and this question remains open.

In this paper, we generalize the work in [6] to give a set of general criteria for the analogue of Amitsur's property to hold over $R[x; \delta]$. We say that a radical \mathfrak{F} satisfies the δ -Amitsur property, if:

$$(1.1) \quad \text{for all rings } R \text{ and for all derivations } \delta \text{ of } R, \\ (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta] = \mathfrak{F}(R[x; \delta]).$$

When possible, we also explicitly describe the resulting ideal

$$\mathfrak{F}(R[x; \delta]) \cap R \trianglelefteq R.$$

Throughout the paper, R will be an arbitrary associative ring, possibly without 1, and δ will be an arbitrary derivation on R . We reserve fractal letters for radicals, capitalized English letters for rings and sets, and lowercase English letters for ring elements or variables. We write $I \trianglelefteq R$ to mean that I is a two-sided ideal of R . When we use the word "radical" we will mean a radical in the sense of Kurosh and Amitsur [7, Definition 2.1.1]. (Two exceptions to this convention are when we speak of the "Wedderburn radical" and "bounded nilradical,"

whose names have been established in the literature, but these are not technically radicals.) In order to be precise, \mathfrak{F} is a radical if it assigns to each ring R an ideal $\mathfrak{F}(R) \trianglelefteq R$ satisfying the following three conditions.

- (R1) If $\mathfrak{F}(R) = R$ and $R \rightarrow S$ is a surjective ring homomorphism, then $\mathfrak{F}(S) = S$.
- (R2) We have $\mathfrak{F}(\mathfrak{F}(R)) = \mathfrak{F}(R)$, and, if $I \trianglelefteq R$ with $\mathfrak{F}(I) = I$, then $\mathfrak{F}(I) \subseteq \mathfrak{F}(R)$.
- (R3) The equality $\mathfrak{F}(R/\mathfrak{F}(R)) = 0$ always holds.

As usual, we say the radical \mathfrak{F} is *hereditary* if it satisfies the further condition:

- (R4) If $I \trianglelefteq R$, then $\mathfrak{F}(I) = \mathfrak{F}(R) \cap I$.

2. Preliminaries on rings with derivations. In this short section, we collect a few definitions and results that are important when working with skew polynomials of derivation type. Given a derivation δ on a ring R , following standard nomenclature we say that a subset $S \subseteq R$ is a δ -subset if $\delta(S) \subseteq S$. When $I \trianglelefteq R$ is an ideal and a δ -subset, we simply say that I is a δ -ideal.

The next lemma, the proof of which is omitted, relates the ideals of R and $R[x; \delta]$.

Lemma 2.1. *Let R be a ring and δ a derivation of R .*

- (1) *If I is a right ideal of R , then $I[x; \delta]$ is a right ideal of $R[x; \delta]$ and $I[x; \delta] \cap R = I$.*
- (2) *If I is a δ -ideal of R , then $I[x; \delta]$ is an ideal of $R[x; \delta]$.*
- (3) *If J is an ideal of $R[x; \delta]$, then $J \cap R$ is an ideal of R .*

Given

$$f(x) = \sum_{i=0}^n a_i x^i \in R[x; \delta],$$

with $a_i \in R$ for each $0 \leq i \leq n$, we abuse notation by writing

$$\delta^j(f(x)) = \sum_{i=0}^n \delta^j(a_i) x^i.$$

(Note that δ is not a derivation on $R[x; \delta]$.) We will find many occasions to make use of the following extremely useful result, which capitalizes on the startlingly pretty formula $xf(x) - f(x)x = \delta(f(x))$ that recursively leads to

$$(2.1) \quad \delta^j(f(x)) = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} x^i f(x) x^{j-i}.$$

Lemma 2.2. *If J is an ideal of $R[x; \delta]$ and $f(x) \in J$, then*

$$R\delta^j(f(x))R \subseteq J$$

for every $j \geq 0$. Moreover, if J is closed under multiplication by x , then $\delta^j(f(x)) \in J$ for all $j \geq 0$ and $J \cap R$ is a δ -ideal of R .

Proof. Given $r, s \in R$, we obtain $r\delta^j(f(x))s \in J$ by multiplying (2.1) on the left by r and on the right by s . In order to prove the last sentence, note that $J \cap R$ is clearly an ideal in R , and, when J is closed under multiplication by x , then (2.1) proves the δ -invariance claims. \square

3. The prime radical. The prime radical $\mathfrak{P}(R)$ of a ring R has many equivalent definitions:

- The lower radical described by the class of nilpotent rings, and thus, the limit of the (higher) Wedderburn radicals.
- The intersection of all of the prime ideals.
- The set of strongly nilpotent elements.
- The limit of the (higher) left (or right) T -nilpotent radideals, see [8].
- The limit of the (higher) bounded nilradicals.

If we write

$$\mathfrak{P}_\delta(R) := \mathfrak{P}(R[x; \delta]) \cap R,$$

the ideal $\mathfrak{P}_\delta(R) \trianglelefteq R$ can similarly be described as a limit of δ -Wedderburn ideals [6, Theorem 2.1, Corollary 2.2], as the intersection of δ -prime δ -ideals [5, Theorem 1.1], and as the set of strongly δ -nilpotent elements [11, Proposition 1.11]. In this section, we will pursue an analogue of the fourth bullet point.

