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SOME VARIATIONS OF MULTIPLE ZETA VALUES

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ABSTRACT. In this paper, we build some variations of multiple zeta values and investigate their relations. Among other things, we prove that

$$\sum_{\substack{|\boldsymbol{\alpha}|=m+r\\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1}k_2^{-\alpha_2}\cdots k_r^{-\alpha_r}(k_r-p)^{-1}$$

can be evaluated as a linear combination of $\zeta(r),\zeta(r-1), \ldots,\zeta(r-p+1)$ for $r\geq p+1$. In particular, for $r\geq 2$,

$$\sum_{\substack{|\alpha|=m+r\\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1}k_2^{-\alpha_2}\cdots k_r^{-\alpha_r}(k_r-1)^{-1}=\zeta(r),$$

which may be compared to the well-known sum formula. A similar discussion leads to the twisted sum formula.

1. Introduction. A *multiple zeta value* or *r-fold Euler sum* is defined [**3, 6, 7**] by

$$\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)=\sum_{1\leq k_1<k_2<\cdots<k_r} k_1^{-\alpha_1}k_2^{-\alpha_2}\cdots k_r^{-\alpha_r},$$

with positive integers $\alpha_1,\alpha_2,\ldots,\alpha_r$, and $\alpha_r\geq 2$ for the sake of convergence. The number r and weight $|\boldsymbol{\alpha}|=\alpha_1+\alpha_2+\cdots+\alpha_r$ are the depth and the weight of $\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)$, respectively.

Due to Kontsevich [**5**], multiple zeta values can be represented by iterated integrals or Drinfel'd integrals over simplices of weight dimension:

$$\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)=\int_{E_{|\boldsymbol{\alpha}|}}\Omega_1\Omega_2\cdots\Omega_{|\boldsymbol{\alpha}|}$$

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with $E_{|\boldsymbol{\alpha}|} : 0 < t_1 < t_2 < \cdots < t_{|\boldsymbol{\alpha}|} < 1$ and

$$\Omega_j = \begin{cases} \frac{dt_j}{1-t_j} & \text{if } j = 1, \alpha_1+1, \alpha_1+\alpha_2+1, \ldots, \alpha_1+\cdots+\alpha_{r-1}+1; \\ \frac{dt_j}{t_j} & \text{otherwise.} \end{cases}$$

Now, if we replace some of $dt_j/(1-t_j)$, respectively, dt_j/t_j by $dt_j/(1-t_j)^{p_j+1}$, respectively, $dt_j/t_j^{p_j+1}$, the resulting iterated integral may still converge but represents something else. For example,

$$\int_{E_4} \frac{dt_1}{(1-t_1)^2} \frac{dt_2}{t_2} \frac{dt_3}{t_3} \frac{dt_4}{t_4} = \zeta(3) = \int_{E_4} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3} \frac{dt_4}{t_4^2}$$

and

$$\int_{E_4} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \frac{dt_4}{t_4^2} = \zeta(2) - \zeta(3).$$

On the other hand, both integrals

$$\int_{E_2} \frac{dt_1}{(1-t_1)^2} \frac{dt_2}{t_2} \quad \text{and} \quad \int_{E_2} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2^2}$$

are divergent, and they fail to represent any multiple zeta value.

The zeta value $\zeta(2)$ can be represented by the double integral

$$\int_{E_2} \frac{dt_1 dt_2}{(1-t_1)t_2}.$$

However, there is another integral representation of $\zeta(2)$, represented as

$$\int_{E_3} \frac{dt_1}{(1-t_1)^2} \frac{dt_2}{t_2} \frac{dt_3}{t_3}.$$

In general, for positive integers p and r with $r \geq p+1$, the integral

$$\int_{E_{r+1}} \frac{p!\, dt_1}{(1-t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}$$

can be evaluated as

$$\sum_{k=1}^{\infty} \frac{(k+1)(k+2)\cdots(k+p-1)}{k^{r+1}}.$$

Thus, it is a linear combination of single zeta values.

In order to state our conclusion more precisely, we need some notation. For a positive integer n, we write $\boldsymbol{\lambda} \vdash n$ to denote that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_g)$ is a partition of n, i.e., $1 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_g$ and $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \cdots + \lambda_g = n$. Also, we let $\{a\}^m$ be the m repetitions of a. When $\boldsymbol{\lambda} = (\{1\}^{m_1}, \{2\}^{m_2}, \dots, \{k\}^{m_k})$, we let

$$\mu_{\boldsymbol{\lambda}} = 1^{m_1} m_1! 2^{m_2} m_2! \cdots k^{m_k} m_k!.$$

Main Theorem A. *For a pair of positive integers p, r with $r \ge p+1$ and nonnegative integers m, n, we have*

$$\begin{aligned} p! \sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1 \le k_1 < k_2 < \cdots < k_r}} & k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r} (k_r - p)^{-1} \sum_{\boldsymbol{\lambda} \vdash n} \mu_{\boldsymbol{\lambda}}^{-1} h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_g} \\ &= \binom{n+r-1}{n} \sum_{k=1}^{\infty} \frac{1}{k^{n+r}} \sum_{\substack{c+d=m \\ 0 \le c \le p-1}} \frac{(-1)^c}{c!} f_{k,p}^{(c)}(0) \\ &\qquad \times \Bigg(\sum_{\boldsymbol{\lambda}' \vdash d} \mu_{\boldsymbol{\lambda}'}^{-1} H_{\lambda_1'} H_{\lambda_2'} \cdots H_{\lambda_{g'}'} \Bigg), \end{aligned}$$

where the summation ranges over all partitions $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_g)$ of n and all partitions $\boldsymbol{\lambda}' = (\lambda_1', \lambda_2', \dots, \lambda_{g'}')$ of d, and

$$h_j = \sum_{\ell=k_1}^{k_r - p} \frac{1}{\ell^j}, \qquad H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}$$

and

$$f_{k,p}(a) = \prod_{j=1}^{p-1} (k + j + a).$$

When $p = 1$, $f_{k,1}(a) = 1$, we have the following.

Corollary 1.1. *For nonnegative integers m, n, r with $r \geq 2$, we have*

$$\sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r} (k_r-1)^{-1} \sum_{\boldsymbol{\lambda}\vdash n} \mu_{\boldsymbol{\lambda}}^{-1} h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_g} = \binom{n+r-1}{n} \zeta(n+r),$$

where the summation ranges over all partitions $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_g)$ of n and $h_j = \sum_{\ell=k_1}^{k_r-1} (1/\ell^j)$.

