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LOCALIZATION OPERATORS FOR THE WINDOWED FOURIER TRANSFORM ASSOCIATED WITH SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. We introduce the windowed Fourier transform connected with some singular partial differential operators defined on the half plane $[0, +\infty[\times \mathbb{R}$. Then, we investigate localization operators and show that these operators are not only bounded but also in the Shatten-von Neumann class. We give a trace formula when the symbol function is a nonnegative function.

1. Introduction. Time-frequency localization operators were first introduced by Daubechies **[8, 9, 10]**. She pointed out the role of these operators to localize a signal simultaneously in time and frequency.

This class of operator occurs in various branches of mathematics and has been studied by many authors. Indeed, many applications have been discovered to time-frequency analysis, for example, in the areas of differential equations, quantum mechanics and signal processing **[5, 11, 12, 18, 27, 32]**. In the literature, they are also known as anti-Wick operators, wave packets, Toeplitz operators or Gabor multipliers **[4, 6, 12, 14]**. In **[31]**, Wong showed that the localization operators introduced by Daubechies are examples of Weyl transforms which enjoy good mapping properties as compact operators from $L^2(\mathbb{R}^n)$ into itself. He **[22]** also studied these operators for which he gave the Shatten-von Neumann properties and trace formula.

Motivated by their impact in real-life signals, in this paper, localization operators are defined by means of the most used time-frequency representation that is the windowed transform connected with singular partial differential operators also known as the short-time Fourier

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transform introduced in several settings, for example, **[7, 16, 17]**. For this, we consider the singular partial differential operators:

$$\begin{cases} \Delta_1 = \frac{\partial}{\partial x}, \\ \\ \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2} \qquad (r,x) \in\,]0,+\infty[\, \times \mathbb{R}, \quad \alpha \geqslant 0. \end{cases}$$

We associate to Δ_1 and Δ_2 the Riemann-Liouville transform $\mathscr{R}_\alpha$ defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, with respect to the first variable as well), by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2},\ x+rt) & \\ \quad \cdot(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}dt\,ds & \text{if } \alpha > 0, \\ \frac{1}{\pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(dt/\sqrt{1-t^2}) & \text{if } \alpha = 0\,. \end{cases}$$

The transform $\mathscr{R}_\alpha$ generalizes the mean operator defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)\,d\theta.$$

The mean operator $\mathscr{R}_0$ and its dual play an important role and have many applications, for example, in the image processing of so-called synthetic aperture radar (SAR) data **[1, 23]** or in the linearized inverse scattering problem in acoustics **[13]**.

In **[2]**, we defined a convolution product and a Fourier transform $\mathscr{F}_\alpha$ associated with $\mathscr{R}_\alpha$, and we established many harmonic analysis results (inversion formula, Paley-Wiener and Plancherel theorems, etc.). In **[20]**, Hamadi and Rachdi introduced the windowed Fourier transform associated with the Riemann-Liouville operator, which is a generalization of the classical windowed Fourier transform. Many harmonic analysis results related to the Riemann-Liouville operator have already been proved, for example, **[2, 3, 20, 21]**.

In this paper, we will define a type of localization operator associated to the Riemann-Liouville operator and show that this operator is not only bounded but also contained in the Shatten-Von Neumann class. In addition, a trace formula is given when the symbol is a nonnegative function.

2. Riemann-Liouville transform associated with the operators Δ_1 and Δ_2. In this section, we recall some properties of the

Riemann-Liouville transform that we will use in the following sections. For more details, see [**2**].

For all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$, the system

$$\begin{cases} \Delta_1 u(r,x) = -i\lambda u(r,x), \\ \Delta_2 u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1, \ (\partial u)/(\partial r)(0,x) = 0 \quad \text{for all } x \in \mathbb{R}, \end{cases}$$

admits a unique solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})\exp(-i\lambda x), \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(s) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind with index α, see [**24, 30**]. Moreover, we have

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1 \quad \text{if and only if } (\mu,\lambda) \in \Upsilon,$$

where Υ is the set defined by

$$\Upsilon = \mathbb{R}^2 \cup \{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ |\mu| \leqslant |\lambda|\}. \tag{2.2}$$

Proposition 2.1. *The eigenfunction $\varphi_{\mu,\lambda}$ given by* (2.1) *has the following Mehler integral representation*

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^1\int_{-1}^1 \cos(\mu r s\sqrt{1-t^2})e^{-i\lambda(x+rt)} \\ \quad \cdot(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}dt\,ds & \text{if } \alpha > 0, \\ \frac{1}{\pi}\int_{-1}^1 \cos(r\mu\sqrt{1-t^2})e^{-i\lambda(x+rt)}(dt/\sqrt{1-t^2}) & \text{if } \alpha = 0. \end{cases}$$

This result shows that

$$\varphi_{\mu,\lambda}(r,x) = \mathscr{R}_\alpha(\cos(\mu\cdot)\exp(-i\lambda\cdot))(r,x),$$

where $\mathscr{R}_\alpha$ is the Riemann-Liouville transform associated with the operators Δ_1 and Δ_2, given in the introduction.