A subset $S \subseteq R$ is *left T -nilpotent* if, for every sequence of elements $s_1, s_2, \dots \in S$, there is some index $n \geq 1$ such that $s_1 s_2 \cdots s_n = 0$. Such subsets are quite well behaved, as evidenced by the next two lemmas.

Lemma 3.1. *Let R be a ring, $I \subseteq R$ and $J \trianglelefteq R$. If J is left T -nilpotent and \bar{I} is left T -nilpotent in R/J , then $I + J$ is left T -nilpotent.*

Proof. This is a slight strengthening of [8, Lemma 4.2], with the same proof, mutatis mutandis. □

Lemma 3.2. *Let R be a ring, $I \subseteq R$ and J a one-sided ideal of R .*

- (1) *If J is left T -nilpotent, then the two-sided ideal generated by J is left T -nilpotent.*
- (2) *If I and J are left T -nilpotent, then $I + J$ is left T -nilpotent.*

Proof. This is [8, Proposition 4.3]. □

The *left T -nilpotent radical* is defined by

$$\mathfrak{T}_\ell(R) := \{a \in R : aR \text{ is left } T\text{-nilpotent}\},$$

or equivalently, as the set

$$\{a \in R : \text{the ideal generated by } a \text{ is left } T\text{-nilpotent}\}.$$

For more information and basic facts, see [8, Section 4]. This radical-like ideal satisfies the δ -Amitsur property, and moreover, we can explicitly describe the derived ideal as follows.

Theorem 3.3. *Given a ring R with a derivation δ , then*

$$\mathfrak{T}_\ell(R[x; \delta]) = \mathfrak{T}_{\ell, \delta}(R)[x; \delta],$$

where

$$(3.1) \quad \mathfrak{T}_{\ell, \delta}(R) := \left\{ a \in R : \sum_{j=0}^{\infty} \delta^j(a)R \text{ is left } T\text{-nilpotent} \right\} = \mathfrak{T}_\ell(R[x; \delta]) \cap R.$$

Proof. Define $\mathfrak{T}_{\ell,\delta}(R)$ as in (3.1), and note that this is an ideal of R by Lemma 3.2 (1), and hence, a δ -ideal. We first show that

$$\mathfrak{T}_{\ell}(R[x; \delta]) \subseteq \mathfrak{T}_{\ell,\delta}(R)[x; \delta].$$

Fix $f(x) \in \mathfrak{T}_{\ell}(R[x; \delta])$, and write

$$f(x) = \sum_{i=0}^n a_i x^i$$

with $a_i \in R$ for each $0 \leq i \leq n$. Set

$$J_i = \sum_{j=0}^{\infty} \delta^j(a_i)R$$

for $0 \leq i \leq n$. We will show that J_n is left T -nilpotent.

Fix a sequence of elements $r_1, r_2, \dots \in R$ and a sequence of non-negative integers i_1, i_2, \dots , and set

$$t_k := \delta^{i_1}(a_n)r_1\delta^{i_2}(a_n)r_2 \cdots \delta^{i_k}(a_n)r_k$$

for each $k \geq 1$. Each of the elements

$$\delta^{i_1}(f(x))r_1\delta^{i_2}(f(x))r_2, \delta^{i_3}(f(x))r_3\delta^{i_4}(f(x))r_4, \dots$$

belongs to $\mathfrak{T}_{\ell}(R[x; \delta])$ by Lemma 2.2, and thus, there exists some index k such that

$$\delta^{i_1}(f(x))r_1\delta^{i_2}(f(x))r_2 \cdots \delta^{i_k}(f(x))r_k = 0.$$

The degree nk coefficient in this product is exactly t_k , and thus, $t_k = 0$.

We now show that J_m is left T -nilpotent for any $0 \leq m \leq n$. By a recursive argument, we may assume that $J_{m+1}, J_{m+2}, \dots, J_n$ are left T -nilpotent, and thus, the two-sided ideal J generated by $J_{m+1} + J_{m+2} + \cdots + J_n$ is a left T -nilpotent ideal by Lemma 3.2. Lemma 3.1 tells us that, in order to prove J_m is left T -nilpotent, we can pass to the quotient ring R/J ; thus, we may assume that m is the leading index of $f(x)$. However, then, the methods of the previous paragraph apply, and thus, J_m is left T -nilpotent as desired, which proves the inclusion $\mathfrak{T}_{\ell}(R[x; \delta]) \subseteq \mathfrak{T}_{\ell,\delta}(R)[x; \delta]$.

We now show the opposite inclusion $\mathfrak{T}_\ell(R[x; \delta]) \supseteq \mathfrak{T}_{\ell, \delta}(R)[x; \delta]$. Fix

$$f(x) = \sum_{i=0}^n a_i x^i \in \mathfrak{T}_{\ell, \delta}(R)[x; \delta],$$

and also fix a sequence of polynomials $g_1(x), g_2(x), \dots \in R[x; \delta]$. Every coefficient in the product $f(x)g_1(x)f(x)g_2(x) \cdots f(x)g_k(x)$ is a \mathbb{Z} -linear combination of terms of the form

$$(3.2) \quad \delta^{j_1}(a_{i_1})\delta^{j'_1}(r_1)\delta^{j_2}(a_{i_2})\delta^{j'_2}(r_2) \cdots \delta^{j_k}(a_{i_k})\delta^{j'_k}(r_k),$$

where $j_1, j'_1, j_2, j'_2, \dots, j_k, j'_k$ are non-negative integers; for each $m \geq 1$, we have that r_m is a coefficient of $g_m(x)$, that $a_{i_m} \in \{a_0, a_1, \dots, a_n\}$ and that

$$j_m, j'_m < \sum_{p=1}^m \deg(g_p(x)) + m \deg(f(x)) + 1.$$

In particular, note that there are only *finitely* many choices for each j_m, j'_m, i_m and r_m (given that we have already chosen the sequence $g_1(x), g_2(x), \dots$).