The special case $n = 0$ gives an analogue of the sum formula

$$\sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r} (k_r-1)^{-1} = \zeta(r). \tag{1.1}$$

On the other hand, the case $p = 2$, $n = 0$, gives the following.

Corollary 1.2. *For nonnegative integers m, r with $r \geq 3$, we have*

$$\sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1} k_2^{-\alpha_2} \cdots k_r^{-\alpha_r} (k_r-2)^{-1} = \left(1 - \frac{1}{2^{m+1}}\right) \zeta(r-1) + \frac{1}{2^{m+1}} \zeta(r). \tag{1.2}$$

Remark 1.3. Formulae (1.1) and (1.2) appearing in Corollaries 1.1 and 1.2 may be compared with the well-known sum formula **[9, 10]**

$$\sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\leq k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1} \cdots k_{r-1}^{-\alpha_{r-1}} k_r^{-\alpha_r-1} = \zeta(m+r+1).$$

However, they are much more complicated. In the case where $r = 3$ and $m = 2$, identity (1.1) appears to be

$$[\zeta(1,4) + \zeta(2,3) + \zeta(3,2)] - [\zeta(1,1,3) + \zeta(1,2,2) + \zeta(2,1,2)] + [\zeta(1,3) + \zeta(2,2)] - \zeta(1,1,2) + \zeta(1,2) = \zeta(3).$$

It is equivalent to $\zeta(1,2) = \zeta(3)$.

The outline of this paper is as follows. Section 2 introduces multiple zeta values with parameters. Specifically, as a starting point, we will give a simple proof of the sum formula of multiple zeta values. Section 3 presents a useful recursive relation between complete homogeneous symmetry functions and power sum symmetric functions. Section 4 gives a proof of Main Theorem A. In Section 5, we first raise t_1 to a power q to consider the integral

$$\int_{E_{r+1}} \left(\frac{1-t_{r+1}}{1-t_1}\right)^a \frac{p!t_1^q\,dt_1}{(1-t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}.$$

Then, a similar discussion leads to a twisted sum formula.

Main Theorem B. *Suppose that p, q, r are nonnegative integers with $r \geq p+1$. Then the following hold.*

(1) *Let $0 \leq q \leq p-1$. Then, for any nonnegative integer n,*

$$\begin{aligned} p!q! \sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1\leq \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r} \prod_{i=0}^{q}(\ell_r-p+i)^{-1} \\ = \sum_{k=1}^{\infty} \frac{\Gamma(k+q)}{(k+q)^r\Gamma(k)} \sum_{\substack{c+d=n \\ 0\leq c\leq p-q-1}} \frac{(-1)^c f_{k,p,q}^{(c)}(0)}{c!} \\ + \sum_{\boldsymbol{\lambda}\vdash d} \mu_{\boldsymbol{\lambda}}^{-1} H_{\lambda_1}H_{\lambda_2}\cdots H_{\lambda_g}. \end{aligned}$$

Here, the summation ranges over all of the partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_g)$ of d,

$$f_{k,p,q}(a) = \prod_{j=q+1}^{p-1}(k+a+j) \quad \text{and} \quad H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}.$$

(2) *Let $q > p-1$. Then, for any nonnegative integer n,*

$$p!q! \sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1\leq \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r} \prod_{i=0}^{q}(\ell_r-p+i)^{-1}$$

$$= \sum_{k=1}^{\infty} \frac{\Gamma(k+p)}{(k+q)^{r+1}\Gamma(k)} \sum_{c+d=n} \sum_{\boldsymbol{\lambda}' \vdash c} \mu_{\boldsymbol{\lambda}'}^{-1} h_{\lambda'_1} h_{\lambda'_2} \cdots h_{\lambda'_{g'}}$$
$$+ \sum_{\boldsymbol{\lambda} \vdash d} \mu_{\boldsymbol{\lambda}}^{-1} H_{\lambda_1} H_{\lambda_2} \cdots H_{\lambda_g}.$$

Here, the summation ranges over all of the partition $\boldsymbol{\lambda}' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{g'})$*, respectively,* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_g)$*, of* c*, respectively,* d*, and*

$$h_j = \sum_{\ell=p}^{q} \frac{1}{(k+\ell)^j}, \quad \textit{respectively, } H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}.$$

2. Multiple zeta values with parameters. Multiple zeta values with parameters [**4**] were first introduced in order to simplify the proof of the sum formula as well as of the restricted sum formula.

The zeta value $\zeta(2)$ is the simplest multiple zeta value which can be represented by the double integral

$$\int_{E_2} \frac{dt_1}{(1-t_1)} \frac{dt_2}{t_2}.$$

If we attach the factor

$$\left(\frac{1-t_2}{1-t_1}\right)^a \left(\frac{t_1}{t_2}\right)^b$$

with $a > -1$, $b > -1$ to the integral, it becomes

$$\int_{E_2} \left(\frac{1-t_2}{1-t_1}\right)^a \left(\frac{t_1}{t_2}\right)^b \frac{dt_1 dt_2}{(1-t_1)t_2},$$

and it can be evaluated as

$$\sum_{k=1}^{\infty} \frac{1}{(k+a)(k+b)}$$

in light of the expansion

$$\frac{1}{(1-t)^{a+1}} = \sum_{k=1}^{\infty} \frac{\Gamma(k+a)}{\Gamma(k)\Gamma(a+1)} t^{k-1}, \quad |t| < 1.$$

Therefore, for $a, b > -1$, we have the identity

$$\int_{E_2} \left(\frac{1-t_2}{1-t_1}\right)^a \left(\frac{t_1}{t_2}\right)^b \frac{dt_1 dt_2}{(1-t_1)t_2} = \sum_{k=1}^{\infty} \frac{1}{(k+a)(k+b)}.$$

Now, applying the differential operator

$$\frac{(-1)^{m+n}}{m!n!} \left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)$$

to both sides of the identity and then set $a = b = 0$, we obtain that

$$\frac{1}{m!n!} \int_{E_2} \left(\log \frac{1-t_1}{1-t_2}\right)^m \left(\log \frac{t_2}{t_1}\right)^n \frac{dt_1 dt_2}{(1-t_1)t_2} = \zeta(m+n+2).$$