We denote

- $d\nu_\alpha(r,x)$ as the measure defined on $[0,+\infty[\times\mathbb{R}$, by
$$d\nu_\alpha(r,x) = c_\alpha r^{2\alpha+1} dr\otimes dx,$$
with $c_\alpha = 1/(\sqrt{2\pi}2^\alpha\Gamma(\alpha+1))$.
- $L^p(d\nu_\alpha)$ as the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, satisfying

$$\begin{aligned}\|f\|_{p,\nu_\alpha} &= \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{1/p} < +\infty \quad &&\text{if } p\in[1,+\infty[;\\ \|f\|_{\infty,\nu_\alpha} &= \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}} |f(r,x)| < +\infty &&\text{if } p=+\infty.\end{aligned}$$

- $\gamma_\alpha(\mu,\lambda)$ as the measure defined on Υ, by

$$\begin{aligned}\iint_\Gamma f(\mu,\lambda)\,d\gamma_\alpha(\mu,\lambda) = c_\alpha\bigg\{&\int_0^{+\infty}\int_{\mathbb{R}} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu\,d\mu\,d\lambda\\ &+\int_0^{|\lambda|}\int_{\mathbb{R}} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha\mu\,d\mu\,d\lambda\bigg\}.\end{aligned}$$

- $L^p(\gamma_\alpha),\ p\in[1,+\infty]$, as the space of measurable functions on Υ, satisfying

$$\begin{aligned}\|f\|_{p,\gamma_\alpha} &= \left(\iint_\Upsilon |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p} < +\infty \quad &&\text{if } p\in[1,+\infty[,\\ \|f\|_{\infty,\gamma_\alpha} &= \operatorname*{ess\,sup}_{(\mu,\lambda)\in\Upsilon} |f(\mu,\lambda)| < +\infty &&\text{if } p=+\infty.\end{aligned}$$

Definition 2.2.

(i) The translation operator associated with the Riemann-Liouville transform is defined on $L^1(d\nu_\alpha)$, for all $(r,x),\ (s,y)\in[0,+\infty[\times\mathbb{R}$, by

$$\tau_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+(1/2))}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{2\alpha}\theta\,d\theta.$$

(ii) The convolution product associated with the Riemann-Liouville transform of f and $g\in L^1(d\nu_\alpha)$ is defined, for all $(r,x)\in[0,+\infty[\times\mathbb{R}$, by

$$f*g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \tau_{(r,-x)}\check{f}(s,y)\,g(s,y)\,d\nu_\alpha(s,y),$$

where $\check{f}(s,y) = f(s,-y)$.

The next properties follow from Definition 2.2.

- The product formula

$$\tau_{(r,x)}\varphi_{\mu,\lambda}(s,y)=\varphi_{\mu,\lambda}(r,x)\ \varphi_{\mu,\lambda}(s,y).$$

- Let f be in $L^1(d\nu_\alpha)$. Then, for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}}\tau_{(s,y)}f(r,x)\,d\nu_\alpha(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\,d\nu_\alpha(r,x).$$

- If $f\in L^p(d\nu_\alpha),\ 1\leqslant p\leqslant+\infty$, then, for all $(s,y)\in[0,+\infty[\times\mathbb{R}$, the function $\tau_{(s,y)}f$ belongs to $L^p(d\nu_\alpha)$, and we have

$$\left\|\tau_{(s,y)}f\right\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}. \tag{2.3}$$

- For $f,g\in L^1(d\nu_\alpha),\ f*g$ belongs to $L^1(d\nu_\alpha)$, and the convolution product is commutative and associative.
- For $f\in L^1(d\nu_\alpha),\ g\in L^p(d\nu_\alpha),\ 1<p\leqslant+\infty$, the function $f*g\in L^p(d\nu_\alpha)$ and

$$\|f*g\|_{p,\nu_\alpha}\leqslant\|f\|_{1,\nu_\alpha}\|g\|_{p,\nu_\alpha}.$$

- Let $p,q,r\in[1,+\infty]$ be such that $1/p+1/q=1+1/r$. Then, for all f in $L^p(d\nu_\alpha)$ and g in $L^q(d\nu_\alpha)$, the function $f*g$ belongs to the space $L^r(d\nu_\alpha)$, and we have

$$\|f*g\|_{r,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}\|g\|_{q,\nu_\alpha}.$$

Definition 2.3. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$, for all $(\mu,\lambda)\in\Upsilon$, by

$$\mathscr{F}_\alpha(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\varphi_{\mu,\lambda}(r,x)\,d\nu_\alpha(r,x),$$

where Υ is the set defined by relation (2.2).

