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A HIERARCHY FOR CLOSED n-CELL COMPLEMENTS

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ABSTRACT. Let C and D be a pair of crumpled n-cubes and h a homeomorphism of $\mathrm{Bd}\, C$ to $\mathrm{Bd}\, D$ for which there exists a map $f_h : C \to D$ such that $f_h \mid \mathrm{Bd}\, C = h$ and $f_h^{-1}(\mathrm{Bd}\, D) = \mathrm{Bd}\, C$. In our view, the presence of such a triple (C, D, h) suggests that C is "at least as wild as" D. The collection $\mathscr{W}_n$ of all such triples is the subject of this paper. If $(C, D, h) \in \mathscr{W}_n$, but there is no homeomorphism such that D is at least as wild as C, we say that C is "strictly wilder than" D. The latter concept imposes a partial order on the collection of crumpled n-cubes. Here, we study features of these wildness comparisons, and we present certain attributes of crumpled cubes that are preserved by the maps arising when $(C, D, h) \in \mathscr{W}_n$. This effort may be viewed as an initial way of classifying the wildness of crumpled cubes.

1. Introduction. The existence of wildly embedded spheres in the n-sphere S^n has been recognized since the 1920s, with the publication of the famous Alexander horned sphere **[1]** and a related 2-sphere wildly embedded in S^3 presented by Antoine **[2]**. Later, in the 20th century, there was an extensive study of conditions under which an $(n-1)$-sphere in S^n is locally flat, and hence, standardly embedded. Little has been done, however, to classify or organize the rich variety of wildly embedded objects. This paper strives to initiate that organizational effort.

To that end, we consider triples (C, D, h) consisting of a pair of crumpled n-cubes C and D and a homeomorphism h from the boundary of the first to the boundary of the second, and we name the subcollection

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$\mathscr{W}_n$ consisting of all such triples (C, D, h) for which there exists a map $f_h : C \to D$ extending h such that $f_h^{-1}(\mathrm{Bd}\, D) = \mathrm{Bd}\, C$. We call f_h a map associated with the triple. Given $(C, D, h) \in \mathscr{W}_n$, we think of C as being at least as wild as D. Of course, this wildness measure $\mathscr{W}_n$ heavily depends upon the homeomorphism h; thus, we regard C as being at least as wild as D, provided there is some homeomorphism h for which $(C, D, h) \in \mathscr{W}_n$.

Several results established here offer justification for this measure as a rating of wildness. For instance, when $(C, D, h) \in \mathscr{W}_n$, f_h must induce an epimorphism

$$\pi_1(\mathrm{Int}\, C) \longrightarrow \pi_1(\mathrm{Int}\, D).$$

Any homotopy taming set for C must be sent to a homotopy taming set for D; as a consequence, the wild set of D, that is, the set of points at which $\mathrm{Bd}\, D$ fails to be locally collared in D, must lie in the image under h of the wild set of C.

Given C, we describe a standard flattening away from a closed subset X of $\mathrm{Bd}\, C$ that produces a new crumpled n-cube C_X and a homeomorphism

$$h_X : \mathrm{Bd}\, C \longrightarrow \mathrm{Bd}\, C_X$$

such that $(C, C_X, h_X) \in \mathscr{W}_n$ and $\mathrm{Bd}\, C_X$ is locally flat in C_X at all points of $h_X(\mathrm{Bd}\, C - X) = h_X(\mathrm{Bd}\, C) - X$. Moreover, when X is the closure of an open subset of the wild set for C, then $h_X(X) = X$ equals the wild set for C_X. The standard flattening technique furnishes an efficient method for presenting unusual examples.

We also introduce a notion of "strictly wilder than," stating that a crumpled n-cube C is strictly wilder than another crumpled cube D if there exists a homeomorphism h such that $(C, D, h) \in \mathscr{W}_n$ but there is no homeomorphism

$$H : \mathrm{Bd}\, D \longrightarrow \mathrm{Bd}\, C$$

such that $(D, C, H) \in \mathscr{W}_n$. We study this partial order briefly in Section 5. If the crumpled n-cube C contains a spot at which its boundary is locally flat, or if $\mathrm{Bd}\, C$ has a finitely generated fundamental group, we show that C cannot be a maximal element in this partial order. We suspect that there are no maximal elements whatsoever but have been unable to confirm the suspicion. The preservation of "at least as wild as" under different operations, such as suspension and spin, is discussed in Section 4. In Section 6, the sewing space of crumpled cubes is shown to have some nice properties whenever the "at least as wild as" condition prevails.

Maps such as f_h have been used by Wang [**23**] and others to impose a partial order on knots in S^3.

2. Definitions and notation. The symbol Cl A is used to denote the closure of A; the boundary and interior of A are denoted as $\mathrm{Bd}\, A$ and $\mathrm{Int}\, A$; the symbol $\mathbb{1}$ is the identity map.

Definition 2.1. A *crumpled n-cube C* is a space homeomorphic to the union of an $(n-1)$-sphere Σ in S^n and one of its complementary domains. The sphere Σ is the *boundary* of C, written $\mathrm{Bd}\, C$, and $C - \Sigma$ is the *interior* of C, written $\mathrm{Int}\, C$.

Definition 2.2. A *closed n-cell-complement* is a crumpled n-cube C embedded in S^n so that $S^n - \mathrm{Int}\, C$ is an n-cell.

Every crumpled n-cube admits such an embedding [**7, 10, 18, 20**]. This concept arises since, when dealing with the possible wildness in S^n of a compact subset of $\mathrm{Bd}\, C$, it is useful to treat C as a closed n-cell complement in order to preclude any wildness complications arising from the other crumpled cube $S^n - \mathrm{Int}\, C$.

Definition 2.3. A subset T of the boundary of a crumpled cube C is a *homotopy taming set* for C if every map

$$m : I^2 \longrightarrow C$$

can be approximated by a map

$$m' : I^2 \longrightarrow C$$

such that $m'(I^2) \subset T \cup \mathrm{Int}\, C$.

Every crumpled n-cube has a one-dimensional homotopy taming set [**6**]. All crumpled 3-cubes have zero-dimensional homotopy taming sets; it is unknown whether the same is true for crumpled n-cubes when $n > 3$.

Crumpled cubes with particularly nice homotopy taming sets are referred to as follows:

Definition 2.4. A crumpled n-cube C is *Type* 1 if there exists a zero-dimensional homotopy taming set T in $\mathrm{Bd}\, C$ such that T is a countable union of Cantor sets that are tame relative to $\mathrm{Bd}\, C$.

Definition 2.5. The *inflation* of a crumpled n-cube C is

$$\mathrm{Infl}\,(C,d) = \{\langle c,t\rangle \ \in C \times \mathbb{R}^1 \mid c \in C \text{ and } |t| \leq d(c)\},$$

where $d : C \to [0,1]$ is a map such that $d^{-1}(0) = \mathrm{Bd}\, C$ [**9,** page 270]. Neither the homeomorphism type nor the embedding type of $\mathrm{Infl}\,(C,d)$ depends upon the choice of map d, so ordinarily we suppress reference to d.

Definition 2.6. Let C be a crumpled n-cube. A point $p \in \mathrm{Bd}\, C$ is a *piercing point* of C if there exists an embedding ξ of C in the n-sphere S^n such that $\xi(\mathrm{Bd}\, C)$ can be pierced with a tame arc at $\xi(p)$.

All boundary points of crumpled n-cubes are piercing points when $n > 3$. McMillan [**22**] has shown that boundary points p of crumpled 3-cubes C are piercing points of C if and only if $C-p$ is locally, simply connected at p.

Definition 2.7. A proper map

$$f : M \longrightarrow \widetilde{M}$$

between connected, orientable n-manifolds has *degree* 1 if f induces an isomorphism of (cohomology groups with compact supports)

$$H^n_c(\widetilde{M}) \longrightarrow H^n_c(M).$$

3. Some basic properties for the collection $\mathscr{W}_n$. The fundamental aim here is the attempt to measure or compare the wildness of two given crumpled n-cubes using $\mathscr{W}_n$. The obvious, but basic, features worth noting are:

(1) $(C, C, \mathbb{1}) \in \mathscr{W}_n$,
(2) (C, C', h) and (C', C'', h') in $\mathscr{W}_n$ implies that $(C, C'', h'h)$ is in $\mathscr{W}_n$.

Loosely speaking, we think of C as being wilder than D if there exists a $(C, D, h) \in \mathscr{W}_n$. This comparison, however, raises the following question: for (C, D, h) in $\mathscr{W}_n$, does there ever exist a $(D, C, H) \in \mathscr{W}_n$? The basic fact that $(C, C, \mathbb{1}) \in \mathscr{W}_n$ supplies an affirmative answer indicates that this "wilder than" language is misleading. Accordingly, we phrase the concept more conservatively as follows.

Definition 3.1. C is *at least as wild as* D if and only if there exists a homeomorphism h such that $(C, D, h) \in \mathscr{W}_n$.