This is precisely the well-known sum formula [**9, 10**] since the integral represents the sum of multiple zeta values [**6, 7, 8**]

$$\sum_{|\boldsymbol{\alpha}|=m+n+1} \zeta(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}+1).$$

For a multiple zeta value $\zeta(\alpha_1, \alpha_2, \ldots, \alpha_r)$ with iterated integral representation

$$\int_{E_{|\boldsymbol{\alpha}|}} \Omega_1 \Omega_2 \cdots \Omega_{|\boldsymbol{\alpha}|},$$

we can add the factors

$$\frac{1}{(1-t_1)^a}, \quad \left(\frac{1-t_{|\boldsymbol{\alpha}|}}{1-t_1}\right)^b, \quad \left(\frac{t_1}{t_{|\boldsymbol{\alpha}|}}\right)^c \quad \text{or} \quad (t_{|\boldsymbol{\alpha}|})^d$$

with $a, b, c, d > -1$ to the iterated integral to form multiple zeta values with various parameters. All of these iterated integrals with parameters may be evaluated like the standard multiple zeta values. Here, we evaluate the integral

$$G_p(a,b) = \int_{E_{r+1}} \left(\frac{1-t_{r+1}}{1-t_1}\right)^a \left(\frac{t_1}{t_{r+1}}\right)^b \frac{p!\, dt_1}{(1-t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}$$

and its dual

$$G_p^{\vee}(a,b) = \int_{E_{r+1}} \left(\frac{1-u_{r+1}}{1-u_1}\right)^b \left(\frac{u_1}{u_{r+1}}\right)^a \left(\prod_{j=1}^{r} \frac{du_j}{1-u_j}\right) \frac{p!\, du_{r+1}}{u_{r+1}^{p+1}}.$$

Proposition 2.1. *With the notation fixed as above, we then have*

$$G_p(a,b) = \sum_{k=1}^{\infty} \frac{1}{(k+b)^r} \cdot \frac{\Gamma(k+p+a)\Gamma(a+1)\Gamma(p+1)}{\Gamma(k+1+a)\Gamma(p+a+1)}.$$

Proof. As a first step, expand $1/(1-t_1)^{p+a+1}$ as

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+p+a)}{\Gamma(k)\Gamma(p+a+1)} t_1^{k-1}, \quad |t_1| < 1$$

so that

$$\int\limits_{0<t_1<t_2<\cdots<t_r<t_{r+1}} \frac{t_1^b\, dt_1}{(1-t_1)^{p+a+1}} \prod_{j=2}^{r} \frac{dt_j}{t_j} = \sum_{k=1}^{\infty} \frac{\Gamma(k+p+a)}{\Gamma(k)\Gamma(p+a+1)} \cdot \frac{t_{r+1}^{k+b}}{(k+b)^r}.$$

In the next step, which is integration with respect to t_{r+1}, the integral is

$$\int_0^1 t_{r+1}^{k-1}(1-t_{r+1})^a dt_{r+1},$$

which is a β-integral with the value

$$\frac{\Gamma(k)\Gamma(a+1)}{\Gamma(k+1+a)}.$$

Thus, our assertion follows. □

In the same manner, we have the following.

Proposition 2.2. *With the notation as above, we then have*

$$G_p^{\vee}(a,b) = \sum_{1\le k_1<k_2<\cdots<k_r} \frac{1}{(k_1+a)\cdots(k_r+a)(k_r-p)} \times \frac{p!\Gamma(k_1+b)\Gamma(k_r-p+1)}{\Gamma(k_1)\Gamma(k_r-p+b+1)}.$$

Combining the above propositions yields the following.

Proposition 2.3. *For a pair of positive integers p, r with $r \ge p+1$, we have, for real numbers a, b with $a, b > -1$,*

$$\sum_{k=1}^{\infty} \frac{1}{(k+b)^r} \cdot \frac{\Gamma(k+p+a)\Gamma(a+1)}{\Gamma(k+1+a)\Gamma(p+a+1)}$$
$$= \sum_{1 \le k_1 < k_2 < \cdots < k_r} \frac{1}{(k_1+a)(k_2+a)\cdots(k_r+a)(k_r-p)} \times \frac{\Gamma(k_1+b)\Gamma(k_r-p+1)}{\Gamma(k_1)\Gamma(k_r-p+b+1)}.$$

3. The digamma function and preliminaries. The digamma function [**1**] is defined as

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

for $x > 0$. The functional equation of the gamma function

$$\Gamma(x+1) = x\Gamma(x)$$

then implies that

$$\psi(x+1) = \frac{1}{x} + \psi(x),$$

and hence, for any positive integer n,

$$\psi(x+n) = \sum_{j=0}^{n-1} \frac{1}{j+x} + \psi(x).$$

On the other hand, $\psi(x)$ comes from the Kronecker limit formula for the Hurwitz zeta function $\zeta(s;x)$:

$$\lim_{s \to 1^+} \left[\zeta(s;x) - \frac{1}{s-1} \right] = -\psi(x).$$

It follows that

$$\sum_{j=0}^{\infty} \left(\frac{1}{j+1} - \frac{1}{j+x} \right) = \psi(x) - \psi(1) = \psi(x) + \gamma,$$

with γ the Euler constant defined by

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right).$$

Therefore,

$$\frac{d}{dx}[\psi(x)+\gamma]=\zeta(2;x),$$

and, for any positive integer n,

$$\frac{d^n}{dx^n}[\psi(x)+\gamma]=(-1)^{n-1}n!\zeta(n+1;x).$$

Now, we shall evaluate the differentiation

$$\theta_n=\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[\frac{\Gamma(k_1+x)\Gamma(k_2)}{\Gamma(k_1)\Gamma(k_2+x)}\right]\bigg|_{x=0}$$

for positive integers k_1 and k_2 with $k_1<k_2$.