The next properties follow from Definition 2.3.

- Let f be in $L^1(d\nu_\alpha)$. For all $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have for all $(\mu,\lambda)\in\Upsilon$,

$$\mathscr{F}_\alpha(\tau_{(r,-x)}f)(\mu,\lambda)=\varphi_{\mu,\lambda}(r,x)\mathscr{F}_\alpha(f)(\mu,\lambda). \tag{2.4}$$

- For $f, g \in L^1(d\nu_\alpha)$, we have for all $(\mu, \lambda) \in \Upsilon$,

$$\mathscr{F}_\alpha(f * g)(\mu, \lambda) = \mathscr{F}_\alpha(f)(\mu, \lambda)\mathscr{F}_\alpha(g)(\mu, \lambda). \tag{2.5}$$

- For $f \in L^1(d\nu_\alpha)$, we have for all $(\mu, \lambda) \in \Upsilon$,

$$\mathscr{F}_\alpha(f)(\mu, \lambda) = B \circ \widetilde{\mathscr{F}_\alpha}(f)(\mu, \lambda),$$

where

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu, \lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r, x) j_\alpha(r\mu) \exp(-i\lambda x)\, d\nu_\alpha(r, x),$$

$(\mu, \lambda) \in \mathbb{R}^2$, and, for all $(\mu, \lambda) \in \Upsilon$,

$$Bf(\mu, \lambda) = f(\sqrt{\mu^2 + \lambda^2}, \lambda).$$

- For $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f) \in L^1(\gamma_\alpha)$, we have the inversion formula for $\mathscr{F}_\alpha$, for almost every $(r, x) \in [0, +\infty[\times\mathbb{R}$,

$$f(r, x) = \iint_\Upsilon \mathscr{F}_\alpha(f)(\mu, \lambda)\overline{\varphi}_{\mu,\lambda}(r, x)\gamma_\alpha(\mu, \lambda).$$

We denote by (see **[2, 19, 25]**)

- $\mathscr{S}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, as well as with respect to the first variable;
- $\mathscr{S}_*(\Upsilon)$ the space of functions $f : \Upsilon \to \mathbb{C}$ infinitely differentiable, as well as with respect to the first variable and rapidly decreasing, together with all their derivatives, i.e., for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Upsilon} (1 + |\mu|^2 + |\lambda|^2)^{k_1} \left| \left(\frac{\partial}{\partial\mu}\right)^{k_2} \left(\frac{\partial}{\partial\lambda}\right)^{k_3} f(\mu, \lambda)\right| < +\infty,$$

where

$$\frac{\partial f}{\partial\mu}(\mu, \lambda) = \begin{cases} \frac{\partial}{\partial r}(f(r, \lambda)) & \text{if } \mu = r \in \mathbb{R} \\ \frac{1}{i}\frac{\partial}{\partial t}(f(it, \lambda)) & \text{if } \mu = it,\ |t| \leqslant |\lambda|. \end{cases}$$

Each of these spaces is equipped with its usual topology.

Remark 2.4. From **[2]**, the Fourier transform $\mathscr{F}_\alpha$ is an isomorphism from $\mathscr{S}_*\left(\mathbb{R}^2\right)$ onto $\mathscr{S}_*\left(\Upsilon\right)$. The inverse mapping is given for all

$(r,x) \in \mathbb{R}^2$ by

$$\mathscr{F}_\alpha^{-1}(f)(r,x) = \iint_{\Upsilon} f(\mu,\lambda)\overline{\varphi}_{\mu,\lambda}(r,x)\gamma_\alpha(\mu,\lambda).$$

3. The windowed Fourier transform associated with the Riemann-Liouville operator.

Definition 3.1. The windowed Fourier transform associated with the Riemann-Liouville operator is the mapping V defined on $\mathscr{S}_*(\mathbb{R}^2) \times \mathscr{S}_*(\mathbb{R}^2)$ for all $((r,x),(\mu,\lambda)) \in \mathbb{R}^2 \times \Upsilon$ by