Theorem 3.2. *For every crumpled n-cube C and every homeomorphism h from* $\operatorname{Bd} C$ *to* $\operatorname{Bd} B^n$*, $(C, B^n, h) \in \mathscr{W}_n$.*

Proof. Since B^n is an absolute retract, the homeomorphism h extends to a map

$$f_h : C \longrightarrow B^n.$$

Think of B^n as the unit ball. Restrict the metric on C so that C has diameter ≤ 1. Treat $f_h(x)$ as a vector from the origin through the image point $f_h(x)$. Modify f_h by sending any x in C to the vector

$$(1 - \operatorname{dist}(x, \operatorname{Bd} C)) \cdot f_h(x),$$

i.e., the scalar product. Now,

$$f_h^{-1}(\operatorname{Bd} B^n) = \operatorname{Bd} C. \qquad \square$$

Theorem 3.3. *Suppose C and D are crumpled n-cubes such that* $\operatorname{Bd} C$ *and* $\operatorname{Bd} D$ *have closed neighborhoods U_C and U_D in C and D, respectively, that are homeomorphic via*

$$H : U_C \longrightarrow U_D,$$

and suppose that $\pi_1(\operatorname{Int} D) = 1$*. Then,*

$$(C, D, H \mid \operatorname{Bd} C) \in \mathscr{W}_n.$$

Proof. Here, $\operatorname{Int} D$ is homologically and homotopically trivial, implying that H extends to a map

$$f_H : C \longrightarrow D \quad \text{with} \quad f_H(C - U_C) \subset \operatorname{Int} D.$$

Clearly, f_H ensures that $(C, D, H \mid \operatorname{Bd} C) \in \mathscr{W}_n$. $\square$

Next, we outline an example showing the existence of $(C, C, h) \in \mathscr{W}_n$ where no associated map $f_h : C \to C$ can be a homeomorphism. It is meant to suggest that a reflexivity aspect of the "at least as wild as" relation occasionally holds for complicated reasons. In the next section, we will present an example showing that the relation actually fails to be asymmetric, and hence, does not determine a partial order on the collection of crumpled n-cubes.

Example 3.4. A triple $(C, C, h) \in \mathscr{W}_3$ is such that every associated map

$$f_h : C \longrightarrow C$$

induces a homomorphism

$$\pi_1(\operatorname{Int} C) \longrightarrow \pi_1(\operatorname{Int} C)$$

with nontrivial kernel. Consider the closed 3-cell complement C bounded by Alexander's horned sphere. Wipe out the wildness in one of the two primary horns, but leave the other horn unchanged. The result is a new, crumpled 3-cube D, simpler than the original in that some of the wildness has been eliminated, nevertheless (by inspection) embedded exactly like the original. We continue to differentiate the two using different names, despite the fact that C and D are equivalent. There is a homeomorphism h from $\operatorname{Bd} C$ to $\operatorname{Bd} D$, sending one of the primary horns of C onto the wild horn of D (the entire wild part of D) and sending the other primary horn of C into the flattened part of D. It takes little effort to show that h extends to the appropriate kind of map from C to D; however, that is quite similar to showing that $(C, B^n, h) \in \mathscr{W}_n$. Let

$$J \subset \operatorname{Bd} C$$

be a simple closed curve separating the two horns of C, and note that any associated map f_h must send loops in $\operatorname{Int} C$ near J to homotopically nonessential loops in $\operatorname{Int} D$.

Lemma 3.5. *Suppose that* $(C, D, h) \in \mathscr{W}_n$*,*

$$f_h : C \longrightarrow D$$

is a map associated with h*,* W *is a connected open subset of* D *such that*

$W \cap \operatorname{Bd} D$ *is connected and* Y *is the component of* $f_h^{-1}(W)$ *containing*

$$f_h^{-1}(W \cap \operatorname{Bd} D) = h^{-1}(W \cap \operatorname{Bd} D).$$

Then, f_h *induces an epimorphism*

$$\pi_1(Y \cap \operatorname{Int} C) \longrightarrow \pi_1(W \cap \operatorname{Int} D).$$

Proof. Treat C and D as closed n-cell complements. Extend f_h to a proper map

$$F_h : S^n \longrightarrow S^n,$$

which restricts to a homeomorphism between $S^n - \operatorname{Int} C$ and $S^n - \operatorname{Int} D$. Since it is a homeomorphism over some open subset, F_h must have geometric degree 1. Then, by **[15]**,

$$f_h \mid Y \cap \operatorname{Int} C = F_h \mid Y \cap \operatorname{Int} C : Y \cap \operatorname{Int} C \longrightarrow W \cap \operatorname{Int} D$$

also has degree 1, which implies that it induces an epimorphism of fundamental groups. □

Corollary 3.6. *If* $(C, D, h) \in \mathscr{W}_n$*, and*

$$f_h : C \longrightarrow D$$

is an associated map extending h *with*

$$f_h(\operatorname{Int} C) \subset \operatorname{Int} D,$$

then f_h *induces an epimorphism of* $\pi_1(\operatorname{Int} C)$ *to* $\pi_1(\operatorname{Int} D)$.

Proof. Apply Lemma 3.5 with $W = D$ and $Y = C$. □

Corollary 3.7. *If* $(C, D, h) \in \mathscr{W}_n$ *and* $\operatorname{Int} C$ *is an open* n*-cell, then* $\operatorname{Int} D$ *is an open* n*-cell.*

Proof. According to **[21]**, $\operatorname{Int} D$ is an open n-cell if (and only if) $\operatorname{Int} D$ is simply connected at ∞; in other words, given one neighborhood U of $\operatorname{Bd} D$ in D, a smaller neighborhood V must be produced there of $\operatorname{Bd} D$ such that each loop in $V \cap \operatorname{Int} D$ is null-homotopic in $U \cap \operatorname{Int} D$. Pull back to C. First, find a connected neighborhood V' of $\operatorname{Bd} C$ in C such that not only is $V' \subset f_h^{-1}(U)$ but also all loops in $V' \cap \operatorname{Int} C$ are null-homotopic in $f_h^{-1}(U) \cap \operatorname{Int} C$. Next, locate a connected neighborhood V

of $\mathrm{Bd}\, D$ in D with $f_h^{-1}(V) \subset V'$. From Lemma 3.5, every loop γ in $V \cap \mathrm{Int}\, D$ is homotopic to the image of a loop γ' in

$$f_h^{-1}(V) \cap \mathrm{Int}\, C \subset V' \cap \mathrm{Int}\, C,$$

and γ' in turn is null-homotopic in

$$f_h^{-1}(U) \cap \mathrm{Int}\, C.$$

Finally, apply f_h to see that γ itself is null-homotopic in $U \cap \mathrm{Int}\, D$. $\square$

Theorem 3.8. *If $(C, D, h) \in \mathscr{W}_n$ and T is a homotopy taming set for C, then $h(T)$ is a homotopy taming set for D.*

Proof. Consider any map $\phi : I^2 \to D$ and $\epsilon > 0$. From Lemma 3.5, for any $x \in \mathrm{Bd}\, D$, there exist a small connected neighborhood N_x of $x \in D$ and a small connected neighborhood M_x of $h^{-1}(x)$ in $f_h^{-1}(N_x)$ such that f_h induces an epimorphism of

$$\pi_1(M_x \cap \mathrm{Int}\, C) \longrightarrow \pi_1(N_x \cap \mathrm{Int}\, D).$$

Perform this such that every loop in an M_x contracts in a subset of C, whose image under f_h is an $(\epsilon/2)$-subset of D. Note that, if L is a loop in $M_x \cap \mathrm{Int}\, C$, then its image under f_h contracts in an $(\epsilon/2)$-subset of $h(T) \cup \mathrm{Int}\, D$. Produce a (Lebesgue number) $\delta \in (0, \epsilon/2)$ such that any δ-subset of D within δ of $\mathrm{Bd}\, D$ lies in some N_x.

We define a new map

$$\phi' : I^2 \longrightarrow D$$

such that ϕ' is ϵ-close to ϕ and

$$\phi'(I^2) \subset h(T) \cup \mathrm{Int}\, D.$$

First, impose a triangulation $\mathscr{T}$ of I^2 with a mesh so fine that the diameter of each $\phi(\Delta)$, where Δ denotes a 2-simplex of $\mathscr{T}$, is less than δ. Next, approximate ϕ by a map (still called ϕ) such that the image of the 1-skeleton of $\mathscr{T}$ avoids $\mathrm{Bd}\, D$; this may be accomplished without affecting any of the size controls achieved to this point. Then, note that, for those 2-simplices $\Delta \in \mathscr{T}$ such that $\phi(\Delta)$ meets $\mathrm{Bd}\, D$, $\phi(\partial\Delta)$ is homotopic in some N_x to the image under f_h of a loop $L \subset M_x$, and $f_h \mid L$ bounds a singular disk in an $(\epsilon/2)$-subset of $h(T) \cup \mathrm{Int}\, D$. Of course, ϕ' and ϕ agree on those $\Delta \in \mathscr{T}$ such that $\phi(\Delta) \cap \mathrm{Bd}\, D = \emptyset$. $\square$

The following corollaries apply when $(C, D, h) \in \mathscr{W}_n$.

Corollary 3.9. *If C is Type 1, so is D.*

Corollary 3.10. *If C has the disjoint disks property, so does D.*

Corollary 3.11. *If C has a zero-dimensional homotopy taming set, so does D.*

Remark 3.12. If T is any homotopy taming set for the crumpled cube C, then the wild set of C is contained in the closure of T. Essentially, by definition of the homotopy taming set, $\operatorname{Int} C$ is locally simply connected at all points of $\operatorname{Bd} C - T$, which ensures local flatness there [**3**], [**12,** subsection 7.6], [**16**].