Proposition 3.1. *For positive integers k_1 and k_2 with $k_1<k_2$, we then have*

$$n\theta_n=\sum_{j=0}^{n-1}\theta_j h_{n-j},$$

with

$$h_m=\sum_{\ell=k_1}^{k_2-1}\frac{1}{\ell^m}.$$

Proof. Let

$$g(x)=\frac{\Gamma(k_1+x)\Gamma(k_2)}{\Gamma(k_1)\Gamma(k_2+x)}.$$

Then,

$$\log g(x)=\log\Gamma(k_1+x)+\log\Gamma(k_2)-\log\Gamma(k_1)-\log\Gamma(k_2+x),$$

and hence,

$$g'(x)=-g(x)[\psi(k_2+x)-\psi(k_1+x)]=-g(x)h_1(x),$$

with

$$h_m(x)=\sum_{\ell=k_1}^{k_2-1}\frac{1}{(\ell+x)^m}.$$

Therefore, for $n \geq 1$,

$$\begin{aligned} g^{(n)}(x) &= -\frac{d^{n-1}}{dx^{n-1}}[g(x)h_1(x)] \\ &= -\sum_{p+q=n-1} \frac{(n-1)!}{p!q!} g^{(p)}(x) h_1^{(q)}(x), \\ &= -\sum_{p+q=n-1} \frac{(n-1)!}{p!q!} g^{(p)}(x) \cdot (-1)^q q! h_{q+1}(x), \end{aligned}$$

and hence,

$$n\theta_n = \sum_{p+q=n-1} \theta_p h_{q+1} = \sum_{j=0}^{n-1} \theta_j h_{n-j}. \qquad \square$$

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a set of indeterminates. The *complete homogeneous symmetric functions* T_r are defined by

$$T_0 = 1; \qquad T_r = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad 1 \leq r \leq n.$$

On the other hand, the *power sum symmetric functions* S_r are defined by

$$\begin{aligned} &S_0 = 1; \qquad S_r = \sum_{i=1}^{n} x_i^r, \quad r \geq 1; \\ &S_{\boldsymbol{\lambda}} = S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_g} \quad \text{if } \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_g). \end{aligned}$$

The next proposition will be used to evaluate θ_n.

Proposition 3.2 (**[11, 12]**). *For a positive integer m, the complete homogeneous symmetric and the power sum symmetric functions satisfy the recursive relation*:

$$mT_m = \sum_{j=1}^{m} T_{m-j} S_j.$$

Moreover, we can solve the recursive relation as

$$T_m = \sum_{\boldsymbol{\lambda} \vdash m} \mu_{\boldsymbol{\lambda}}^{-1} S_{\boldsymbol{\lambda}}.$$

Corollary 3.3. *For positive integers k_1 and k_2 with $k_1 < k_2$, let*

$$\theta_n = \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \left[\frac{\Gamma(k_1 + x)\Gamma(k_2)}{\Gamma(k_1)\Gamma(k_2 + x)} \right] \Bigg|_{x=0}.$$

Then,

$$\theta_n = \sum_{\boldsymbol{\lambda} \vdash n} \mu_{\boldsymbol{\lambda}}^{-1} h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_g}$$

with

$$h_m = \sum_{\ell = k_1}^{k_2 - 1} \frac{1}{\ell^m}.$$

4. Proof of Main Theorem A. Now, we are ready to prove our main theorem.

Proof of Main Theorem A. We begin with the iterated integral

$$\int_{E_{r+1}} \frac{p!\, dt_1}{(1 - t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}.$$

Attach the factor

$$\left(\frac{1 - t_{r+1}}{1 - t_1} \right)^a \left(\frac{t_1}{t_{r+1}} \right)^b$$

with $a, b > -1$ to the iterated integral to form the following integral:

$$G_p(a, b) = \int_{E_{r+1}} \left(\frac{1 - t_{r+1}}{1 - t_1} \right)^a \left(\frac{t_1}{t_{r+1}} \right)^b \frac{p!\, dt_1}{(1 - t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}.$$

From Proposition 2.1, we have

$$G_p(a, b) = \sum_{k=1}^{\infty} \frac{1}{(k + b)^r} \cdot \frac{\Gamma(k + p + a)\Gamma(a + 1)\Gamma(p + 1)}{\Gamma(k + 1 + a)\Gamma(p + a + 1)}.$$

Under the change of variables

$$u_1 = 1 - t_{r+1}, \qquad u_2 = 1 - t_r, \ldots, u_{r+1} = 1 - t_1,$$

the integral is transformed into the following integral:

$$G_p^{\vee}(a, b) = \int_{E_{r+1}} \left(\frac{1 - u_{r+1}}{1 - u_1} \right)^b \left(u_1 u_{r+1} \right)^a \left(\prod_{j=1}^{r} \frac{du_j}{1 - u_j} \right) \frac{p!\, du_{r+1}}{u_{r+1}^{p+1}}.$$

From Proposition 2.2, we have

$$G_p^{\vee}(a,b) = \sum_{1\le k_1<k_2<\cdots<k_r} \frac{1}{(k_1+a)\cdots(k_r+a)(k_r-p)} \times \frac{p!\Gamma(k_1+b)\Gamma(k_r-p+1)}{\Gamma(k_1)\Gamma(k_r-p+b+1)}.$$

Of course, we have $G_p(a,b) = G_p^{\vee}(a,b)$ for $a,b>-1$. Therefore, for nonnegative integers m and n,

$$\frac{(-1)^{m+n}}{m!n!}\left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)G_p(a,b)\bigg|_{a=b=0} = \frac{(-1)^{m+n}}{m!n!}\left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)G_p^{\vee}(a,b)\bigg|_{a=b=0}.$$

In the expression of $G_p(a,b)$, the quotient

$$\frac{\Gamma(k+p+a)}{\Gamma(k+1+a)} = \prod_{j=1}^{p-1}(k+j+a) := f_{k,p}(a)$$

is a polynomial in a of degree $p-1$. This yields

$$\frac{(-1)^{m+n}}{m!n!}\left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)G_p(a,b) = \binom{n+r-1}{n}\sum_{k=1}^{\infty}\frac{1}{(k+b)^{n+r}} \times \frac{(-1)^m}{m!}\frac{\partial^m}{\partial a^m}\left[f_{k,p}(a)\frac{\Gamma(a+1)\Gamma(p+1)}{\Gamma(a+p+1)}\right],$$

and, in light of Corollary 3.3,

$$\frac{(-1)^{m+n}}{m!n!}\left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)G_p(a,b)\bigg|_{a=b=0} = \binom{n+r-1}{n}\sum_{k=1}^{\infty}\frac{1}{k^{n+r}}\sum_{\substack{c+d=m\\ 0\le c\le p-1}}\frac{(-1)^c}{c!}f_{k,p}^{(c)}(0) \times \left(\sum_{\boldsymbol{\lambda}'\vdash d}\mu_{\boldsymbol{\lambda}'}^{-1}H_{\lambda_1'}H_{\lambda_2'}\cdots H_{\lambda_g'}\right),$$