(3.1)
$$V(f,g)((r,x),(\mu,\lambda)) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\varphi_{\mu,\lambda}(s,y)\tau_{(r,x)}g(s,y)\, d\nu_\alpha(s,y).$$

Remark 3.2. By means of relations (2.4) and (2.5), the transform V can also be written as

(i) $V(f,g)((r,x),(\mu,\lambda)) = \mathscr{F}_\alpha(f\tau_{(r,x)}g)(\mu,\lambda)$.
(ii) $V(f,g)((r,x),(\mu,\lambda)) = \check{g} * (\varphi_{\mu,\lambda}f)(r,-x)$, where $\check{g}(s,y) = g(s,-y)$ and $*$ is the convolution product given in Definition 2.2.

We denote by

- $\mathscr{S}_*(\mathbb{R}^2 \times \mathbb{R}^2)$ the space of infinitely differentiable functions $f((r,x),(s,y))$ on $\mathbb{R}^2 \times \mathbb{R}^2$, as well as with respect to the variables r and s, and rapidly decreasing, together with all their derivatives;
- $\mathscr{S}_*(\mathbb{R}^2 \times \Upsilon)$ the space of infinitely differentiable functions $f((r,x),(\mu,\lambda))$ on $\mathbb{R}^2 \times \Upsilon$, as well as with respect to the variables r and μ, and rapidly decreasing, together with all their derivatives;
- $L^p(d\nu_\alpha \otimes d\gamma_\alpha), 1 \leqslant p \leqslant +\infty$, the space of measurable functions on $([0,+\infty[\, \times \mathbb{R}) \times \Upsilon$, verifying for $p \in [1,+\infty[$,

$$\|f\|_{p,\nu_\alpha\otimes\gamma_\alpha} = \left(\int_0^{+\infty}\int_{\mathbb{R}}\iint_{\Upsilon} |f((r,x),(\mu,\lambda))|^p d\nu_\alpha(r,x)\gamma_\alpha(\mu,\lambda)\right)^{1/p} < +\infty,$$

and, for $p = +\infty$,

$$\|f\|_{\infty,\nu_\alpha\otimes\gamma_\alpha} = \operatorname*{ess\ sup}_{(r,x),(\mu,\lambda)\in([0,+\infty[\times\mathbb{R})\times\Upsilon} |f((r,x),(\mu,\lambda))| < +\infty.$$

Proposition 3.3.

(i) *The windowed Fourier transform V is a bilinear, continuous mapping from $\mathscr{S}_*(\mathbb{R}^2)\times\mathscr{S}_*(\mathbb{R}^2)$ into $\mathscr{S}_*(\mathbb{R}^2\times\Upsilon)$.*

(ii) *For $p\in\,]1,2]$, we have*

$$\|V(f,g)\|_{p',\nu_\alpha\otimes\gamma_\alpha} \leqslant \|f\|_{p,\nu_\alpha}\|g\|_{p',\nu_\alpha}.$$

The transform V can be extended to a continuous bilinear operator, also denoted by V, from $L^p(d\nu_\alpha)\times L^{p'}(d\nu_\alpha)$ into $L^{p'}(d\nu_\alpha\otimes d\gamma_\alpha)$, where $p' = p/(p-1)$, is the conjugate exponent of p.

For more details about the windowed Fourier transform associated with the Riemann-Liouville operator, the reader is referred to **[20]**.

Now, we are able to define the localization operators associated with the windowed Fourier transform, associated with Riemann-Liouville operators.

Definition 3.4. Let σ be in $L^1(d\nu_\alpha\otimes d\gamma_\alpha)+L^{+\infty}(d\nu_\alpha\otimes d\gamma_\alpha)$, and let f,g be in $L^2(d\nu_\alpha)$ such that $\|g\|_{2,\nu_\alpha}=1$. The localization operator $L_g(\sigma)$ associated with the windowed Fourier transform is defined by

$$\begin{aligned} L_g(\sigma)(f)(r,x) = \int_0^{+\infty}\int_{\mathbb{R}}\iint_\Upsilon &\sigma((s,y),(\mu,\lambda))V(f,g)((s,y),(\mu,\lambda)) \\ &\times \overline{\varphi}_{\mu,\lambda}(r,x)\overline{\tau_{s,y}g(r,x)}\,d\nu_\alpha(s,y)\,d\gamma_\alpha(\mu,\lambda), \end{aligned}$$

where σ is called the *symbol function* and g is called the *windowed function.*

Using relation (3.1), $L_g(\sigma)$ can be written as

$$L_g(\sigma)(f)(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} f(z,t)k((z,t),(r,x))\,d\nu_\alpha(z,t),$$

where k is the reproducing kernel given by

$$\begin{aligned} k((z,t),(r,x)) = \int_0^{+\infty}\int_{\mathbb{R}}\iint_\Upsilon &\sigma((s,y),(\mu,\lambda))\varphi_{\mu,\lambda}(z,t)\tau_{(s,y)}\,g(z,t) \\ &\times \overline{\varphi}_{\mu,\lambda}(r,x)\overline{\tau_{(s,y)}g(r,x)}\,d\nu_\alpha(s,y)\,d\gamma_\alpha(\mu,\lambda). \end{aligned}$$