Corollary 3.13. *Suppose that W_C and W_D are the wild sets in* $\operatorname{Bd} C$ *and* $\operatorname{Bd} D$*, respectively. Then, $W_D \subset h(W_C)$.*

Proof. Take any homotopy taming set T for C. Note that singular disks in C may be adjusted, fixing points that are sent to W_C while moving the image off of $\operatorname{Bd} C - W_C$, i.e., $T \cap W_C$ is another homotopy taming set for C. Then, since $h(T \cap W_C)$ is a homotopy taming set for D, we have

$$W_D \subset \operatorname{Cl} \mathrm{h}(T \cap W_C) \subset \operatorname{Cl} \mathrm{h}(T) \cap h(W_C) \subset h(W_C). \qquad \square$$

Remark 3.14. Even when C and D are locally flat modulo wild sets W_C and W_D that are tame in space, and there is a homeomorphism

$$h : \operatorname{Bd} C \longrightarrow \operatorname{Bd} D,$$

with $h(W_C) = W_D$, we cannot infer that C is at least as wild as D. In order to see why not, we consider a pair of different crumpled 3-cubes, each of which is locally flat modulo two points. The first might have simply connected interior and the second might be non-simply connected. For higher-dimensional cases, spins or suspensions can be applied to obtain different crumpled n-cubes, each locally flat modulo an $(n-3)$-cell or a pair of $(n-3)$-spheres that are tame in S^n.

Corollary 3.15. *If (C, D, h) and (D, C, h^{-1}) both belong to $\mathscr{W}_n$, then h sends the wild set of C onto the wild set of D.*

Theorem 3.16. *Suppose that C and D are closed n-cell complements, $(C, D, h) \in \mathscr{W}_n$, X is a compact subset of* $\operatorname{Bd} C$ *with* $\dim X < n-2$ *and X is* 1-LCC *embedded in S^n. Then, $h(X)$ is* 1-LCC *embedded in S^n.*

Proof. Extend

$$f_h : C \longrightarrow D$$

to a map

$$F_h : S^n \longrightarrow S^n,$$

with

$$F_h \mid S^n - C : S^n - C \longrightarrow S^n - D$$

a homeomorphism.

Let V be a neighborhood of $h(x) \in h(X)$. Find a smaller connected neighborhood W of $h(x)$ with $W \cap \operatorname{Bd} D$ connected such that every loop in $F_h^{-1}(W)$ contracts in $F_h^{-1}(V)$. Let Y denote the component of $F_h^{-1}(W)$ containing x. Note that, due to the dimension restriction, X does not separate Y. From the same argument as that for Lemma 3.5, F_h induces an epimorphism

$$\pi_1(Y - X) \longrightarrow \pi_1(W - h(X)).$$

Consider any loop α in $W - h(X)$. It is the image of a loop α' from $Y - X$. By design, α' is null-homotopic in $F_h^{-1}(V)$; even better, since X is 1-LCC embedded, α' is null-homotopic in $F_h^{-1}(V) - X$. Application of F_h demonstrates that α is null-homotopic in $V - h(X)$. □

Corollary 3.17. *If C and D are closed n-cell complements, $(C, D, h) \in \mathscr{W}_n$ and* $\operatorname{Bd} C$ *is locally flat modulo a Cantor set tamely embedded in S^n, $n \geq 5$, then* $\operatorname{Bd} D$ *is locally flat modulo a Cantor set tamely embedded in S^n.*

Corollary 3.18. *Let C and D be closed n-cell complements. If $(C, D, h) \in \mathscr{W}_n$ and* $\operatorname{Bd} C$ *is locally flat modulo a codimension* 3 *subset*

W_C of $\operatorname{Bd} C$ *that is embedded in space as a tame polyhedron, then D is also locally flat modulo a tame subset.*

Corollary 3.19. *No closed n-cell complement that is locally flat modulo a tame subset of codimension* 3 *or greater can be at least as wild as that which is locally flat modulo a wild set.*

Theorem 3.20. *If $(C, D, h) \in \mathscr{W}_3$ and p is a piercing point of C, then $h(p)$ is a piercing point of D.*

Proof. By [**22**], a point x in the boundary of a crumpled cube C is a piercing point if and only if C has a homotopy taming set T such that $x \notin T$. Consequently, the existence of such a T with $p \notin T$ implies that $h(p) \notin h(T)$, which in turn, implies that $h(p)$ is a piercing point of D. □

Definition 3.21. The boundary Σ of a crumpled n-cube C can be *carefully almost approximated* from $\operatorname{Int} C$ provided that, for each $\epsilon > 0$, there exists a locally flat embedding θ of Σ in S^n within ϵ of the inclusion

$$\Sigma \longrightarrow S^n$$

such that each component of $\theta(\Sigma) - \operatorname{Int} C$ has diameter $< \epsilon$ and $\Sigma \cap \theta(\Sigma)$ is covered by the interiors of a finite collection of pairwise disjoint $(n-1)$-cells in Σ, each of diameter $< \epsilon$.

Theorem 3.22. *Suppose that C and D are crumpled n-cubes,* $\operatorname{Bd} C$ *can be carefully almost approximated from* $\operatorname{Int} C$*, and $(C, D, h) \in \mathscr{W}_n$. Suppose also that*

$$m : B^2 \longrightarrow \operatorname{Bd} D$$

is a map and δ is a positive number. Then, there exists a map

$$m' : B^2 \longrightarrow D$$

such that

$$\rho(m', m) < \delta, m' \mid \partial B^2 = m \mid \partial B^2$$

and

$$m'(B^2) \cap \operatorname{Bd} D \subset N_\delta(m(\partial B^2)).$$

Proof. From [**8,** Lemma 5.2], the same conclusion holds for the map

$$h^{-1}m : B^2 \longrightarrow \mathrm{Bd}\, C.$$

The properties in C readily transfer to D via f_h. □

4. Preservation of wildness comparisons under certain operations. In this section, we shall show the "at least as wild as" property is preserved under suspension, rounded product and spin operations, but is not preserved under the inflation operation.

Theorem 4.1. *If $(C, D, h) \in \mathscr{W}_n$, then so is $(\Sigma(C), \Sigma(D), \Sigma(h))$, where Σ denotes the suspension operator.*

Proof. The proof is elementary: suspend an associated map f_h. □

The next example shows that the inflation operator does not preserve the "at least as wild as" property.

Example 4.2. If C is at least as wild as D and Infl (C) is a crumpled cube (equivalently for $n > 4$, C has the disjoint disks property), then Infl (C) may not be at least as wild as Infl (D). Suppose that D is a crumpled cube, the boundary of which is everywhere wild, and its inflation is also a crumpled cube. Then, the only crumpled cube C is D itself for which Infl (C) is at least as wild as Infl (D). If

$$H : \mathrm{Bd}\,\mathrm{Infl}\,(\mathrm{C}) \longrightarrow \mathrm{Bd}\,\mathrm{Infl}\,(\mathrm{D})$$

is a homeomorphism, H must send the wild set of Infl (C) to cover the wild set of Infl (D), that is,

$$H(\mathrm{Bd}\, C \times \{0\}) \supset \mathrm{Bd}\, D \times \{0\}.$$

Since no proper subset of an $(n-1)$-sphere can cover another $(n-1)$-sphere, it follows that

$$H(\mathrm{Bd}\, C \times \{0\}) = \mathrm{Bd}\, D \times \{0\}.$$

As a result, H must send either of the obvious copies of C in the boundary of the first inflation onto a copy of D in the second.

Definition 4.3. Given a crumpled n-cube C, we define its *rounded product* by I, denoted $\mathrm{Round}(C \times I)$, as the crumpled $(n+1)$-cube in

$\mathbb{R}^{n+1}$ bounded by $\lambda(\mathrm{Bd}\,(C \times I))$, where λ is an embedding that agrees with inclusion on $(\mathrm{Bd}\,C)\times[1/3,\,2/3]$, is locally flat elsewhere, and where the image of λ misses $\mathrm{Int}\,C \times [1/3,\,2/3]$. Equivalently, $\mathrm{Round}(C \times I)$ is obtained by attaching $(n+1)$-cells B_+ and B_- to $C \times [1/3,\,2/3]$ along $C \times \{2/3\}$ and $C \times \{1/3\}$, respectively, with the requirement that both $\mathrm{Bd}\,\mathrm{B}_+ - (\mathrm{Int}\,C \times \{2/3\})$ and $\mathrm{Bd}\,B_- - (\mathrm{Int}\,C \times \{1/3\})$ be n-cells.