with

$$H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}.$$

On the other hand,

$$\begin{aligned}
&\frac{(-1)^{m+n}}{m!n!}\left(\frac{\partial^m}{\partial a^m}\right)\left(\frac{\partial^n}{\partial b^n}\right)G_p^\vee(a,b) \\
&\quad = \sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\le k_1<k_2<\cdots<k_r}} (k_1+a)^{-\alpha_1}(k_2+a)^{-\alpha_2}\cdots(k_r+a)^{-\alpha_r}(k_r-p)^{-1} \\
&\qquad\qquad \times \frac{(-1)^n}{n!}\frac{\partial^n}{\partial b^n}\left[\frac{p!\Gamma(k_1+b)\Gamma(k_r-p+1)}{\Gamma(k_1)\Gamma(k_r-p+b+1)}\right],
\end{aligned}$$

and hence, when restricted to $a = b = 0$, the quotient is

$$p! \sum_{\substack{|\boldsymbol{\alpha}|=m+r \\ 1\le k_1<k_2<\cdots<k_r}} k_1^{-\alpha_1}k_2^{-\alpha_2}\cdots k_r^{-\alpha_r}(k_r-p)^{-1}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_g}$$

with

$$h_j = \sum_{\ell=k_1}^{k_r-p} \frac{1}{\ell^j}.$$

This proves our assertion. □

Here, we consider another example with $\zeta(\{1\}^{r-1},2)$ expressed as

$$\int_{E_{r+2}} \frac{dt_1 dt_2}{(1-t_1)^2 t_2}\left(\prod_{j=3}^{r+1}\frac{dt_j}{1-t_j}\right)\frac{dt_{r+2}}{t_{r+2}} \tag{4.1}$$

instead of the usual iterated integral representation

$$\int_{E_{r+1}}\left(\prod_{j=1}^{r}\frac{dt_j}{1-t_j}\right)\frac{dt_{r+1}}{t_{r+1}}.$$

Proposition 4.1. *For nonnegative integers n and r with $r \geq 2$, we have*

$$\sum_{|\boldsymbol{\alpha}|=n+r} \alpha_1 \zeta(\alpha_1, \ldots, \alpha_{r-1}, \alpha_r + 1) = \sum_{1 \leq \ell_1 < \ell_2} \frac{1}{\ell_1^r \ell_2 (\ell_2 - 1)} \sum_{\boldsymbol{\lambda} \vdash n} \mu_{\boldsymbol{\lambda}}^{-1} h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_g},$$

where

$$h_m = \sum_{\ell=\ell_1}^{\ell_2 - 1} \frac{1}{\ell^m}.$$

Proof. Attach the factor

$$\left(\frac{t_1}{t_{r+2}} \right)^a$$

to the integral representation of $\zeta(\{1\}^{r-1}, 2)$ as given in (4.1). This leads to the identity

$$\int_{E_{r+2}} \left(\frac{t_1}{t_{r+2}} \right)^a \frac{dt_1 dt_2}{(1-t_1)^2 t_2} \left(\prod_{j=3}^{r+1} \frac{dt_j}{1-t_j} \right) \frac{dt_{r+2}}{t_{r+2}} = \sum_{1 \leq k_1 < k_2 < \cdots < k_r} \frac{k_1}{(k_1 + a)^2 (k_2 + a) \cdots (k_r + a) k_r}.$$

With the change of variables

$$u_1 = 1 - t_{r+2}, \qquad u_2 = 1 - t_{r+1}, \ldots, u_{r+1} = 1 - t_2, \qquad u_{r+2} = 1 - t_1,$$

the integral is transformed into the integral

$$\int_{E_{r+2}} \left(\frac{1 - u_{r+2}}{1 - u_1} \right)^a \frac{du_1}{1 - u_1} \left(\prod_{j=2}^{r} \frac{du_j}{u_j} \right) \frac{du_{r+1} du_{r+2}}{(1 - u_{r+1}) u_{r+2}^2},$$

which can be evaluated as

$$\sum_{1 \leq \ell_1 < \ell_2} \frac{1}{\ell_1^r \ell_2 (\ell_2 - 1)} \cdot \frac{\Gamma(\ell_1 + a) \Gamma(\ell_2)}{\Gamma(\ell_1) \Gamma(\ell_2 + a)}.$$

Therefore, we conclude that

$$\sum_{1\le k_1<k_2<\cdots<k_r} \frac{k_1}{(k_1+a)^2(k_2+a)\cdots(k_r+a)k_r}$$
$$= \sum_{1\le \ell_1<\ell_2} \frac{1}{\ell_1^r\ell_2(\ell_2-1)}\cdot\frac{\Gamma(\ell_1+a)\Gamma(\ell_2)}{\Gamma(\ell_1)\Gamma(\ell_2+a)}.$$

Applying the differential operator

$$\frac{(-1)^n}{n!}\frac{d^n}{da^n}$$

to both sides of the above identity and then setting $a=0$, we obtain our assertion. □

Example 4.2. In the case $r=3$, $n=2$, the corresponding identity from Proposition 4.1 is

$$\zeta(3,1,2)+\zeta(2,1,3)+\zeta(2,2,2)=\zeta(4,2)+\zeta(3,3),$$

which can be numerically verified.