Theorem 3.5. *Let σ be in $L^1(d\nu_\alpha \otimes d\gamma_\alpha)$. Then,*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a bounded operator, and we have $\|L_g(\sigma)\| \leqslant \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}$.

Proof. Let f and $h \in L^2(d\nu_\alpha)$. From Proposition 3.3 (ii) and using the density of $\mathscr{S}_*(\mathbb{R}^2)$ in $L^2(d\nu_\alpha)$, we obtain that $L_g(\sigma)$ belongs to $L^2(d\nu_\alpha)$. Then,

(3.2)
$$\langle L_g(\sigma)(f),\ h\rangle_{2,\nu_\alpha} = \iint_\Upsilon \int_0^{+\infty}\int_{\mathbb{R}} \sigma((s,y),(\mu,\lambda))V(f,g)((s,y),(\mu,\lambda)) \times \overline{V}(h,g)((s,y),(\mu,\lambda))\, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda).$$

Now, using Hölder's inequality and relation (2.3), we get

$$|V(f,g)((s,\ y),(\mu,\lambda)))\overline{V}(h,g)((s,y),(\mu,\lambda))| \leqslant \|f\|_{2,\nu_\alpha}\|\tau_{(s,y)}g\|^2_{2,\nu_\alpha}\|h\|_{2,\nu_\alpha} \leqslant \|f\|_{2,\nu_\alpha}\|h\|_{2,\nu_\alpha}. \tag{3.3}$$

From relations (3.2) and (3.3), we obtain

$$|\langle L_g(\sigma)(f),\ h\rangle_{2,\nu_\alpha}| \leqslant \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}\|f\|_{2,\nu_\alpha}\|h\|_{2,\nu_\alpha}.$$

Thus, $\|L_g(\sigma)\| \leqslant \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}$. □

Theorem 3.6. *Let $\sigma \in L^{+\infty}(d\nu_\alpha \otimes d\gamma_\alpha)$. Then,*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a bounded operator, and we have $\|L_g(\sigma)\| \leqslant \|\sigma\|_{\infty,\nu_\alpha\otimes\gamma_\alpha}$.

Proof. Let f and $h \in L^2(d\nu_\alpha)$. Using Hölder's inequality and relation (3.3), we obtain

$$\begin{aligned}
&|\langle L_g(\sigma)(f), h\rangle_{2,\nu_\alpha}| \\
&\leqslant \|\sigma\|_{\infty,\nu_\alpha\otimes\gamma_\alpha}\left(\iint_\Upsilon\int_0^{+\infty}\int_{\mathbb{R}}|V(f,g)((s,y),(\mu,\lambda))|^2 d\nu_\alpha(s,y)\gamma_\alpha(\mu,\lambda)\right)^{1/2} \\
&\quad\times\left(\iint_\Upsilon\int_0^{+\infty}\int_{\mathbb{R}}|\overline{V}(h,g)((s,y),(\mu,\lambda))|^2 d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda)\right)^{1/2} \\
&\leqslant \|\sigma\|_{\infty,\nu_\alpha\otimes\gamma_\alpha}\|f\|_{2,\nu_\alpha}\|h\|_{2,\nu_\alpha}.
\end{aligned}$$

This completes the proof. □

Now, by the Riesz-Thorin theorem **[29]**, Theorem 3.5 and Theorem 3.6, we obtain

Theorem 3.7. *Let* $\sigma \in L^p(d\nu_\alpha \otimes d\gamma_\alpha)$, $1 \leqslant p \leqslant +\infty$. *Then*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a bounded operator, and $\|L_g(\sigma)\| \leqslant \|\sigma\|_{p,\nu_\alpha \otimes \gamma_\alpha}$.

4. Compactness of the localization operator. In this section, we introduce the notation of the Shatten-von Neumann class S_p (for more details, see **[26]**), and we will use it to study the compactness of the localization operator.