Theorem 4.4. *If $(C, D, h) \in \mathscr{W}_n$, then*

$$(\mathrm{Round}(C \times I), \mathrm{Round}(D \times I), \mathrm{Round}(h \times \mathbb{1})) \in \mathscr{W}_{n+1},$$

where $\mathrm{Round}(h \times \mathbb{1})$ *denotes any homeomorphisms between the boundaries that extend*

$$h \times \mathbb{1} : \mathrm{Bd}\,C \times \left[\frac{1}{3}, \frac{2}{3}\right] \longrightarrow \mathrm{Bd}\,D \times \left[\frac{1}{3}, \frac{2}{3}\right].$$

Proof. Since $(C, D, h) \in \mathscr{W}_n$, we have a typical associated map

$$f_h : C \longrightarrow D$$

extending h. Define

$$F : \mathrm{Round}(C \times I) \longrightarrow \mathrm{Round}(D \times I)$$

as

$$f_h \times \mathbb{1} : C \times \left[\frac{1}{3}, \frac{2}{3}\right] \longrightarrow D \times \left[\frac{1}{3}, \frac{2}{3}\right];$$

extend to the $(n+1)$-cells B_+ and β_+ attached along $C \times \{2/3\}$ and $D \times \{2/3\}$, respectively, so that no point of $\mathrm{Int}\,B_+$ is sent to a boundary point of β_+, and do the same for the $(n+1)$-cells attached at the $1/3$-levels. □

Here, we introduce a method for spinning a crumpled n-cube C that sometimes produces a crumpled $(n+k)$-cube. It is closely related to the method of spinning a decomposition described in [**9,** Section 28], and we will use the results from that section. The procedure depends upon the choice of an $(n-1)$-cell β in $\mathrm{Bd}\,C$. For simplicity, we tolerate using only those cells β that are standardly embedded in $\mathrm{Bd}\,C$. For $k > 0$, the *k-spin* of C relative to β is the decomposition space $\mathrm{Sp}^k(C, \beta) = C \times S^k/\mathscr{G}_\beta$, where $\mathscr{G}_\beta$ is the decomposition whose nondegenerate elements are $\{c \times S^k \mid c \in \beta\}$. This is a generalized

$(n+k)$-manifold with a boundary, and its boundary is the image of $(\mathrm{Bd}\, C - \mathrm{Int}\, \beta) \times S^k$, the k-spin of the $(n-1)$-cell $\mathrm{Bd}\, C - \mathrm{Int}\, \beta$, which is an $(n+k-1)$-sphere. As a result, $\mathrm{Sp}^k(C, \beta)$ is a crumpled $(n+k)$-cube if and only if it embeds in S^{n+k}.

Given crumpled n-cubes C and D plus a homeomorphism h of $\mathrm{Bd}\, C$ to $\mathrm{Bd}\, D$, a naturally defined homeomorphism $\mathrm{Sp}^k(h)$ is derived between the boundaries of certain k-spins. Specifically, let

$$q_C : (\mathrm{Bd}\, C - \mathrm{Int}\, \beta) \times S^k \longrightarrow \mathrm{Bd}\, \mathrm{Sp}^k(C, \beta)$$

and

$$q_D : (\mathrm{Bd}\, D - \mathrm{Int}\, h(\beta)) \times S^k \longrightarrow \mathrm{Bd}\, \mathrm{Sp}^k(D, h(\beta))$$

denote the decomposition maps, appropriately restricted. Define

$$\mathrm{Sp}^k(h) : \mathrm{Bd}\, \mathrm{Sp}^k(C, \beta) \longrightarrow \mathrm{Bd}\, \mathrm{Sp}^k(D, h(\beta))$$

as $q_D(h \times \mathbb{1})(q_C)^{-1}$, where

$$h \times \mathbb{1} : (\mathrm{Bd}\, C - \mathrm{Int}\, \beta) \times S^k \longrightarrow (\mathrm{Bd}\, D - \mathrm{Int}\, h(\beta)) \times S^k.$$

There is another effective method of studying $\mathrm{Sp}^k(C, \beta)$. Attach an exterior collar $\lambda(\mathrm{Bd}\, C \times [0,1])$ to C, with $\lambda(c, 0) = c$ for all $c \in \mathrm{Bd}\, C$. The union of C and the collar is an n-cell B^n. Let G be the decomposition of B^n consisting of points and the arcs $\lambda(c \times [0,1])$, $c \in \beta$. (Admissibility is satisfied.) The k-spin of B^n is topologically S^{n+k}, and G gives rise to a cell-like decomposition $\mathrm{Sp}^k(G)$. The decomposition space $\mathrm{Sp}^k(B^n)/\mathrm{Sp}^k(G)$ contains $\mathrm{Sp}^k(C, \beta)$. Hence, if $\mathrm{Sp}^k(B^n)/\mathrm{Sp}^k(G)$ is the $(n+k)$-sphere, then $\mathrm{Sp}^k(C, \beta)$, bounded by an $(n+k-1)$-sphere, must be a crumpled $(n+k)$-cube. According to **[9,** Theorem 28.9], when $n+k \geq 5$, $\mathrm{Sp}^k(B^n)/\mathrm{Sp}^k(G)$ is topologically S^{n+k} if and only if every pair of maps

$$\mu_1, \mu_2 : I^2 \longrightarrow B^n/G$$

can be arbitrarily closely approximated by the maps

$$\mu_1', \mu_2' : I^2 \longrightarrow B^n/G$$

such that

$$(*) \qquad \mu_1'(I^2) \cap \mu_2'(I^2) \cap \pi_G(\partial B^n) = \emptyset,$$

where π_G denotes the decomposition map

$$\pi_G : B^n \longrightarrow B^n/G.$$

In this case, $(*)$ can be replaced with

$$(**) \qquad \mu_1'(I^2) \cap \mu_2'(I^2) \cap \pi_G(\beta) = \emptyset,$$

since all nondegenerate elements of G meet β, meaning that both singular disks $\mu_1'(I^2)$ and $\mu_2'(I^2)$ can be adjusted to avoid $\pi_G(\partial B^n) - \pi_G(\beta)$.

In summary, $\mathrm{Sp}^k(C, \beta)$ is a crumpled $(n+k)$-cube if and only if it satisfies the disjoint disks property at β, a property defined by condition $(**)$. Of course, the usual disjoint disk property is sufficient to ensure that $(**)$ holds.

Lemma 4.5. *Suppose that $(C, D, h) \in \mathscr{W}_n$ and that β is an $(n-1)$-cell standardly embedded in $\mathrm{Bd}\, C$ such that $\mathrm{Sp}^k(C, \beta)$ is a crumpled $(n+k)$-cube. Then, $\mathrm{Sp}^k(D, h(\beta))$ is also a crumpled $(n+k)$-cube.*

Proof. Since $\mathrm{Sp}^k(C, \beta)$ is a crumpled cube, C contains homotopy taming sets T_1 and T_2 such that $T_1 \cap T_2 \cap \beta = \emptyset$. Then, $h(T_1)$ and $h(T_2)$ are homotopy taming sets for D and

$$h(T_1) \cap h(T_2) \cap h(\beta) = h(T_1 \cap T_2 \cap \beta) = \emptyset,$$

which ensures that $\mathrm{Sp}^k(D, h(\beta))$ is a crumpled $(n+k)$-cube. $\square$

Theorem 4.6. *Suppose $(C, D, h) \in \mathscr{W}_n$ and that β is an $(n-1)$-cell standardly embedded in $\mathrm{Bd}\, C$ such that $\mathrm{Sp}^{n+k}(C, \beta)$ is a crumpled $(n+k)$-cube. Then, $(\mathrm{Sp}^k(C, \beta), \mathrm{Sp}^k(D, h(\beta)), \mathrm{Sp}^k(h)) \in \mathscr{W}_{n+k}$.*

Proof. Let

$$p_C : C \times S^k \longrightarrow \mathrm{Sp}^k(C, \beta)$$

and

$$p_D : D \times S^k \longrightarrow \mathrm{Sp}^k(D, h(\beta))$$

denote the decomposition maps. Let

$$f_h : C \longrightarrow D$$

be a map associated with (C, D, h). Define

$$F : \mathrm{Sp}^k(C, \beta) \longrightarrow \mathrm{Sp}^k(D, h(\beta))$$

as $p_D(f_h \times \mathbb{1})(p_C)^{-1}$, where

$$f_h \times \mathbb{1} : C \times S^k \longrightarrow D \times S^k. \qquad \square$$

5. Strict wildness considerations. As mentioned in Section 3, the definition of "at least as wild as" does not provide a partial order on the collection of crumpled n-cubes. In order to show that the relation fails to be antisymmetric, we present a pair of crumpled n-cubes with non-homeomorphic wild sets, therefore ensuring the two are topologically distinct, each at least as wild as the other.

For greater clarity, we shall introduce a standard flattening technique to construct such an example.

Definition 5.1. Given a crumpled n-cube C and a compact set $X \subset \mathrm{Bd}\, C$, the crumpled n-cube C_X is a *standard flattening* relative to X if there exists a homeomorphism h_X of $\mathrm{Bd}\, C$ to $\mathrm{Bd}\, C_X$ such that

$$(C, C_X, h_X) \in \mathscr{W}_n, \qquad h_X \mid X = \mathbb{1} \mid X, \qquad \mathrm{Bd}\, C_X - X$$

is locally collared in C_X, $C \subset C_X$, and C_X admits a strong deformation retraction

$$r : C_X \longrightarrow C$$

such that $rh_X = \mathbb{1} \mid \mathrm{Bd}\, C$ and all nontrivial point preimages under r are sent to $\mathrm{Bd}\, C - X$.

5.1. In this subsection, we show the existence of standard flattening for every crumpled cube C and closed subset X of $\mathrm{Bd}\, C$. Treat C as a closed n-cell complement. Name an embedding λ of $\mathrm{Bd}\, C \times I$, giving a collar $\mathrm{Bd}\, C$ in that n-cell $S^n - \mathrm{Int}\, C$ with $\lambda(s, 0) = s$ for all $s \in \mathrm{Bd}\, C$. Find a continuous function

$$\mu : \mathrm{Bd}\, C \longrightarrow [0, 1]$$

such that $X = \mu^{-1}(0)$. Define C_X as

$$C \cup \{\lambda(s \times [0, \mu(s)]) \text{ where } s \in \mathrm{Bd}\, C\}.$$

The retraction

$$r : C_X \longrightarrow C$$

deforms each $\lambda(s \times [0, \mu(s)])$ to $\lambda(s \times 0)$; it should be obvious why r is a strong deformation retraction. For later convenience, further restrict μ such that

$$\operatorname{diam} \lambda(s' \times [0, \mu(s')]) < \operatorname{dist}(s', X)$$

for all $s' \in \operatorname{Bd} C - X$.