Suppose, by way of contradiction, that, for each $k \geq 1$, there is a term as in (3.2) which is nonzero. By an application of König's tree lemma (see, for instance, its application in [8]), we can fix sequences $i_1, i_2, \dots, j_1, j_2, \dots, j'_1, j'_2, \dots \in \mathbb{N}$ and $r_1, r_2, \dots \in R$ such that

$$s_k := \delta^{j_1}(a_{i_1})\delta^{j'_1}(r_1)\delta^{j_2}(a_{i_2})\delta^{j'_2}(r_2) \cdots \delta^{j_k}(a_{i_k})\delta^{j'_k}(r_k) \neq 0$$

for every $k \geq 1$. On the other hand, some a_i must occur infinitely often in the sequence a_{i_1}, a_{i_2}, \dots , and, since $a_i \in \mathfrak{T}_{\ell, \delta}(R)$, we must have $s_k = 0$ for k large enough, giving the needed contradiction. □

Example 3.4. The ideal $\mathfrak{T}_{\ell, \delta}(R)$ is not always the maximal δ -ideal contained in $\mathfrak{T}_\ell(R)$. Indeed, let

$$R = \mathbb{F}_2[x_0, x_1, \dots : x_i^2 = 0]$$

with derivation $\delta(x_i) = x_{i+1}$. We see that $\mathfrak{T}_\ell(R) = \mathfrak{P}(R) = (x_0, x_1, \dots)$ is already a δ -ideal, but $x_0 \notin \mathfrak{T}_{\ell, \delta}(R)$ since

$$x_0 \delta(x_0) \delta^2(x_0) \cdots \delta^k(x_0) = x_0 x_1 x_2 \cdots x_k \neq 0$$

for each $k \geq 1$.

Following [8, Section 5], set $\mathfrak{T}_\ell^{(0)} = (0)$ and recursively define the higher left T -nilpotent radideals

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \{a \in R : a + \mathfrak{T}_\ell^{(\beta)}(R) \in \mathfrak{T}_\ell(R/\mathfrak{T}_\ell^{(\beta)}(R))\},$$

if α is the successor of β and, if α is a limit ordinal,

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \bigcup_{\beta < \alpha} \mathfrak{T}_\ell^{(\beta)}(R).$$

We can now extend Theorem 3.3 to the higher left T -nilpotent radideals by a simple transfinite induction.

Corollary 3.5. *The higher left T -nilpotent radideals $\mathfrak{T}_\ell^{(\alpha)}$ satisfy the δ -Amitsur property (1.1). Thus, we can write*

$$\mathfrak{T}_\ell^{(\alpha)}(R[x; \delta]) = \mathfrak{T}_{\ell, \delta}^{(\alpha)}(R)[x; \delta]$$

for a unique δ -ideal $\mathfrak{T}_{\ell, \delta}^{(\alpha)}(R) \trianglelefteq R$.

Proof. Assume that the statement is true for all ordinals $\beta < \alpha$. Note that we have a natural surjection

$$R[x; \delta] \longrightarrow R[x; \delta]/\mathfrak{T}_{\ell, \delta}^{(\beta)}(R)[x; \delta],$$

and a natural isomorphism

$$R[x; \delta]/\mathfrak{T}_{\ell, \delta}^{(\beta)}(R)[x; \delta] \cong (R/\mathfrak{T}_{\ell, \delta}^{(\beta)}(R))[x; \bar{\delta}],$$

where $\bar{\delta}$ is the derivation induced on the factor ring by δ since $\mathfrak{T}_{\ell, \delta}^{(\beta)}(R)$ is a δ -ideal.

First, consider the case where α is the successor of some ordinal β . By Theorem 3.3, we know that \mathfrak{T}_ℓ satisfies the δ -Amitsur property. Thus, the elements of $\mathfrak{T}_\ell((R/\mathfrak{T}_{\ell, \delta}^{(\beta)}(R))[x; \bar{\delta}])$ are determined by the constant polynomials in this ideal. Lifting these constant polynomials through the natural isomorphism and surjection above, we obtain the desired conclusion.

Finally, if α is a limit ordinal, we have

$$\begin{aligned} \mathfrak{T}_\ell^{(\alpha)}(R[x; \delta]) &= \bigcup_{\beta < \alpha} \mathfrak{T}_\ell^{(\beta)}(R[x; \delta]) = \bigcup_{\beta < \alpha} \mathfrak{T}_{\ell, \delta}^{(\beta)}(R)[x; \delta] \\ &= \left(\bigcup_{\beta < \alpha} \mathfrak{T}_{\ell, \delta}^{(\beta)}(R) \right) [x; \delta]. \end{aligned}$$

Thus, we can take

$$\mathfrak{T}_{\ell, \delta}^{(\alpha)}(R) = \bigcup_{\beta < \alpha} \mathfrak{T}_{\ell, \delta}^{(\beta)}(R). \quad \square$$

As noted at the beginning of this section, these higher left T -nilpotent radideals stabilize to the prime radical. Thus, we obtain a new characterization of the prime radical of $R[x; \delta]$, and we also recover the result of Ferrero, Kishimoto and Motose that the prime radical satisfies the δ -Amitsur property [6].