Let T be a compact operator from a Hilbert space H into itself; then, the linear operator

$$(T^*T)^{1/2} : H \longrightarrow H$$

is positive and compact. Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H consisting of eigenvectors of $(T^*T)^{1/2}$, and let $s_k(T)$ be the eigenvalue corresponding to the eigenvector ψ_k. We say that the compact operator T is in the Shatten-von Neumann class S_p, if

$$\sum_k s_k(T)^p < +\infty.$$

The set of all bounded linear operators is denoted by S_∞ and the set of all compact operators by K. First, we recall two properties of S_1 and S_∞; for more details, see **[32,** Proposition 2.4].

Proposition 4.1. *Let*

$$T : H \longrightarrow H$$

be a bounded linear operator such that

$$\sum_{k=1}^{+\infty} |\langle T\psi_k,\ \psi_k\rangle| < +\infty,$$

for all orthonormal bases $(\psi_k)_{k\in\mathbb{N}}$ *for* H. *Then* T *is in* S_1.

Theorem 4.2. *Let* $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$. *Then the bounded linear operator*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_1 *and*

$$\|L_g(\sigma)\|_{S_1} \leqslant 4\|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}. \tag{4.1}$$

Proof. Let $(\psi)_{k\in\mathbb{N}}$ be any orthonormal basis for $L^2(d\nu_\alpha)$. Then, by Definition 3.4, Fubini's theorem, the Parseval identity and relation (2.3), we obtain

$$\begin{aligned}
&\sum_{k=1}^{+\infty} |\langle L_g(\sigma)(\psi_k), \psi_k\rangle_{2,\nu_\alpha}| \\
&\quad\leqslant \sum_{k=1}^{+\infty} \iint_\Upsilon \int_0^{+\infty}\int_{\mathbb{R}} |\sigma((s,y),(\mu,\lambda))||V(\psi_k,g)((s,y),(\mu,\lambda))| \\
&\quad\times |\overline{V}(\psi_k,g)((s,y),(\mu,\lambda))|\, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\quad= \iint_\Upsilon \int_0^{+\infty}\int_{\mathbb{R}} |\sigma((s,y),(\mu,\lambda))| \\
&\qquad+ \sum_{k=1}^{+\infty} |V(\psi_k,g)((s,y),(\mu,\lambda))|^2 d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\quad= \iint_\Upsilon \int_0^{+\infty}\int_{\mathbb{R}} |\sigma((s,y),(\mu,\lambda))|\|\varphi_{\mu,\lambda}\tau_{(s,y)}g\|_{2,\nu_\alpha}^2\, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\quad\leqslant \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}\|g\|_{2,\nu_\alpha} = \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha} < +\infty.
\end{aligned} \tag{4.2}$$

Hence, by Proposition 4.1,

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_1.

In order to prove estimate (4.1), first let $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$ be a nonnegative function. Then, $(L_g^*(\sigma)L_g(\sigma))^{1/2} = L_g(\sigma)$. Thus, if $(\psi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis for $L^2(d\nu_\alpha)$ consisting of eigenvalues of

$$(L_g^*(\sigma)L_g(\sigma))^{1/2} : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha),$$

using relation (4.2), we have

$$\|L_g(\sigma)\|_{S_1} = \sum_{k=1}^{+\infty} \langle (L_g^*(\sigma)L_g(\sigma))^{1/2}(\psi_k), \psi_k\rangle_{2,\nu_\alpha} \tag{4.3}$$

$$= \sum_{k=1}^{+\infty} \langle L_g(\sigma)(\psi_k), \psi_k \rangle_{2,\nu_\alpha} \leqslant \|\sigma\|_{1,\nu_\alpha \otimes \gamma_\alpha}.$$

Now, if $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$ is a real-valued function, then the result follows from the fact that $\sigma = \sigma_+ - \sigma_-$, where

$$\sigma_+ = \max(\sigma, 0), \qquad \sigma- = -\min(\sigma, 0).$$

In fact, by applying inequality (4.3), we get

$$\begin{aligned} \|L_g(\sigma)\|_{S_1} &= \|L_g(\sigma_+) - L_g(\sigma_-)\|_{S_1} \\ &\leqslant \|L_g(\sigma_+)\|_{S_1} + \|L_g(\sigma_-)\|_{S_1} \\ &\leqslant \|\sigma_+\|_{1,\nu_\alpha\otimes\gamma_\alpha} + \|\sigma_-\|_{1,\nu_\alpha\otimes\gamma_\alpha} \\ &\leqslant 2\|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}. \end{aligned} \tag{4.4}$$