We use a general theorem concerning weighted sums of multiple zeta values. The simplest multiple zeta value of height 2 is of the form $\zeta(\{1\}^{p-1},2,\{1\}^{q-1},2)$ with $p,q\ge 1$, and the usual Drinfel'd integral representation is given by

$$\zeta(\{1\}^{p-1},2,\{1\}^{q-1},2)$$
$$=\int_{E_{p+q+2}}\left(\prod_{j=1}^{p}\frac{dt_j}{1-t_j}\right)\frac{dt_{p+1}}{t_{p+1}}\left(\prod_{k=p+2}^{p+q+1}\frac{dt_k}{1-t_k}\right)\frac{dt_{p+q+2}}{t_{p+q+2}}.$$

We attach the factor $(t_1/t_{p+q+2})^a$ with $a>-1$ to consider the integral

$$I_1(p,q;a)=\int_{E_{p+q+2}}\left(\frac{t_1}{t_{p+q+2}}\right)^a\left(\prod_{j=1}^{p}\frac{dt_j}{1-t_j}\right)\frac{dt_{p+1}}{t_{p+1}}\left(\prod_{k=p+2}^{p+q+1}\frac{dt_k}{1-t_k}\right)\frac{dt_{p+q+2}}{t_{p+q+2}},$$

which can be evaluated as

$$\sum_{1\le k_1<k_2<\cdots<k_{p+q}}\prod_{j=1}^{p}(k_j+a)^{-1}\prod_{i=0}^{q}(k_{p+i}+a)^{-1}k_{p+q}^{-1}.$$

Applying the differential operator $((-1)^n/n!)\,(d^n/da^n)$ and then setting $a=0$, we have

$$\sum_{|\boldsymbol{\alpha}|=p+q+n+1} \alpha_p \zeta(\alpha_1,\alpha_2,\ldots,\alpha_{p+q-1},\alpha_{p+q}+1).$$

Letting $r=p+q\geq 2$, the above can be rewritten as

$$\sum_{|\boldsymbol{\alpha}|=n+r+1} \alpha_p \zeta(\alpha_1,\alpha_2,\ldots,\alpha_{r-1},\alpha_r+1)$$

for $p=1,2,\ldots,r-1$.

On the other hand, the dual of the integral $I_1(p,q;a)$ is

$$I_1^{\vee}(p,q;a) = \int_{E_{p+q+2}} \left(\frac{1-u_{p+q+2}}{1-u_1}\right)^a \times \frac{du_1}{1-u_1}\left(\prod_{j=2}^{q+1}\frac{du_j}{u_j}\right)\frac{du_{q+2}}{1-u_{q+2}}\left(\prod_{k=q+3}^{p+q+2}\frac{du_k}{u_k}\right),$$

which can be evaluated as

$$\sum_{1\leq \ell_1<\ell_2} \frac{1}{\ell_1^{q+1}\ell_2^{p+1}} \cdot \frac{\Gamma(\ell_1+a)\Gamma(\ell_2+1)}{\Gamma(\ell_1)\Gamma(\ell_2+a+1)}.$$

Hence, applying the differential operator $((-1)^n/n!)\,(d^n/da^n)$ and then setting $a=0$, we get

$$\sum_{1\leq \ell_1<\ell_2} \frac{1}{\ell_1^{q+1}\ell_2^{p+1}} \sum_{\boldsymbol{\lambda}\vdash n} \mu_{\boldsymbol{\lambda}}^{-1}\widetilde{h}_{\lambda_1}\widetilde{h}_{\lambda_2}\cdots\widetilde{h}_{\lambda_g},$$

or

$$\sum_{1\leq \ell_1<\ell_2} \frac{1}{\ell_1^{r-p+1}\ell_2^{p+1}} \sum_{\boldsymbol{\lambda}\vdash n} \mu_{\boldsymbol{\lambda}}^{-1}\widetilde{h}_{\lambda_1}\widetilde{h}_{\lambda_2}\cdots\widetilde{h}_{\lambda_g},$$

with

$$\widetilde{h}_m = \sum_{\ell=\ell_1}^{\ell_2} \frac{1}{\ell^m}.$$

We conclude for $1 \le p \le r-1$ that

$$\sum_{|\boldsymbol{\alpha}|=n+r+1} \alpha_p \zeta(\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r + 1) = \sum_{1\le \ell_1<\ell_2} \frac{1}{\ell_1^{r-p+1}\ell_2^{p+1}} \sum_{\boldsymbol{\lambda}\vdash n} \mu_{\boldsymbol{\lambda}}^{-1} \widetilde{h}_{\lambda_1}\widetilde{h}_{\lambda_2}\cdots\widetilde{h}_{\lambda_g}.$$

In addition, we have the following theorem.

Theorem 4.3. *For a pair of nonnegative integers n, r with $r \ge 2$, we have*

$$\sum_{|\boldsymbol{\alpha}|=n+r} \alpha_r \zeta(\alpha_1, \ldots, \alpha_{r-1}, \alpha_r + 1) = \zeta(n+r+1) + \sum_{1\le \ell_1<\ell_2} \frac{1}{\ell_1\ell_2^{r}} \sum_{\boldsymbol{\lambda}\vdash n} \mu_{\boldsymbol{\lambda}}^{-1} (\widetilde{h}_{\lambda_1}\cdots\widetilde{h}_{\lambda_g} - h_{\lambda_1}\cdots h_{\lambda_g}),$$

where

$$\widetilde{h}_m = \sum_{\ell=\ell_1}^{\ell_2} \frac{1}{\ell^m} \quad \text{and} \quad h_m = \sum_{\ell=\ell_1}^{\ell_2-1} \frac{1}{\ell^m}.$$

Proof. For the positive integer p with $1 \le p \le r$ and real number $a > -1$, we consider the integral

$$I_2(p, r; a) = \int_{E_{r+2}} \left(\frac{t_1}{t_{r+2}}\right)^a \frac{1}{1-t_p} \left(\prod_{j=1}^{r} \frac{dt_j}{1-t_j}\right) \frac{dt_{r+1}dt_{r+2}}{t_{r+1}t_{r+2}},$$

which can be evaluated as

$$\sum_{1\le k_1<k_2<\cdots<k_r} \frac{k_p - k_{p-1}}{(k_1+a)\cdots(k_{r-1}+a)(k_r+a)^2 k_r}$$

with $k_0 = 0$. Also, its dual is given by

$$I_2^{\vee}(p, r; a) = \int_{E_{r+2}} \left(\frac{1-u_{r+2}}{1-u_1}\right)^a \frac{1}{u_{r-p+3}} \cdot \frac{du_1 du_2}{(1-u_1)(1-u_2)} \prod_{j=3}^{r+2} \frac{du_j}{u_j},$$

which can be evaluated as

$$\sum_{1\le \ell_1<\ell_2} \frac{1}{\ell_1\ell_2^{r-p+1}(\ell_2-1)^p}\cdot\frac{\Gamma(\ell_1+a)\Gamma(\ell_2)}{\Gamma(\ell_1)\Gamma(\ell_2+a)}.$$