Proposition 3.6. *Given a ring R with a derivation δ , then we have $\mathfrak{P}(R[x; \delta]) = \mathfrak{P}_\delta(R)[x; \delta]$, where*

$$\mathfrak{P}_\delta(R) = \mathfrak{P}(R[x; \delta]) \cap R$$

is the limit of the δ -ideals $\mathfrak{T}_{\ell, \delta}^{(\alpha)}(R)$. In particular, the prime radical satisfies the δ -Amitsur property.

4. An alternate characterization of the δ -Amitsur property.

Perhaps the most well known and utilized condition for checking the Amitsur property is that of Krempa [10, Theorem 1], which states that a radical \mathfrak{F} has Amitsur's property if and only if, for every ring R ,

$$\mathfrak{F}(R[x]) \cap R = 0 \implies \mathfrak{F}(R[x]) = 0.$$

This equivalence holds for differential polynomial rings with only minimal changes to the proofs, as we will now show. In order to begin, we first need the basic fact that radicals are closed under multiplications in unital extensions.

Lemma 4.1. *Let \mathfrak{F} be a radical, and let R be a ring. If $I \trianglelefteq R$, then $\mathfrak{F}(I) \trianglelefteq R$, and hence, $\mathfrak{F}(I) \subseteq \mathfrak{F}(R)$.*

Proof. This is [2, Theorem 1]. □

In order to apply this to unital extensions, we make the following definition. If R is unital, we set $R^1 = R$; otherwise, we let $R^1 = R \oplus \mathbb{Z}$ be the Dorroh extension of R by \mathbb{Z} , where addition is component-wise and multiplication is given by the rule $(r, m)(s, n) = (rs + ms + nr, mn)$ for all $r, s \in R$ and $m, n \in \mathbb{Z}$. Note that $R \trianglelefteq R^1$.

Corollary 4.2. *Let \mathfrak{F} be a radical, let R be a ring and let δ be a derivation on R . Then the following occurs:*

$$x\mathfrak{F}(R[x; \delta]) + \mathfrak{F}(R[x; \delta])x \subseteq \mathfrak{F}(R[x; \delta]).$$

Consequently, $\mathfrak{F}(R[x; \delta]) \cap R$ is a δ -ideal of R .

Proof. If R contains 1, then this result is trivial. If R does not contain 1, let R^1 be the Dorroh extension as above. We extend δ to R^1 in the only possible manner which preserves the fact that δ is a derivation by making δ act trivially on \mathbb{Z} . It is easy to verify that $R[x; \delta] \trianglelefteq R^1[x; \delta]$. Thus, by Lemma 4.1, $\mathfrak{F}(R[x; \delta])$ is an ideal in $R^1[x; \delta]$, and, in particular, is closed under multiplication by x . Finally, apply Lemma 2.2. □

Note that this corollary holds for more general extensions (such as Ore extensions, see [11] for the definition), not merely those of derivation type. This fact may be useful in studying radicals of such rings, but we will not make use of such generality here.

With Corollary 4.2 in place, we are now in a position to prove that Krempa’s characterization holds for differential polynomial rings.

Proposition 4.3. *Let \mathfrak{F} be a radical. This radical has the δ -Amitsur property (1.1) if and only if, for every ring R and every derivation δ on R , the following occurs:*

$$(4.1) \quad \mathfrak{F}(R[x; \delta]) \cap R = 0 \implies \mathfrak{F}(R[x; \delta]) = 0.$$

Proof. The proof follows by modifying the proof of [10, Theorem 1]; however, we include it for completeness. The forward direction is clear, which leaves only the reverse.

Set

$$A = \mathfrak{F}(R[x; \delta]) \cap R,$$

which is a δ -ideal of R , and thus, $A[x; \delta]$ is a well-defined ring. By Corollary 4.2,

$$A[x; \delta] \subseteq \mathfrak{F}(R[x; \delta]).$$

Thus,

$$\mathfrak{F}(R[x; \delta]/A[x; \delta]) = \mathfrak{F}(R[x; \delta])/A[x; \delta]$$

by standard radical arguments.

On the other hand,

$$R[x; \delta]/A[x; \delta] \cong \overline{R}[x; \overline{\delta}],$$

where $\overline{R} = R/A$, and $\overline{\delta}$ is the induced derivation on the factor ring (which exists since A is a δ -ideal). This yields a string of isomorphisms:

$$\begin{aligned} \mathfrak{F}(\overline{R}[x; \overline{\delta}]) \cap \overline{R} &\cong \mathfrak{F}(R[x; \delta]/A[x; \delta]) \cap R/A \\ &\cong (\mathfrak{F}(R[x; \delta])/A[x; \delta]) \cap ((R + A[x; \delta])/A[x; \delta]) \\ &= (\mathfrak{F}(R[x; \delta]) \cap (R + A[x; \delta]))/A[x; \delta] \\ &= (\mathfrak{F}(R[x; \delta]) \cap R + A[x; \delta])/A[x; \delta] \\ &= (A + A[x; \delta])/A[x; \delta] = 0. \end{aligned}$$