Next, apply the proof of the homotopy extension theorem to the pair $(C - X, \operatorname{Bd} C - X)$ and ANR $C_X - X$. Consider the inclusion of $C - X$ in the target $C_X - X$ and the homotopy of $\operatorname{Bd} C - X$, starting with the inclusion and ending with the homeomorphism

$$h_X : \operatorname{Bd} C - X \longrightarrow \operatorname{Bd} C_X - X,$$

sending $s = \lambda(s, 0)$ to $\lambda(s, \mu(s))$. The track of this homotopy at s has diameter $< \operatorname{dist}(s, X)$. There is a neighborhood N of

$$((C - X) \times \{0\}) \cup ((\operatorname{Bd} C - X) \times I)$$

over which the partial homotopy extends into $C_X - X$. Call this extension ψ. Each $s \in \operatorname{Bd} C - X$ has a neighborhood O_s such that $O_s \times I \subset N$, and $\operatorname{diam} \psi(s \times I)$ has diameter $< \operatorname{dist}(s, X)$. Let O be the union of all of the O_s. Find a Urysohn function

$$u : C_X \longrightarrow [0, 1],$$

with $u(\operatorname{Bd} C - X) = \{1\}$ and $u(C - O) = \{0\}$. Then, define

$$\Psi : (C - X) \times I \longrightarrow C_X - X \quad \text{as } \Psi(s, t) = \psi(s, t \cdot u(s)).$$

The claim is that Ψ extends via projection to X on $X \times I$ to provide a map

$$C \times I \longrightarrow C_X.$$

This function Ψ is continuous at X: for points y within ϵ of X, either $\Psi(y \times I) = y$ or $\operatorname{diam} \Psi(y \times I) < \operatorname{dist}(y, X) < \epsilon$ such that $\operatorname{dist}(\Psi(y, t), x) < 2\epsilon$, assuring continuity.

Thus, the function h_X extends via the identity on X to a homeomorphism

$$h_X : \operatorname{Bd} C \longrightarrow \operatorname{Bd} C_X.$$

Note that $rh_X = \mathbb{1} \mid \operatorname{Bd} C$. The desired map

$$f : C \longrightarrow C_X$$

is approximately $f(c) = \Psi(c, 1)$, where $c \in C$. This type of map extends the homeomorphism h_X between the boundaries. The only problem is that f may send some point of $\operatorname{Int} C$ to a point of $\operatorname{Bd} C_X - X$. This may be fixed similarly to improvement of the map $C \to B^n$ to ensure that no point of $\operatorname{Int} C$ is sent to $\operatorname{Bd} B^n$. Note that, with any standard flattening C_X of C, X may be regarded as a subset of $\operatorname{Bd} C_X$.

Remark 5.2. A standard flattening (C, C_X, h_X) can easily result in a relatively uninteresting example in which C_X turns out to be an n-cell. That happens whenever C has a homotopy taming set T that misses X. Consequently, when C is the sort of crumpled cube for which any countable dense subset J of $\operatorname{Bd} C$ is a homotopy taming set, then $\dim X \leq n - 2$ implies C_X is an n-cell. However, if X is the closure of some open subset of C's wild set, then C_X is truly wild at each point of X.

Lemma 5.3. *Suppose that C is a crumpled n-cube, $\{X, Y\}$ a pair of compact sets with $Y \subset X \subset \operatorname{Bd} C$, (C, C_X, h_X) and (C, C_Y, h_Y) are standard flattenings of C with respect to X and Y, respectively, and $(C_X, (C_X)_Y, (h_X)_Y)$ is the standard flattening of C_X with respect to Y. Then, it follows that $(C_X)_Y = C_Y$ and $(h_X)_Y h_X = h_Y$.*

Proof. The map

$$\mu : \operatorname{Bd} C \longrightarrow [0, 1]$$

producing the standard flattening C_X should be chosen so that $\mu(\operatorname{Bd} C) \subset [0, 1)$. Then, there exists a collar $\lambda_X(\operatorname{Bd} C_X \times [0, 1])$ on $\operatorname{Bd} C_X$ in C_X such that

$$\lambda_X(h_X(s) \times [0, 1]) \subset \lambda(s \times [\mu(s), 1]),$$

whenever $s \in \operatorname{Bd} C$. The conclusion follows. □

Theorem 5.4. *Let C and D be crumpled n-cubes with wild sets W_C and W_D, respectively, and let*

$$h : \operatorname{Bd} C \longrightarrow \operatorname{Bd} D$$

be a homeomorphism such that $W_D \subset h(W_C)$. *Set* $X = h^{-1}(W_D)$. *Then,* $(C, D, h) \in \mathscr{W}_n$ *if and only if* $(C_X, D, hh_X^{-1}) \in \mathscr{W}_n$.

Proof. The reverse implication follows from Definition 5.1, Lemma 5.3 and the transitivity property for appropriate pairs of triples belonging to $\mathscr{W}_n$.

Turning to the forward implication, we consider $(C, D, h) \in \mathscr{W}_n$ and name a map

$$f_h : C \longrightarrow D$$

associated with h. There is a retraction

$$r : C_X \longrightarrow C \subset C_X$$

which restricts to h^{-1} on $\operatorname{Bd} C_X$ and otherwise satisfies $r(C_X - C) \subset \operatorname{Bd} C - X$. Then,

$$f_h r : C_X \longrightarrow D$$

restricts to hh_X^{-1} on $\operatorname{Bd} C_X$. The only problem with

$$f_h r : C_X \longrightarrow D$$

is that it sends some points of $\operatorname{Int} C_X - \operatorname{Int} C$ to $\operatorname{Bd} D$. However, the images of those troublesome points miss W_D; therefore, modifications of a now familiar sort can be used to push those images off of $\operatorname{Bd} D$, as required. $\square$

Example 5.5. Here, we show a pair of crumpled n-cubes with non-homeomorphic wild sets such that each is at least as wild as the other.

In $\mathbb{R}^{n-1}$, identify a countable collection of round $(n-2)$-spheres, no two of which intersect, plus a point to which these spheres converge. Let Z be the union, and let Z^* be Z with one of the spheres removed. It may still be shown that there exists a homeomorphism h of $\mathbb{R}^{n-1}$ to itself such that $h(Z) = Z^*$.

Next, label the $(n-1)$-balls bounded by the spheres in Z as

$$B_1, B_2, B_3, \ldots, B_i, \ldots,$$

with the understanding that B_1 misses Z^*, and that homeomorphism h of $\mathbb{R}^{n-1}$ to itself sends B_i to B_{i+1}. Extend h to a homeomorphism H of $\mathbb{R}^{n-1} \times [0, \infty)$ to itself that takes each $p \times [0, \infty)$ to $h(p) \times [0, \infty)$ and carries $B_i \times [0, 1/i]$ onto $B_{i+1} \times [0, 1/(i+1)]$.

Find a crumpled cube C whose boundary is locally flat modulo a simple closed curve J standardly embedded in $\operatorname{Bd} C$. Replace each n-cell $B_i \times [0, 1/i]$ with a copy of $C \subset B_i \times [0, 1/i]$, i.e., embedding a copy of C in $B_i \times [0, 1/i]$ so that the image of C contains all of this n-cell's boundary not in $\mathbb{R}^{n-1} \times \{0\}$. For later reference, denote

$$\left(\operatorname{Bd} B_i \times \left[0, \frac{1}{i}\right]\right) \cup \left(B_i \times \left\{\frac{1}{i}\right\}\right)$$

as β_i. Make sure the image of J misses β_i. Once this replacement has been completed for the first $B_1 \times [0, 1]$, it should be done in such a way that is compatible with H in the remainder, that is, perform it so that the homeomorphism H restricted to β_i extends to a homeomorphism from the copy of C in $B_i \times [0, 1/i]$ to the copy of C in $B_{i+1} \times [0, 1/(i+1)]$.

Let K be the subset of $\mathbb{R}^{n-1} \times [0, \infty)$ obtained by deleting the cells $B_i \times [0, 1/i]$ from Z and replacing with the copies of C, and let K^* be the space obtained when $B_1 \times I$ is left as it is. Replace all of the others. H extends to give a homeomorphism of K onto K^*. The one point compactifications of K and K^* are crumpled cubes, and H extends to give a homeomorphism H^* of those crumpled cubes. We use $\overline{K}$ and $\overline{K^*}$ as the names for these crumpled cubes, the compactifications.

Let W denote all of the wild set of $\overline{K}$ except the interior of an open subarc A of the copy of J in the first replacement C. Thus, W consists of a point, an arc and a sequence of simple closed curves whose diameters go to 0. The standard flattening gives $(\overline{K}, \overline{K}_W, h_W) \in \mathscr{W}_n$. Let W' denote all of the wild set of $\overline{K}$ that misses $B_1 \times I$. Looking at J in another way, in that first replacement, W' is the portion of the wild set of $\overline{K}_W$ except for the component that is an arc. We have a standard flattening of $\overline{K}_W$ relative to W', which means that

$$(\overline{K}_W, (\overline{K}_W)_{W'}, h_{W'}) \in \mathscr{W}_n.$$

Here, $\overline{K^*}$ can be regarded as a standard flattening of $\overline{K}$ relative to all of the wild set outside of that copy of J in the first replacement crumpled cube. Standard flattenings with respect to the same subset are homeomorphic. Thus, $(\overline{K}_W)_{W'}$, which is flattening first relative to W and then relative to $W' \subset W$, is the same as flattening relative to W'. As a result, $(\overline{K}_W)_{W'}$ is homeomorphic to $\overline{K^*}$, and we know that $\overline{K^*}$ is homeomorphic to $\overline{K}$. This verifies that $\overline{K}$ is at least as wild as $\overline{K}_W$, and $\overline{K}_W$ is at least as wild as $(\overline{K}_W)_{W'} \cong \overline{K}$.