Summing over all p with $1\le p\le r$, we obtain

$$\sum_{p=1}^{r} I_2(p,r;a)=\sum_{1\le k_1<k_2<\cdots<k_r}\frac{1}{(k_1+a)\cdots(k_{r-1}+a)(k_r+a)^2}$$

and

$$\sum_{p=1}^{r} I_2^{\vee}(p,r;a)=\sum_{1\le \ell_1<\ell_2}\left[\frac{1}{\ell_1(\ell_2-1)^r}-\frac{1}{\ell_1\ell_2^r}\right]\frac{\Gamma(\ell_1+a)\Gamma(\ell_2)}{\Gamma(\ell_1)\Gamma(\ell_2+a)}.$$

This leads to the identity

$$\sum_{1\le k_1<k_2<\cdots<k_r}\frac{1}{(k_1+a)\cdots(k_{r-1}+a)(k_r+a)^2} =\sum_{1\le \ell_1<\ell_2}\left[\frac{1}{\ell_1(\ell_2-1)^r}-\frac{1}{\ell_1\ell_2^r}\right]\frac{\Gamma(\ell_1+a)\Gamma(\ell_2)}{\Gamma(\ell_1)\Gamma(\ell_2+a)}.$$

Applying the differential operator $((-1)^n/n!)\,(d^n/da^n)$ to both sides of the above identity and then setting $a=0$ such that corresponding to the left hand side is the weighted sum

$$\sum_{|\boldsymbol{\alpha}|=n+r}\alpha_r\zeta(\alpha_1,\ldots,\alpha_{r-1},\alpha_r+1).$$

Corresponding to the right hand side is the difference

$$\sum_{1\le \ell_1<\ell_2}\frac{1}{\ell_1(\ell_2-1)^r}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_g}-\sum_{1\le \ell_1<\ell_2}\frac{1}{\ell_1\ell_2^r}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_g}.$$

According to $\ell_1=\ell_2-1$ or $\ell_1<\ell_2-1$, the first sum is rewritten as

$$\zeta(n+r+1)+\sum_{1\le \ell_1<\ell_2}\frac{1}{\ell_1\ell_2^r}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}\widetilde{h}_{\lambda_1}\widetilde{h}_{\lambda_2}\cdots\widetilde{h}_{\lambda_g},$$

where $\widetilde{h}_m=\sum_{\ell=\ell_1}^{\ell_2}\frac{1}{\ell^m}$. □

5. Twisted sum formulae. For nonnegative integers p, q and r with $r \geq p+1$ and real number $a > -1$, consider the integral with parameter a given by

$$\int_{E_{r+1}} \left(\frac{1-t_{r+1}}{1-t_1}\right)^a \frac{p!t_1^q dt_1}{(1-t_1)^{p+1}} \prod_{j=2}^{r+1} \frac{dt_j}{t_j}.$$

It can be evaluated as

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+q)}{(k+q)^r \Gamma(k)} \left[\frac{\Gamma(k+p+a)}{\Gamma(k+q+a+1)}\right] \frac{\Gamma(p+1)\Gamma(a+1)}{\Gamma(p+a+1)}.$$

In addition, its dual is given by

$$\int_{E_{r+1}} \left(\frac{u_1}{u_{r+1}}\right)^a \left(\prod_{j=1}^{r} \frac{du_j}{1-u_j}\right) \frac{p!(1-u_{r+1})^q du_{r+1}}{u_{r+1}^{p+1}}$$

and can be evaluated as

$$p!q! \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_r} [(\ell_1+a)(\ell_2+a)\cdots(\ell_r+a)]^{-1} \prod_{i=0}^{q} (\ell_r - p + i)^{-1}.$$

The above consideration then leads to the following conclusion.

Proposition 5.1. *For nonnegative integers p, q and r with $r \geq p+1$ and real number $a > -1$, we have*

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+q)}{(k+q)^r \Gamma(k)} \left[\frac{\Gamma(k+p+a)}{\Gamma(k+q+a+1)}\right] \frac{\Gamma(p+1)\Gamma(a+1)}{\Gamma(p+a+1)}$$
$$= p!q! \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_r} [(\ell_1+a)(\ell_2+a)\cdots(\ell_r+a)]^{-1} \prod_{i=0}^{q} (\ell_r - p + i)^{-1}.$$

Next, we differentiate both sides of the formula in Proposition 5.1 with respect to the parameter a and give a proof of our Main Theorem B.

Proof of Main Theorem B. When $0 \leq q \leq p-1$, the quotient

$$\frac{\Gamma(k+p+a)}{\Gamma(k+q+a+1)} = \prod_{j=q+1}^{p-1} (k+j+a) := f_{k,p,q}(a)$$

is a polynomial function in a of degree $p-q-1$ so that the identity is rewritten as

$$\sum_{k=1}^{\infty}\frac{\Gamma(k+q)}{(k+q)^r\Gamma(k)}f_{k,p,q}(a)\frac{\Gamma(p+1)\Gamma(a+1)}{\Gamma(p+a+1)}$$
$$= p!q!\sum_{1\le\ell_1<\ell_2<\cdots<\ell_r}[(\ell_1+a)(\ell_2+a)\cdots(\ell_r+a)]^{-1}\prod_{i=0}^{q}(\ell_r-p+i)^{-1}.$$

After both sides are differentiated n times with respect to a and setting $a=0$, we obtain the first part of the conclusion.

When $q>p-1$, the quotient

$$\frac{\Gamma(k+p+a)}{\Gamma(k+q+a+1)}$$

is no longer a polynomial function. Instead, it is a rational function in a. The identity is rewritten as

$$\sum_{k=1}^{\infty}\frac{\Gamma(k+p)}{(k+q)^{r+1}\Gamma(k)}\left[\frac{\Gamma(k+p+a)\Gamma(k+q+1)}{\Gamma(k+p)\Gamma(k+q+a+1)}\right]\frac{\Gamma(p+1)\Gamma(a+1)}{\Gamma(p+a+1)}$$
$$= p!q!\sum_{1\le\ell_1<\ell_2<\cdots<\ell_r}[(\ell_1+a)(\ell_2+a)\cdots(\ell_r+a)]^{-1}\prod_{i=0}^{q}(\ell_r-p+i)^{-1}.$$

After both sides are differentiated n times with respect to a and setting $a=0$, we obtain the second part of the conclusion. □

Some particular cases are of special interest and are worth mentioning here. When $q=p-1$, the polynomial function $f_{k,p,q}(a)=1$, we then have the following.