Assuming the implication in the statement of Proposition 4.3 yields

$$0 = \mathfrak{F}(\overline{R}[x; \overline{\delta}]) \cong \mathfrak{F}(R[x; \delta])/A[x; \delta],$$

and hence,

$$\mathfrak{F}(R[x; \delta]) = A[x; \delta] = (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta];$$

in other words, \mathfrak{F} has the δ -Amitsur property. □

5. Two different attacks on radicals. There are two distinct ways of showing that the Jacobson radical \mathfrak{J} satisfies Amitsur's property. The first derives from Amitsur's original characterization, found in [1], of the Jacobson radical as

$$\mathfrak{J}(R[x]) = N[x]$$

for some nil ideal $N \trianglelefteq R$ and involves a clever isomorphism trick. The second method has its roots in the unpublished work of Bergman [3] and utilizes the fact that the Jacobson radical behaves well with respect to, what are called in the literature, *finite centralizing extensions* (see [15, subsection 10.1]). We will now describe how to transfer these

arguments to the situation of skew polynomial extensions of derivation type, which allows us to show that many radicals have the δ -Amitsur property.

Both arguments begin in the same way. Let \mathfrak{F} be a radical, and assume that \mathfrak{F} does not have the δ -Amitsur property. Thus, we may fix a ring R and a derivation δ on R such that there exists a polynomial

$$f(x) \in \mathfrak{F}(R[x; \delta]) \setminus (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta]$$

with $n := \deg(f(x)) \geq 1$ minimal among all choices of R and δ . After passing to a factor ring, if necessary, we may assume

$$\mathfrak{F}(R[x; \delta]) \cap R = 0,$$

and hence, this minimal degree n occurs over a ring where the differential version of Krempa's criterion (4.1) fails. Write

$$f(x) = \sum_{i=0}^n a_i x^i \in \mathfrak{F}(R[x; \delta])$$

with each $a_i \in R$. Now, it may easily be verified that the map sending

$$h(x) \longrightarrow h(x+1)$$

is an automorphism of $R[x; \delta]$ (even in the case where R is non-unital), and radicals are invariant under automorphisms; thus,

$$f(x+1) \in \mathfrak{F}(R[x; \delta]).$$

However, since $\deg(f(x+1) - f(x)) < \deg(f(x))$, minimality of n implies $f(x+1) - f(x) = 0$.

Expansion yields

$$f(x+1) - f(x) = na_n x^{n-1} + \text{lower order terms},$$

and thus, $na_n = 0$. If either $n = 1$ or R is a \mathbb{Q} -algebra, we have a contradiction (to the fact that $a_n \neq 0$), so we may assume $n \geq 2$. Letting $m > 1$ be the smallest integer with $ma_n = 0$ and fixing a prime $p \mid m$, we can replace $f(x)$ by $(m/p)f(x)$, and thus, $pa_n = 0$. However, $pf(x) \in \mathfrak{F}(R[x; \delta])$; thus, by minimality of degree, we in fact have that $pf(x) = 0$.

It is at this juncture that the arguments of Amitsur and Bergman diverge; thus, we first describe Amitsur's argument. We want to reduce

to the case $pR = 0$. In order to facilitate such a reduction, we make the additional assumption that \mathfrak{F} is hereditary. Then, letting

$$R_p = \{r \in R : pr = 0\} \trianglelefteq R,$$

the hereditary assumption provides

$$(5.1) \quad \mathfrak{F}(R_p) = \mathfrak{F}(R) \cap R_p.$$

From the more difficult of the two inclusions in (5.1) we get

$$f(x) \in \mathfrak{F}(R) \cap R_p \subseteq \mathfrak{F}(R_p),$$

and, from the (easier) other inclusion, we see that there are no nonzero polynomials of smaller degree in $\mathfrak{F}(R_p)$. Thus, after replacing R by R_p , if necessary, we may reduce to the case $pR = 0$.

In the usual polynomial case it may be shown that, since $f(x + 1) = f(x)$, we have $f(x) = g(x^p - x)$ for some polynomial $g(x) \in R[x]$. Unsurprisingly, in the differential polynomial case, it may be shown that $f(x)$ is a left polynomial in the variable $t := x^p - x$, see [1] for a quick argument. Note that, for any $r \in R$, we have $(x^p - x)r = r(x^p - x) + (\delta^p - \delta)r$ since $pR = 0$. Moreover, since R is an \mathbb{F}_p -algebra, it is straightforward to check that δ^p is a derivation on R , and thus, $\delta^p - \delta$ is a derivation as well. Hence,

$$g(x^p - x) = g(t) \in R[t; \delta^p - \delta],$$

and clearly, $\deg(g) < \deg(f)$. In the standard polynomial case, it may be argued that $g(t)$ belongs to $\mathfrak{F}(R[t])$ (often, by appealing to additional assumptions on the radical \mathfrak{F}), and then, since $R[t] \cong R[x]$, we have $g(x) \in \mathfrak{F}(R[x])$, contradicting the minimality of n . Unfortunately, there is, in general, no degree preserving isomorphism between $R[x; \delta]$ and $R[t; \delta^p - \delta]$. Thus, Amitsur's argument often breaks at this point for differential polynomial rings.