Finally, if $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$ is a complex-valued function, then we write $\sigma = \sigma_1 + i\sigma_2$, where σ_1 and σ_2 are the real and imaginary parts of σ, respectively. Formula (4.4) then yields

$$\begin{aligned} \|L_g(\sigma)\|_{S_1} &= \|L_g(\sigma_1) + iL_g(\sigma_2)\|_{S_1} \\ &\leqslant \|L_g(\sigma_1)\|_{S_1} + \|L_g(\sigma_2)\|_{S_1} \\ &\leqslant 2(\|\sigma_1\|_{1,\nu_\alpha\otimes\gamma_\alpha} + \|\sigma_2\|_{1,\nu_\alpha\otimes\gamma_\alpha}) \\ &\leqslant 4\|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}. \end{aligned}$$

This completes the proof. □

Hereafter, we ameliorate the constant given in Theorem 4.2.

Theorem 4.3. *Let $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$. Then the bounded linear operator*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_1 and

$$\|L_g(\sigma)\|_{S_1} \leqslant \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}.$$

Proof. Since $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$, by Theorem 4.2, $L_g(\sigma)$ is in S_1. Using [**32,** Theorem 2.2], an orthonormal basis $\{\upsilon_k,\ k = 1, 2, \ldots\}$ exists for $N(L_g(\sigma))^\perp$, the orthogonal complement of the kernel of $L_g(\sigma)$, consisting of eigenvectors of $|L_g(\sigma)|$ and $\{\omega_k,\ k = 1, 2, \ldots\}$ an orthonormal set in $L^2(d\nu_\alpha)$, such that

$$L_g(\sigma)(f) = \sum_{k=1}^{+\infty} s_k \langle f \mid \upsilon_k\rangle_{\nu_\alpha} \omega_k,$$

where $s_k,\ k = 1, 2, \dots$ are the positive singular values of $L_g(\sigma)$ corresponding to υ_k. Then, we obtain

$$\|L_g(\sigma)\|_{S_1} = \sum_{k=1}^{+\infty} s_k = \sum_{k=1}^{+\infty} \langle L_g(\sigma)(\upsilon_k) \mid \omega_k\rangle_{\nu_\alpha}.$$

Thus, by Fubini's theorem, Schwartz's inequality and Bessel's inequality, we obtain

$$\begin{aligned}
\|L_g(\sigma)\|_{S_1} &= \sum_{k=1}^{+\infty} \langle L_g(\sigma)(\upsilon_k) \mid \omega_k\rangle_{\nu_\alpha} \\
&= \sum_{k=1}^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}} \iint_{\Upsilon} \sigma((s,y),(\mu,\lambda)) V(\upsilon_k, g)((s,y),(\mu,\lambda)) \\
&\quad \times \overline{V}(\omega_k, g)((s,y),(\mu,\lambda))\, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\leqslant \int_0^{+\infty} \int_{\mathbb{R}} \iint_{\Upsilon} |\sigma((s,y),(\mu,\lambda))| \left(\sum_{k=1}^{+\infty} |V(\upsilon_k,g)((s,y),(\mu,\lambda))|^2 \right)^{1/2} \\
&\quad \times \left(\sum_{k=1}^{+\infty} |V(\omega_k, g)((s,y),(\mu,\lambda))|^2 \right)^{1/2} d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\leqslant \int_0^{+\infty} \int_{\mathbb{R}} \iint_{\Upsilon} |\sigma((s,y),(\mu,\lambda))|\, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda) \\
&\leqslant \|\sigma\|_{1,\nu_\alpha \otimes \gamma_\alpha}.
\end{aligned}$$

□

Theorem 4.4. *Let $\sigma \in L^p(d\nu_\alpha \otimes d\gamma_\alpha)$, $1 \leqslant p < +\infty$. Then, the bounded linear operator*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is compact.

Proof. Let $\sigma \in L^p(d\nu_\alpha \otimes d\gamma_\alpha)$, $1 \leqslant p < +\infty$. Then, there exists a sequence

$$(\sigma_k)_k \in S_*(\mathbb{R}^2 \times \Upsilon)$$

such that $(\sigma_k)_k$ converges to σ in $L^p(d\nu_\alpha \otimes d\gamma_\alpha)$. It follows from Theorem 3.7 that

$$\|L_g(\sigma_k) - L_g(\sigma)\|_{S_1} \leqslant \|\sigma_k - \sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha} \longrightarrow 0,$$

when $k \to +\infty$. This means that

$$L_g(\sigma_k) \longrightarrow L_g(\sigma) \quad \text{in } \mathfrak{B}(L^2(d\nu_\alpha)) \text{ as } k \to +\infty.$$

From Theorem 4.2,

$$L_g(\sigma_k) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a linear operator in S_1, and hence, compact. It follows that

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is compact. □

Now, we show the Shatten-von Neumann property.