Definition 5.6. C is *strictly wilder than* D if and only there exists a homeomorphism h such that $(C, D, h) \in \mathscr{W}_n$, but there is no homeomorphism H such that $(D, C, H) \in \mathscr{W}_n$.

Definition 5.6 imposes a strict partial order on the collection of all crumpled n-cubes, up to homeomorphism.

Theorem 5.7. *Suppose that D is a closed n-cell complement whose wild set W is a proper subset of* $\operatorname{Bd} D$. *Then, there exists a closed n-cell complement C strictly wilder than D such that* $\operatorname{Bd} C$ *is wild at each of its points.*

Proof. Triangulate $\operatorname{Bd} D - W$, and list the $(n-1)$-simplices

$$\Delta_1, \Delta_2, \ldots, \Delta_k, \ldots$$

of this triangulation. (If W is topologically a polyhedron tamely embedded in $\operatorname{Bd} D$, this can be a finite list and the Δ_i can be allowed to touch W in their boundaries; otherwise, however, require the diameters of the $\Delta_k \to 0$ as $k \to \infty$.)

Since $\operatorname{Bd} D$ is locally flat at all points of $\operatorname{Int} \Delta_i$, Δ_i can be thickened to an n-cell B_i in D such that Δ_i is a standardly embedded subset of $\operatorname{Bd} B_i$ and $B_i \cap B_j \subset \Delta_i \cap \Delta_j$ for all $i \neq j$.

Let K be a crumpled n-cube whose boundary is locally flat modulo an $(n-1)$-cell A standardly embedded in $\operatorname{Bd} K$. Also require that $\operatorname{Bd} A$ be tame in space, i.e., some homotopy taming set for K misses $\operatorname{Bd} A$. (In order to obtain such a K, modify Bing's construction of [**4**] to generate a crumpled 3-cube which is wild at the points of a 2-cell with tame boundary, and suspend as often as needed.) Then, addition to K of a tapered (exterior) collar on $\operatorname{Int} A$ produces an n-cell B^n containing A with $\operatorname{Cl}(B^n - K) - \operatorname{Bd} A$ equal to that tapered collar. Equate each B_i with a copy of B^n so as to identify a copy K_i of K in each B_i; do this such that $K_i \cap \operatorname{Cl}(D - B_i)$ corresponds to $\operatorname{Cl}(\operatorname{Bd} K - A)$. It follows that $(D - \cup_i B_i) \cup K_i$ is a crumpled n-cube C; in other words, C results from D by deleting all of the tapered collars and taking the closure of what remains. It should be immediately obvious that $\operatorname{Bd} C$ is everywhere wild and that $\operatorname{Bd} C$ contains W.

Build an exterior collar on C by first appending the tapered collars in B_i on the various K_i. The union equals D. When it is combined with an exterior collar on $\operatorname{Bd} D$, we have a collar on C. A standard flattening C_W of C then equals D; thus, we have $(C, D = C_W, h_W) \in \mathscr{W}_n$.

Since $\operatorname{Bd} C$ is everywhere wild and $\operatorname{Bd} D$ is not, C is strictly wilder than D. $\square$

Theorem 5.8. *For any crumpled n-cube D, $n > 3$, there exists another crumpled n-cube C and homeomorphism h with $(C, D, h) \in \mathscr{W}_n$ such that every associated map*

$$f_h : C \longrightarrow D$$

extending h restricts to an epimorphism

$$(f_h)_\# : \pi_1(\operatorname{Int} C) \longrightarrow \pi_1(\operatorname{Int} D)$$

having non-trivial kernel.

Proof. Given any crumpled cube D we can find a Cantor set X which misses some homotopy taming set for D, i.e., X is tame in space when D is embedded in S^n as a closed n-cell complement. We claim that there exists a $(C, D, h) \in \mathscr{W}_n$ such that, for every homotopy taming set T for C, $T \cap h^{-1}(X)$ is nonempty. In other words, $h^{-1}(X)$ fails to be tame in space. The key is to produce an $(n-1)$-sphere S in $X \cup \operatorname{Int} D$ that is locally flat modulo X and which contains X as a standard Cantor set in S. From [**19**], S is standardly embedded in S^n; thus, it bounds two n-cells, B and B', with $B \subset D$. Remove B from D and replace it with a crumpled cube K locally flat modulo a Cantor set Z wild in space but tame in $\operatorname{Bd} K$. Specifically, attach K to $D - \operatorname{Int} B$ via a homeomorphism

$$\theta : \operatorname{Bd} B \longrightarrow \operatorname{Bd} K$$

such that $\theta(X) = Z$. Let C be the result of the replacement. Keep in mind that K can be put into S^n as a closed n-cell complement, such that $S^n = B' \cup K \supset C$; in short, C is a crumpled cube.

Note that $S - X$ is simply connected. Hence,

$$\pi_1(\operatorname{Int} C) \cong \pi_1(\operatorname{Int} C - K) * \pi_1(\operatorname{Int} K).$$

Consider any loop

$$\gamma : \partial I^2 \longrightarrow \operatorname{Int} C$$

with an image in $\operatorname{Int} K$ and any map

$$f_h : C \longrightarrow D$$

associated with the obvious homeomorphism between $\operatorname{Bd} C$ and $\operatorname{Bd} D$. Then, $f_h\gamma$ extends to a map

$$\Gamma : I^2 \longrightarrow D;$$

since T is a homotopy taming set for D, Γ can be approximated by a map Γ' agreeing with Γ on $\operatorname{Bd} I^2$, with

$$\Gamma'(I^2) \subset T \cup \operatorname{Int} D.$$

Set $U = D - f_h(K)$, and let V denote the component of $U - \operatorname{Bd} D$ whose closure contains $\operatorname{Bd} D - X$. Find a disk with holes P in I^2 with $\partial I^2 \subset P$, $\Gamma(P) \subset \operatorname{Int} D$ and $\partial P - \partial I^2 \subset V$. This means that the subgroup of $\pi_1(\operatorname{Int} D)$ carried by $f_h\gamma(\partial I^2)$ is in the normal closure of $\pi_1(V)$ and elements $\alpha_1, \dots, \alpha_m$ determined by the components of $\partial P - \partial I^2$. Let V' denote the component of $f_h^{-1}(V)$ whose closure contains $\operatorname{Bd} C - Z$. From Lemma 3.5, the restriction of f_h induces an epimorphism

$$\pi_1(V') \longrightarrow \pi_1(V);$$

thus, $\pi_1(V')$ contains elements α_i' sent by f_h to α_i $(i = 1, \dots, m)$. Observe that

$$V' \subset \operatorname{Int} C - K.$$

If f_h is also restricted to give an isomorphism

$$\pi_1(\operatorname{Int} C) \longrightarrow \pi_1(\operatorname{Int} D),$$

the subgroup of $\pi_1(\operatorname{Int} C)$ carried by $\gamma(\partial I^2)$ would be in the normal closure of the α_i, and hence, on the normal closure of $\pi_1(\operatorname{Int} C - K)$ with respect to $\pi_1(\operatorname{Int} C)$, an impossibility. □

Given one crumpled cube D, Theorem 5.7 presents a method for constructing another crumpled cube C at least as wild as D; in most circumstances C will be strictly wilder than D by virtue of having a larger wild set, topologically distinct from that of D. Theorem 5.8 accomplishes a similar purpose without changing the topological type of the wild set but instead increasing the wildness of the boundary sphere. Decomposition theory affords a more general technique for creating additional examples.

Example 5.9. Here, a decomposition theory technique is presented for producing a crumpled cube at least as wild as a given crumpled cube D and having a topologically equivalent wild set. Given D, locate an n-cell B in D with $B \cap \operatorname{Bd} D = X$, and consider any cell-like decomposition G of S^n whose nondegenerate elements are subsets of B, each of which meets $\operatorname{Bd} B$ in a single point of X. Let N_G denote the union of those nondegenerate elements. In order to be truly effective, assume that there exists at least one loop γ in $\operatorname{Int} B - N_G$ which is homotopically essential in $S^n - N_G$. Let

$$\pi_G : S^n \longrightarrow S^n/G$$

denote the decomposition map. Since

$$\pi_G(N_G) \subset \pi_G(\operatorname{Bd} B)$$

and

$$\pi_G \mid \operatorname{Bd} B$$

is one-to-one, S^n/G is finite-dimensional. Often, G will be shrinkable and S^n/G will be homeomorphic to S^n; this can be ensured by imposing additional restrictions on G. If shrinkability fails, all of the relevant data and examples in higher dimensions can be obtained. When G is shrinkable, set $C = \pi_G(D)$. It is a crumpled cube since $\pi_G \mid \operatorname{Bd} C$ is one-to-one; moreover, the function $(\pi_G)^{-1}$ restricts to a homeomorphism of $\operatorname{Bd} C$ onto $\operatorname{Bd} D$. In order to see that $(C, D, h) \in \mathscr{W}_n$, note that $(\pi_G)^{-1} \mid \pi_G(C - \operatorname{Int} B)$ extends to a map

$$f_h : C \longrightarrow D$$

such that

$$f_h(\pi_G(\operatorname{Int} B)) \subset \operatorname{Int} B,$$

just as in the proof of Theorem 3.2. The special loop γ has an image under π_G that cannot be shrunk in $\operatorname{Int} C$; however, $f_h \pi_G(\gamma) \subset \operatorname{Int} B$ must be contractible in $\operatorname{Int} D$.