We now turn to the other method. Propitiously, in Bergman's argument, there is no roadblock. (Note that we are now not necessarily assuming that \mathfrak{F} is hereditary, although, if that assumption holds, then the reductions in the previous two paragraphs can still be made.) The idea is simply that we need more automorphisms which will give us more control over $f(x)$. In that spirit, let $q > n$ be an integer prime, and let $\zeta = \zeta_q$ be a primitive q th root of unity (from \mathbb{C}). Given any

ring A , define

$$A' := A[\zeta] = A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta].$$

(Here we slightly abuse notation since A may already contain the complex q th roots of unity; thus, it must be remembered that $A[\zeta]$ is shorthand for $A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$.) Note that A sits (isomorphically) as a, possibly non-unital, subring of A' . In our case, we focus on the ring

$$R[x; \delta] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta] \cong R'[x; \delta'],$$

where δ' is the derivation on $R' = R[\zeta]$ determined by the rule $\delta'(r \otimes \alpha) = \delta(r) \otimes \alpha$. In this ring, we have the automorphisms

$$(5.2) \quad \sigma_j(h(x)) = h(x + \zeta^j)$$

for $j \in \mathbb{N}$. Identifying $f(x)$ with

$$f(x) \otimes 1 \in R'[x; \delta'],$$

the constant term of $f(x + \zeta^j) - f(x)$ is exactly

$$\sum_{i=1}^n a_i \otimes \zeta^{ij}.$$

Thus,

$$\begin{aligned} g(x) &:= \sum_{j=0}^{q-1} \zeta^{-nj} (f(x + \zeta^j) - f(x)) \\ &= qa_n \otimes 1 + \sum_{j=0}^{q-1} \sum_{i=1}^{n-1} a_i \otimes \zeta^{j(i-n)} \\ &\quad + \text{higher order terms,} \end{aligned}$$

but, by switching the order of summation and noting that ζ^{i-n} is also a primitive q th root of unity (for $1 \leq i \leq n-1$ since $n < q$), yields

$$\sum_{j=0}^{q-1} \sum_{i=1}^{n-1} a_i \otimes \zeta^{j(i-n)} = \sum_{i=1}^{n-1} \left(a_i \otimes \sum_{j=0}^{q-1} (\zeta^{i-n})^j \right) = \sum_{i=1}^{n-1} a_i \otimes 0 = 0.$$

Thus, qa_n is the constant term in $g(x)$.

We now define a property which holds for many radicals. We say that \mathfrak{F} respects finite cyclotomic extensions when the following occurs:

for all rings A , and all integer primes q ,

$$(5.3) \quad \mathfrak{F}(A) = \mathfrak{F}(A[\zeta_q]) \cap A,$$

with $\zeta_q \in \mathbb{C}$ a primitive q th root of unity and

$$A[\zeta_q] := A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_q].$$

Under this assumption, we see that

$$f(x) \in \mathfrak{F}(R[x; \delta]) \subseteq \mathfrak{F}((R[x; \delta])[\zeta]) = \mathfrak{F}(R'[x; \delta']),$$

and hence, by Lemma 4.1 (to get closure under multiplication by powers of ζ) we have $g(x) \in \mathfrak{F}(R'[x; \delta'])$. However, $\deg(g(x)) < \deg(f(x))$, and thus, the minimality condition on n tells us that every coefficient of $g(x)$ is an element of $\mathfrak{F}(R'[x; \delta'])$. In particular,

$$qa_n \in \mathfrak{F}((R[x; \delta])[\zeta]) \cap R[x; \delta] = \mathfrak{F}(R[x; \delta]).$$

However,

$$pa_n = 0 \in \mathfrak{F}(R[x; \delta])$$

and

$$\gcd(p, q) = 1;$$

hence,

$$a_n \in \mathfrak{F}(R[x; \delta]) \cap R = 0,$$

yielding a contradiction. Combining these results yields the next result.

Theorem 5.1. *If \mathfrak{F} is a radical which respects finite cyclotomic extensions, then \mathfrak{F} has the δ -Amitsur property.*

In the case of hereditary radicals, we can give an even nicer statement by arguing along the lines of [6], as was done in [4] using the stronger notion of normalizing extensions.

Theorem 5.2. *Let \mathfrak{F} be a hereditary radical. Also, assume that \mathfrak{F} respects finite field extensions, meaning: for every \mathbb{F}_p -algebra A , and every integer $m \geq 1$,*

$$(5.4) \quad \mathfrak{F}(A) = \mathfrak{F}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cap A.$$

Then, \mathfrak{F} has the δ -Amitsur property.

Proof. We begin by working contrapositively. Assume that \mathfrak{F} is a hereditary radical which does not have the δ -Amitsur property. As in the argument above, we can reduce to the case where

- (i) R is an \mathbb{F}_p -algebra,
- (ii) $\mathfrak{F}(R[x; \delta]) \cap R = 0$, and
- (iii)

$$f(x) = \sum_{i=0}^n a_i x^i \in \mathfrak{F}(R[x; \delta]) \setminus (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta]$$

has minimal degree $n \geq 2$.