Corollary 5.2. *For positive integers p and r with $r\ge p+1$, we have*

$$p!(p-1)!\sum_{\substack{|\boldsymbol{\alpha}|=n+r\\1\le\ell_1<\ell_2<\cdots<\ell_r}}\ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}\prod_{j=1}^{p}(\ell_r-j)^{-1}$$
$$=\sum_{k=1}^{\infty}\frac{\Gamma(k+p-1)}{(k+p-1)^r\Gamma(k)}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}H_{\lambda_1}H_{\lambda_2}\cdots H_{\lambda_g},$$

where $H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}$. *In particular, when* $p = 1, 2, 3$, *the corresponding identities are*:

(1) $$\sum_{\substack{|\boldsymbol{\alpha}|=n+r\\ 1\le \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}(\ell_r-1)^{-1} = \zeta(r) \text{ for } r \geq 2.$$

(2)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r\\ 1\le \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}[(\ell_r-1)(\ell_r-2)]^{-1}$$
$$= [\zeta(r-1)-\zeta(r)]\left(1-\frac{1}{2^{n+1}}\right) \text{ for } r \geq 3.$$

(3)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r\\ 1\le \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}[(\ell_r-1)(\ell_r-2)(\ell_r-3)]^{-1}$$
$$= \frac{1}{4}[\zeta(r-2)-3\zeta(r-1)+2\zeta(r)]\left(1-\frac{1}{2^n}+\frac{1}{3^{n+1}}\right) \text{ for } r \geq 4.$$

When $p = q$, we have

$$\frac{\Gamma(k+p+a)}{\Gamma(k+p+a+1)} = \frac{1}{k+p+a},$$

such that

$$\sum_{\boldsymbol{\lambda}'\vdash c} \mu_{\boldsymbol{\lambda}'}^{-1} h_{\lambda_1'} h_{\lambda_2'}\cdots h_{\lambda_{g'}'} = \frac{1}{(k+p)^c},$$

This leads to the next corollary.

Corollary 5.3. *For a pair of nonnegative integers* p, r *with* $r \geq p+1$, *we have, for any nonnegative integer* n,

$$(p!)^2 \sum_{\substack{|\boldsymbol{\alpha}|=n+r\\ 1\le \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}[\ell_r(\ell_r-1)\cdots(\ell_r-p)]^{-1}$$

$$= \sum_{k=1}^{\infty} \frac{\Gamma(k+p)}{(k+p)^{r+1}\Gamma(k)} \sum_{c+d=n} \frac{1}{(k+p)^c} \sum_{\boldsymbol{\lambda} \vdash d} \mu_{\boldsymbol{\lambda}}^{-1} H_{\lambda_1} H_{\lambda_2} \cdots H_{\lambda_g}.$$

Here,

$$H_j = \sum_{\ell=1}^{p} \frac{1}{\ell^j}.$$

In particular, when $p = 0, 1, 2$*, the corresponding identities are as follows.*

(1) (*The sum formula*)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1 \le \ell_1 < \ell_2 < \cdots < \ell_r}} \ell_1^{-\alpha_1} \cdots \ell_{r-1}^{-\alpha_{r-1}} \ell_r^{-\alpha_r - 1} = \zeta(n+r+1) \quad \textit{for } r \ge 1.$$

(2)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1 \le \ell_1 < \ell_2 < \cdots < \ell_r}} \ell_1^{-\alpha_1} \ell_2^{-\alpha_2} \cdots \ell_r^{-\alpha_r} [\ell_r(\ell_r - 1)]^{-1} = \zeta(r) - \zeta(n+r+1) \quad \textit{for } r \ge 2.$$

(3)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1 \le \ell_1 < \ell_2 < \cdots < \ell_r}} \ell_1^{-\alpha_1} \ell_2^{-\alpha_2} \cdots \ell_r^{-\alpha_r} [\ell_r(\ell_r - 1)(\ell_r - 2)]^{-1}$$
$$= \frac{1}{2}\zeta(r-1)\left(1 - \frac{1}{2^{n+1}}\right) + \zeta(r)\left(-1 + \frac{1}{2^{n+2}}\right) + \frac{1}{2}\zeta(n+r+1) \quad \textit{for } r \ge 3.$$

Special case $p = 0$ of Main Theorem B provides the evaluation of the sum:

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1 \le \ell_1 < \ell_2 < \cdots < \ell_r}} \ell_1^{-\alpha_1} \ell_2^{-\alpha_2} \cdots \ell_r^{-\alpha_r} [\ell_r(\ell_r + 1) \cdots (\ell_r + q)]^{-1}.$$

Corollary 5.4. *For a pair of nonnegative integers q, r with $r \geq 1$, we have, for any nonnegative integer n,*

$$q! \sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1\leq \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}[\ell_r(\ell_r+1)\cdots(\ell_r+q)]^{-1} = \sum_{k=1}^{\infty}\frac{1}{(k+q)^{r+1}}\sum_{\boldsymbol{\lambda}\vdash n}\mu_{\boldsymbol{\lambda}}^{-1}h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_g},$$

where

$$h_j = \sum_{\ell=0}^{q}\frac{1}{(k+\ell)^j}.$$

Example 5.5. Applying $q = 1, 2$ to Corollary 5.4, the corresponding identities are as follows:

(1)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1\leq \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}\left[\ell_r(\ell_r+1)\right]^{-1} = \zeta(n+r+1)-\zeta(n+r)+\sum_{k=1}^{\infty}\frac{1}{k^n(k+1)^r}.$$

(2)

$$\sum_{\substack{|\boldsymbol{\alpha}|=n+r \\ 1\leq \ell_1<\ell_2<\cdots<\ell_r}} \ell_1^{-\alpha_1}\ell_2^{-\alpha_2}\cdots\ell_r^{-\alpha_r}[\ell_r(\ell_r+1)(\ell_r+2)]^{-1} = \frac{1}{4}\Bigg[\zeta(n+r-1)-3\zeta(n+r)+2\zeta(n+r+1) + \sum_{k=1}^{\infty}\frac{k+1}{k^n(k+2)^r} - 2\sum_{k=1}^{\infty}\frac{k}{(k+1)^n(k+2)^r}\Bigg].$$

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