Theorem 5.8 suggests that there is no maximal element in the "strictly wilder than" partial ordering. We have not established that fact. However, we do have the following corollaries.

Corollary 5.10. *Let C denote a crumpled n-cube such that $\pi_1(\operatorname{Int} C)$ is finitely generated. Then, C is not a maximal element in the "strictly wilder than" partial ordering.*

Proof. By Theorem 5.8, there exist a crumpled n-cube $\widetilde{C}$ and a homeomorphism h such that $(\widetilde{C}, C, h) \in \mathscr{W}_n$ and $\pi_1(\operatorname{Int} \widetilde{C})$ is a non-trivial free product $G * \pi_1(\operatorname{Int} C)$. Hence, by Grushko's theorem, the number of generators of $\pi_1(\operatorname{Int} \widetilde{C})$ must be greater than the rank of $\pi_1(\operatorname{Int} C)$; thus, $\widetilde{C}$ must be strictly wilder than C. □

Corollary 5.11. *Let C denote a crumpled n-cube such that $\pi_1(\operatorname{Int} C)$ is a simple group. Then, C is not a maximal element in the "strictly wilder than" partial ordering.*

Proof. Again, the construction of Theorem 5.8 provides a crumpled n-cube $\widetilde{C}$ at least as wild as C, the fundamental group of which is a non-trivial free product. Here, C cannot be at least as wild as $\widetilde{C}$; there can be no epimorphism of $\pi_1(\operatorname{Int} C)$ to the non-simple group $\pi_1(\operatorname{Int} \widetilde{C})$. □

Corollary 5.12. *If C is a crumpled n-cube such that $\pi_1(\operatorname{Int} C)$ is a torsion group, then C is not a maximal element in the "strictly wilder than" partial ordering.*

The reader may confirm the existence of infinite families totally ordered under the "strictly wilder than" relation.

The n-cell B^n is the unique minimal element in the partial ordering on the collection of all closed-n-cell complements.

Theorem 5.13. *Every non-trivial crumpled n-cube is strictly wilder than the n-cell B^n.*

Proof. That any crumpled n-cube C is at least as wild as n-cell B was established in Theorem 3.2. It suffices to show that (B^n, C, H) is never in $\mathscr{W}_n$, no matter which homeomorphism

$$H : \operatorname{Bd} B^n \longrightarrow \operatorname{Bd} C$$

is under consideration. The empty set is a homotopy taming set for B^n. If $(B^n, C, H) \in \mathscr{W}_n$, then, by Theorem 3.8, $\emptyset$ is also a homotopy taming set for C, in other words, $\operatorname{Int} C$ is 1-ULC. That can only occur when C is an n-cell. $\square$

The results shown below are direct applications of Corollary 3.19 and Theorem 3.20.

Theorem 5.14. *If a closed n-cell complement C is locally flat modulo a wild set, a closed n-cell complement D is locally flat modulo a tame set and C is at least as wild as D, then C is strictly wilder than D.*

Theorem 5.15. *If a crumpled 3-cube C is at least as wild as another crumpled 3-cube D and C has more non-piercing points than D, then C is strictly wilder than D.*

We conclude this section with an open question.

Question 5.16. *Are there any maximal elements in the partial order constructed by "strictly wilder than?"*

6. Sewings of crumpled cubes. The triple $(C, D, h) \in \mathscr{W}_n$ automatically gives rise to a sewing of the two crumpled cubes by identifying each point x of $\operatorname{Bd} C$ with the point $h(x)$ on $\operatorname{Bd} D$. The associated sewing space is denoted $C \cup_h D$. It may be viewed as a decomposition space arising from a cell-like decomposition of S^n into points and the fiber arcs of an n-dimensional annulus. This section focuses on the interplay between (C, D, h) being in $\mathscr{W}_n$ and the sewings h which yield S^n.

Theorem 6.1. *Suppose $(C, D, h) \in \mathscr{W}_3$, that C^* is another crumpled 3-cube and*

$$\theta : \operatorname{Bd} C \longrightarrow \operatorname{Bd} C^*$$

is a homeomorphism such that $C \cup_\theta C^* = S^3$. Then, $D \cup_{\theta h^{-1}} C^* = S^3$.

Proof. By Eaton's characterization of the sewings of crumpled 3-cubes that yield S^3 **[13]**, C and C^* have homotopy taming sets T and T^*, respectively, such that $\theta(T) \cap T^* = \emptyset$. Then, $h(T)$ is a homotopy taming set for D, and clearly, $\theta h^{-1}(h(T)) \cap T^* = \emptyset$. Hence, again by **[13]**, $D \cup_{\theta h^{-1}} C^* = S^3$. $\square$

The same argument fails in higher-dimensional settings since the homotopy taming set mismatch feature is a sufficient, but not a necessary, condition for a sewing of crumpled cubes to yield S^n.

We will use the following controlled homotopy extension theorem.

Theorem 6.2. *For each $\epsilon > 0$, there exists a $\delta > 0$ such that, given any map*

$$f : X \longrightarrow Z$$

of a normal space to Z and any map

$$F_A : A \longrightarrow Z$$

defined on a closed subset A of X which is δ-close to $f \mid A$, then F_A admits a continuous extension

$$F : X \longrightarrow Z$$

which is ϵ-close to f.

Proof. Choose a $\delta > 0$ for which any two δ-close maps are ϵ-homotopic and then reuse the motion control aspect of the proof that standard flattenings exist. □

Theorem 6.3. *Suppose $(C, D, h) \in \mathscr{W}_n$, $n > 4$, that C^* is another crumpled n-cube and*

$$\theta : \operatorname{Bd} C \longrightarrow \operatorname{Bd} C^*$$

is a homeomorphism such that $C \cup_\theta C^* = S^n$. Then, $D \cup_{\theta h^{-1}} C^* = S^n$.

Proof. Since

$$\Sigma = D \cup_{\theta h^{-1}} C^*$$

is the cell-like image of S^n, by Edwards' cell-like approximation theorem **[14]**, or **[9]**, it suffices to prove that this sewing space satisfies the disjoint disks property. In order to begin the process, consider maps

$$\psi_1, \psi_2 : I^2 \longrightarrow \Sigma.$$

We will produce approximating maps with disjoint images in six steps; the only non-routine step, where the hypothesis that $C \cup_\theta C^* = S^n$ comes into play, is the final one.

Step 1. *Approximation to make the preimages* Z_1, Z_2 *of* $\operatorname{Bd} D = \operatorname{Bd} C^*$ *one-dimensional.* Generically, for $e \in \{1,2\}$, $Z_e = \psi_e^{-1}(\operatorname{Bd} D)$ "ought to be" one-dimensional. If not, pushing certain points one at a time into $\Sigma - \operatorname{Bd} D$, subject to convergence controls, we can readily modify ψ_e slightly so that the new map (still called ψ_e) sends a countable, dense subset of I^2 to $\Sigma - \operatorname{Bd} D$, which ensures one-dimensionality.

Step 2. *Approximation to make images of* Z_1, Z_2 *disjoint,* 1-LCC *subsets of* D *and* C^*. Let T and T^* denote σ-compact, homotopy taming sets for D and C^*, respectively. According to **[6]**, T and T^* can be taken to have dimension at most 1. Set $X_e = \psi_e^{-1}(D)$ and $Y_e = \psi_e^{-1}(C^*)$, and note that $Z_e = X_e \cap Y_e$. Approximate each $\psi_e \mid Z_e$ by an embedding

$$\lambda_e : Z_e \longrightarrow \operatorname{Bd} D$$

such that λ_1 and λ_2 have disjoint images in $\operatorname{Bd} D - (T \cup h\theta^{-1}(T^*))$. Require λ_e to be so close to $\psi_e \mid Z_e$ that, by the controlled homotopy extension theorem, λ_e extends to a map

$$\psi'_e : I^2 \longrightarrow \Sigma$$

close to ψ_e. This must be done in two separate operations, one extending λ_e to an approximation of

$$\psi_e \mid X_e : X_e \longrightarrow D,$$

and the other extending λ_e to an approximation of

$$\psi_e \mid Y_e : Y_e \longrightarrow C^*.$$

It should be clear that $\lambda_e(Z_e) = \psi'_e(Z_e)$ is a 1-LCC subset of each crumpled cube; for instance, since T is a homotopy taming set for D, any map $I^2 \to D$ can be approximated by a map

$$I^2 \longrightarrow T \cup \operatorname{Int} D,$$

the image of which avoids $\lambda_e(Z_e)$. It may be worth observing that, unfortunately, Z_e is not the complete preimage of $\operatorname{Bd} D = \operatorname{Bd} C^*$ under ψ'_e.