Now, assume, by way of contradiction, that \mathfrak{F} respects finite field extensions. Setting $R' = R \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}$ with $m \geq n$, we obtain

$$f(x) \in \mathfrak{F}(R[x; \delta] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cong \mathfrak{F}(R'[x; \delta']),$$

where δ' is the natural extended derivation. On the ring $R'[x; \delta']$, the map

$$h(x) \longmapsto h(x + t)$$

is an automorphism for every $t \in \mathbb{F}_{p^m}$. Thus,

$$f(x + t) - f(x) \in \mathfrak{F}(R'[x; \delta'])$$

for every $t \in \mathbb{F}_{p^m}$. Since $\deg(f(x + t) - f(x)) < \deg(f(x))$, the minimality condition on n implies that the constant term of $f(x + t) - f(x)$, which is merely

$$\sum_{i=1}^n a_i t^i,$$

belongs to $\mathfrak{F}(R'[x; \delta'])$. A Vandermonde matrix argument, using the fact that $p^m > n$, tells us that each

$$a_i \in \mathfrak{F}(R'[x; \delta']) \quad \text{for } 1 \leq i \leq n.$$

Hence,

$$f(x) \in (\mathfrak{F}(R[x; \delta] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cap R)[x; \delta] = (\mathfrak{F}(R[x; \delta]) \cap R)[x; \delta],$$

giving the needed contradiction. □

Remark 5.3.

(1) In the argument of Theorem 5.2 we really only need \mathfrak{F} to respect finite field extensions $\mathbb{F}_{p^m}/\mathbb{F}_p$ for a sequence of strictly increasing positive integers

$$m = m_1 < m_2 < \dots .$$

We use this small improvement shortly.

(2) Suppose that A is an \mathbb{F}_p -algebra and \mathfrak{F} is a radical. Let $q \neq p$ be prime, and let $\Phi_q(x) = (x^q - 1)/(x - 1)$ be the q th cyclotomic polynomial. Since $\Phi_q(x)$ is separable modulo p , we can write

$$\Phi_q(x) \equiv \prod_{i=1}^t f_i(x) \pmod{p}$$

where the f_i are relatively prime, monic polynomials which are irreducible modulo p . We now have

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_q] = A \otimes_{\mathbb{F}_p} \mathbb{F}_p[y]/(\Phi_q(y)) = \prod_{i=1}^t (A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^{n_i}}),$$

where $n_i = \deg(f_i)$ for each i . Simple computation now shows that, if \mathfrak{F} respects finite field extensions, then it respects finite cyclotomic extensions (at least when $q \neq p$).

(3) Finite cyclotomic extensions and finite field extensions are merely special cases of finite centralizing extensions. See [15, Section 10] for some further nice examples.

We now give an easy example of where these methods apply. (Note that each of the radicals in the next corollary was already covered by [4].)

Corollary 5.4. *The Jacobson, Levitzki, prime and Brown-McCoy radicals each have the δ -Amitsur property.*

Proof. Let \mathfrak{F} be any of the above radicals. First, suppose that R does not have 1. We then have

$$\mathfrak{F}(R) \subseteq \mathfrak{F}(R^1)$$

since \mathfrak{F} is hereditary. However, $R^1/R \cong \mathbb{Z}$ is \mathfrak{F} -semisimple; thus, we must have the equality $\mathfrak{F}(R) = \mathfrak{F}(R^1)$. A similar argument holds if we replace R by $R[x; \delta]$ (since, in this case, $\mathbb{Z}[x]$ is \mathfrak{F} -semisimple). Therefore, without loss of generality, we can reduce to the case that R is unital. It is well known that each of these radicals respects finite cyclotomic extensions in the case where the ring is unital, see for instance [16, page 454]. Now, merely apply Theorem 5.1. \square

Since the proof that the Brown-McCoy radical respects finite cyclotomic extensions is by no means trivial, we give an alternate proof in that case, which may be of independent interest.

Proposition 5.5. *The Brown-McCoy radical \mathfrak{G} has the δ -Amitsur property.*

Proof. Since \mathfrak{G} is a hereditary radical, by Remark 5.3 (1) and Theorem 5.2 it suffices to show that, whenever F is a finite field, R is an F -algebra, and K/F is a field extension with $[K : F] = 2$, then

$$\mathfrak{G}(R) = \mathfrak{G}(S) \cap R \quad \text{where } S = R \otimes_F K.$$

Further, if R is non-unital and R^* is the Dorroh extension of R by F , then

$$\mathfrak{G}(R) \subseteq \mathfrak{G}(R^*)$$

since R is an ideal in R^* . On the other hand, $R^*/R \cong F$ is Brown-McCoy semisimple. Thus, $\mathfrak{G}(R) = \mathfrak{G}(R^*)$, and hence, we only need to prove that the equality holds in the case when R is a unital F -algebra and S is a unital overring. Fix an F -basis $B = \{1, b\}$ for the extension K/F . Now, we prove the equality $\mathfrak{G}(R) = \mathfrak{G}(S) \cap R$.

(\subseteq). Assume, by way of contradiction, that $r \in \mathfrak{G}(R) \setminus \mathfrak{G}(S)$. Then, $r \notin M$ for some maximal ideal of S . Since S/M is a simple ring,

$$1 - \sum_{i=1}^m s_i r s'_i \in M$$

for some $s_i, s'_i \in S$. Expanding this sum in terms of the basis B , we have

$$\sum_{i=1}^m s_i r s'_i = r_1 + r_2 b$$

for some $r_1, r_2 \in RrR \subseteq \mathfrak{G}(R)$, and also $1 - r_1 - r_2 b \in M$.