Theorem 4.5. *Let $\sigma \in L^p(d\nu_\alpha \otimes d\gamma_\alpha)$, $1 \leqslant p < +\infty$. Then, the bounded linear operator*

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_p and $\|L_g(\sigma)\|_{S_p} \leqslant \|\sigma\|_{p,\nu_\alpha\otimes\gamma_\alpha}$.

Proof. From Theorem 4.3, for $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$, we have

$$\|L_g(\sigma)\|_{S_1} \leq \|\sigma\|_{1,\nu_\alpha\otimes\gamma_\alpha}, \tag{4.5}$$

and, from Theorem 3.6, we obtain

$$\|L_g(\sigma)\|_{S_\infty} = \|L_g(\sigma)\|_{*,L^2(d\nu_\alpha)} \leq \|\sigma\|_{\infty,\nu_\alpha\otimes\gamma_\alpha}. \tag{4.6}$$

By using Theorem 2.2.6 and Theorem 2.2.7 from [**33,** Chapter 2],

$$L^p(d\nu_\alpha \otimes d\gamma_\alpha) = [L^1(d\nu_\alpha \otimes d\gamma_\alpha),\, L^{+\infty}(d\nu_\alpha \otimes d\gamma_\alpha)]_{1/p'},$$
$$S_p = [S_1, S_\infty]_{1/p'},$$

where p' is the conjugate index of p. Thus, by relations (4.5) and (4.6) and the inequality from [**33,** Proof of Theorem 2.2.4], the proof is complete. □

Suppose now that $\sigma \in L^p(d\nu_\alpha \otimes d\gamma_\alpha)$ is a nonnegative function. Then,

$$L_g(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a positive and compact operator. Let $(\psi_k)_k$ be an orthonormal basis for $L^2(d\nu_\alpha)$ consisting of eigenvectors of $L_g(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ and λ_k the eigenvalue corresponding to the eigenvector ψ_k, $k = 1, 2, \ldots$. From Theorem 4.2, $L_g(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is in S_1. Thus,

$$\sum_{k=1}^{+\infty} \lambda_k < \infty.$$

In fact, we give an explicit formula for

$$\sum_{k=1}^{+\infty} \lambda_k.$$

For a positive operator

$$T : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

with a pure spectrum in which the eigenvalues are counted with multiplicities, given by $\{\lambda_k;\ k = 1, 2, \ldots\}$, the trace of T is defined by

$$\mathrm{Tr}(T) = \sum_{k=1}^{+\infty} \lambda_k.$$

Theorem 4.6. *Let $\sigma \in L^1(d\nu_\alpha \otimes d\gamma_\alpha)$ be a nonnegative function. Then,*

$$\mathrm{Tr}(L_g(\sigma)) = \iint_\Upsilon \int_0^{+\infty} \int_{\mathbb{R}} \sigma((s,y),(\mu,\lambda)) \|\varphi_{\mu,\lambda}\tau_{(s,y)}g\|_{2,\nu_\alpha}^2 \, d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda).$$

Proof. Since $\lambda_k = \langle L_g(\sigma)\psi_k, \psi_k\rangle_{2,\nu_\alpha}$,

$$\lambda_k = \iint_\Upsilon \int_0^{+\infty} \int_{\mathbb{R}} \sigma((s,y),(\mu,\lambda)) |V(\psi_k,g)((s,y),(\mu,\lambda))|^2 d\nu_\alpha(s,y)\, d\gamma_\alpha(\mu,\lambda).$$

Hence, by Fubini's theorem, we obtain

$$\mathrm{Tr}(L_g(\sigma)) = \sum_{k=1}^{+\infty} \lambda_k = \iint_\Upsilon \int_0^{+\infty} \int_{\mathbb{R}} \sigma((s,y),(\mu,\lambda))$$

$$+\sum_{k=1}^{+\infty}|V(\psi_k,g)((s,y),(\mu,\lambda))|^2 d\nu_\alpha(s,y)\,d\gamma_\alpha(\mu,\lambda)$$

$$=\iint_{\Upsilon}\int_0^{+\infty}\int_{\mathbb{R}}\sigma((s,y),(\mu,\lambda))\|\varphi_{\mu,\lambda}\tau_{(s,y)}g\|_{2,\nu_\alpha}^2 d\nu_\alpha(s,y)d\gamma_\alpha(\mu,\lambda).$$

This completes the proof. □

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