Step 3. *Approximation to make the image of each* Z_e *disjoint from the other singular disk.* This step uses properties of homotopy taming sets. The maps

$$\psi'_e \mid X_e : X_e \longrightarrow D \quad \text{and} \quad \psi'_e \mid Y_e : Y_e \longrightarrow C^*$$

can be approximated, fixing $\psi'_e \mid Z_e$ by maps

$$\psi^*_e : X_e \cup Y_e \longrightarrow \Sigma$$

such that

$$\psi^*_e(X_e - Z_e) \subset T \cup \operatorname{Int} D \quad \text{and} \quad \psi^*_e(Y_e - Z_e) \subset T^* \cup \operatorname{Int} C^*.$$

It follows that

$$\psi^*_1(Z_1) \cap \psi^*_2(I^2) = \emptyset = \psi^*_1(I^2) \cap \psi^*_2(Z_2).$$

Although this is all we shall discuss explicitly here on this effect, in successive steps, we impose controls for maintaining this disjointness feature.

Step 4. *Approximation to make preimages of* $\operatorname{Bd} D$ *in* $I^2 - Z_e$ *zero-dimensional.* This is a standard operation. Working in D and C^* separately, we use the fact that the boundary is 0-LCC in the crumpled cube to approximate $\psi^*_e \mid X_e - Z_e$ and $\psi^*_e \mid Y_e - Z_e$ by maps

$$\Psi^*_1, \Psi^*_2 : I^2 \longrightarrow \Sigma$$

which put the 1-skeleta of systems of finer and finer triangulations of the respective domains into the interiors of the relevant crumpled cubes.

Step 5. *Covering the preimage of* $\operatorname{Bd} D$ *in* $I^2 - Z_e$ *by a null sequence of pairwise disjoint disks.* Set $P_e = (\psi^*_e)^{-1}(\operatorname{Bd} D) - Z_e$, and express it as a countable union of compact, zero-dimensional sets $K_1, K_2, \ldots$. Cover K_1 by a finite collection $\xi^1_e, \ldots, \xi^k_e$ of small disks (discussed further in the next step) in $I^2 - Z_e$, whose boundaries miss P_e. Then, cover the part of K_2 not covered by the ξ^j_e by a finite collection of much smaller disks. The latter should be pairwise disjoint and disjoint from the first collection. Continue in the same manner.

Step 6. *Approximation to make the intersection of singular disks disjoint.* The heart of the matter is to make the intersections of those disks disjoint from $\operatorname{Bd} D = \operatorname{Bd} C^*$. Let

$$f_h : C \longrightarrow D$$

be a map associated with (C, D, h) being in $\mathscr{W}_n$. Extend f_h to a map

$$F_h : C \cup_\theta C^* \longrightarrow \Sigma = D \cup_{\theta h^{-1}} C^*$$

via the identity on C^*.

The idea is to cover the preimages of $\operatorname{Bd} D$ by null sequences of pairwise disjoint disks ξ_e^j in $I^2 - Z_e$, as in Step 5, to approximately lift (with respect to F_h) the restriction of Ψ_e^* on those disks to maps into

$$C \cup_\theta C^* = S^n,$$

to adjust the lifted maps on these disks to pairwise disjoint embeddings in S^n and to apply F_h.

Lifting singular disks mapped into C^* is no problem since F_h restricts to a homeomorphism over C^*. In this paragraph, we describe how to approximately lift certain singular disks mapped into D. Given $\epsilon > 0$, choose $\delta > 0$ such that subsets A of $C \cup_\theta C^*$ having diameter $< \delta$ are mapped via F_h to sets of diameter $< \epsilon/2$. Next, identify $\delta' > 0$ for which loops in

$$C \subset C \cup_\theta C^*$$

of diameter $< \delta'$ bound singular disks in C of diameter $< \delta$. Build an open cover $U_1, \ldots, U_k$ of $\operatorname{Bd} D$ in D by connected open subsets of D that meet $\operatorname{Bd} D$ in connected sets and for which each $(F_h \mid C)^{-1}(U_i)$ has diameter $< \delta'$. Then, find a compact neighborhood Q of $\operatorname{Bd} D$ in D covered by the U_i, and let $\eta \in (0, \epsilon/4)$ be a Lebesgue number for this cover of Q. By Lemma 3.5 and this choice of η, each η-small loop γ in $\operatorname{Int} D \cap Q$ is homotopic in the intersection of $\operatorname{Int} D$ with one of the U_i to the image of a loop γ' in $\operatorname{Int} C$, and γ' bounds a singular disk ξ' in C whose image under F_h has diameter $< \epsilon/2$. We construct null sequences

$$\xi_e^j \subset I^2 - Z_e, \quad j = 1, 2, \ldots,$$

of pairwise disjoint disks, each of diameter $< \eta$, whose interiors cover the zero-dimensional set P_e. We split each ξ_e^j into an annulus A_e^j and disk E_e^j, the union of which equals ξ_e^j and intersection of which equals $\partial E_e^j \subset \partial A_e^j$. From the construction just described, we see that each

$$\Psi_e^* \mid I^2 - \cup_j \operatorname{Int} \xi_e^j$$

extends over the various ξ_e^j to $F_h \nu_e^j$ on E_e^j and as a short homotopy in

$\Sigma - \operatorname{Bd} D$ on A_e^j, where

$$\nu_e^j : E_e^j \longrightarrow C \cup_\theta C^*.$$

Controls on sizes of ξ_e^j, A_e^j, E_e^j and shortness of the homotopy on A_e^j must be increasingly stringent as $j \to \infty$. Denote the new maps as Ψ_1' and Ψ_2'.

It is a simple matter to perform a general position adjustment in

$$S^n = C \cup_\theta C^*$$

to make the images of the various E_e^j under ν_1 and ν_2 pairwise disjoint, where ν_1 and ν_2 denote the obvious union of maps, that is, approximate these ν_e by ν_e', fixing $\cup_j \partial E_e^j$ so that

$$\nu_1'(\cup_j E_1^j) \cap \nu_2'(\cup_j E_2^j) = \emptyset.$$

Then, the maps

$$\Psi_e : I^2 \longrightarrow \Sigma$$

are defined as $F_h \nu_e'$ on $\cup_j E_e^j$ and as Ψ_e' elsewhere satisfies $\Psi_1(I^2) \cap \Psi_2(I^2) \subset \operatorname{Int} D$, since F_h is one-to-one over C^*. A final general position adjustment affecting only points sent into $\operatorname{Int} D$ eliminates all intersections for the images of Ψ_1, Ψ_2. $\square$

Corollary 6.4. *If* (C_1, D_1, h_1), $(C_2, D_2, h_2) \in \mathscr{W}_n$ *and*

$$\theta : \operatorname{Bd} C_1 \longrightarrow \operatorname{Bd} C_2$$

is a sewing such that $C_1 \cup_\theta C_2 = S^n$, *then*

$$D_1 \cup_{h_2 \theta h_1^{-1}} D_2 = S^n.$$

Theorem 6.5. *If* $(C, D, h) \in \mathscr{W}_n$ *and* $C \cup_h D = S^n$, $n \geq 5$, *then* D *contains disjoint homotopy taming sets* T *and* T'.

Proof. For the proof, we regard D as embedded in S^n via its position as a summand in the sewing space $C \cup_h D = S^n$. The hypothesis that $(C, D, h) \in \mathscr{W}_n$ means that there is a retraction

$$r : S^n \longrightarrow D$$

such that $r^{-1}(\mathrm{Bd}\, D) = \mathrm{Bd}\, D$; simply apply the map

$$f_h : C \longrightarrow D$$

associated with h to the other summand C of the sewing space.

It suffices to show that any two maps

$$\mu_1, \mu_2 : I^2 \longrightarrow D$$

can be approximated by maps with disjoint images. Such maps, regarded as maps into S^n, can be approximated by maps

$$\mu_1', \mu_2' : I^2 \longrightarrow S^n$$

with disjoint images. If these approximations protrude only slightly into $\mathrm{Int}\, C$, $r\mu_1'$ and $r\mu_2'$ will be close to μ_1 and μ_2, respectively. Their images intersect only at points of $\mathrm{Int}\, D$; thus, a final adjustment over $\mathrm{Int}\, D$ eliminates all intersections, similarly as in the proof of Theorem 6.3. □

Corollary 6.6. *If $(C, D, h) \in \mathscr{W}_n$ and $C \cup_{\mathbb{1}} C = S^n$, $n \geq 5$, then*

$$D \cup_{\mathbb{1}} D = S^n.$$

Corollary 6.7. *Suppose that (C, D, h) and $(D, C, h^{-1}) \in \mathscr{W}_n$. Then, $C \cup_h D = S^n$ if and only if h satisfies the mismatch property.*

Proof. It is well known **[5, 11]** that

$$C \cup_h D = S^n,$$

if h satisfies the mismatch property. The reverse implication follows from an argument similar to that given here for Theorem 6.5, due to the existence of retractions of $S^n = C \cup_h D$ to both C and D that are one-to-one over the boundaries. □

Corollary 6.8. *Suppose that G is a* usc *decomposition of S^n such that, for any $p \in \pi(N_G)$ and open set U containing p, there is an open set V such that*

$$p \in V \subset U$$

and $\mathrm{Bd}\, V$ *is an $(n-1)$-sphere which misses $\pi(H_G)$; suppose that*

$$(V, \mathrm{Cl}\,(S^n - V), \mathbb{1})$$

and

$$(\mathrm{Cl}\,(S^n - V), V, \mathbb{1})$$

are in $\mathscr{W}_n$*. Then,* G *is shrinkable and* S^n/G *is homeomorphic to* S^n*.*

Proof. Apply [**17,** main theorem] and Corollary 6.7. □

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