Consider the two-sided ideal

$$R(1 - r_1)R \trianglelefteq R.$$

If this were a proper ideal, it would be contained in a maximal ideal M' of R . However, since $r_1 \in \mathfrak{G}(R)$ we would also have $r_1 \in M'$, and hence, $1 = (1 - r_1) + r_1 \in M'$, which is impossible. Thus, $R(1 - r_1)R = R$, so we can write

$$1 = \sum_{j=1}^n t_j (1 - r_1) t'_j$$

for some elements $t_j, t'_j \in R$. Hence,

$$\sum_{j=1}^n t_j (1 - r_1 - r_2 b) t'_j = 1 - \left(\sum_{j=1}^n t_j r_2 t'_j \right) b = 1 - ub \in M$$

for some element $u \in Rr_2R \subseteq \mathfrak{G}(R)$.

Since K is a finite field, we can fix an integer $k \geq 2$ such that $b^k = 1$. Thus,

$$1 - u^k = (1 - ub)(1 + ub + u^2 b^2 + \dots + u^{k-1} b^{k-1}) \in M.$$

However, $u^k \in \mathfrak{G}(R)$, and thus, by the same argument as at the beginning of the previous paragraph we have $R(1 - u^k)R = R$. Thus, $1 \in M$, giving us the needed contradiction.

(\supseteq). Assume, by way of contradiction, $r \in (\mathfrak{G}(S) \cap R) \setminus \mathfrak{G}(R)$. Fix a maximal ideal I of R , with $r \notin I$. Note that

$$I \otimes_F K = I \oplus Ib$$

is a proper ideal of S , and thus, is contained in a maximal ideal M of S . Since $r \in \mathfrak{G}(S)$ we have $r \in M$. However, $I \subseteq M$ as well, so

$$R = I + RrR \subseteq M.$$

This yields $1 \in M$, a contradiction. □

Remark 5.6. These same techniques fail for the Behrens radical β . From [12, Proposition 3.1], we have $\beta(S) \cap R \subseteq \beta(R)$ for *any* unital finite centralizing extension, and thus, we care only about the reverse inclusion. However, from the Example following Proposition 3.1 in [12], there is a Behrens radical ring R which is an \mathbb{R} -algebra and

$$\beta(R) \not\subseteq \beta(R \otimes_{\mathbb{R}} \mathbb{C}).$$

This shows that β does not even respect quadratic field extensions. One might still hold out hope in the finite field case, but in fact, the example can be modified to disprove that case as follows.

Let K be the field of rational functions in the commuting variables x_i (for $i \in \mathbb{Z}$), with coefficients in $\mathbb{F}_3(y)$. Let σ be the $\mathbb{F}_3(y)$ -automorphism of K which sends

$$x_i \mapsto x_{i+1}.$$

It is easily verified that the skew Laurent polynomial ring (of automorphism type) $K[x, x^{-1}; \sigma]$ is a simple domain with $\mathbb{F}_3(y)$ as the center.

Define $\mathbb{H} = (2, y/F)$ as the quaternion algebra over $F = \mathbb{F}_3(y)$ with generators i, j and relations given by $i^2 = 2, j^2 = y$ and $ij = -ji$. This is a non-split algebra; hence, it is a four-dimensional division algebra with center $\mathbb{F}_3(y)$. The tensor product

$$T = \mathbb{H} \otimes_{\mathbb{F}_3(y)} K[x, x^{-1}; \delta]$$

is also a central simple $\mathbb{F}_3(y)$ -algebra. The remainder of the construction is essentially unchanged from [12] after replacing \mathbb{R} by $\mathbb{F}_3(y)$ and \mathbb{C} by $\mathbb{F}_9(y)$ everywhere; therefore, those details are left to the interested reader. Finally, note that

$$\mathbb{H} \otimes_{\mathbb{F}_3(y)} \mathbb{F}_9(y) \cong \mathbb{M}_2(\mathbb{F}_9(y))$$

since \mathbb{F}_9 contains a square-root of 2, and also note that

$$S = R \otimes_{\mathbb{F}_3(y)} \mathbb{F}_9(y) \cong R \otimes_{\mathbb{F}_3} \mathbb{F}_9.$$

6. Open questions. One radical which is conspicuously missing from Corollary 5.4 is the upper nilradical. Amitsur proved in 1956 that the upper nilradical has Amitsur’s property [1]. It was not until 2014 that a proof was finally found by Smoktunowicz for the fact that the upper nilradical is homogeneous in \mathbb{Z} -graded rings [17]. This work was subsequently extended to gradings over semigroups

in [14]. Unfortunately, these methods are somewhat orthogonal to those employed in this paper, and thus, we ask:

Question 6.1. *Does the δ -Amitsur property hold for the upper nilradical?*

Perhaps even more difficult is the case of the Behrens radical, for we know that this radical behaves poorly with respect to field extensions.

Another set of questions involves element-wise characterizations of the ideal

$$\mathfrak{F}(R[x; \delta]) \cap R,$$

when R has a derivation δ . While much work has been done in the case of the prime radical, there are other basic radicals which might yield to similar analyses. In particular, we ask:

Question 6.2. *If \mathfrak{L} is the Levitzki radical, is there a simple description of the ideal $\mathfrak{L}(R[x; \delta]) \cap R$, for any ring R with a derivation δ ?*

Finally, in Section 3 we gave a list of five characterizations of the prime radical, four of which are now known to generalize extremely well to the differential polynomial case. Thus, we ask:

Question 6.3. *How does the bounded nilradical behave in differential polynomial rings? Is there a simple element-wise description of this ideal in terms of the coefficient ring?*

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