ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 5, 2017

REMARKS ON THE LIFESPAN OF SOLUTIONS TO WAVE EQUATIONS WITH A POTENTIAL

QIDI ZHANG

ABSTRACT. In this paper, we show that the solution to the semilinear wave equation with a potential of quadratic type blows up in finite time. We also give an upper bound estimate for the lifespan of the solution.

1. Introduction. The paper is devoted to studying blow up of the solution to the following semi-linear wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + V(x)u = |u|^p & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varepsilon f(x), \qquad \partial_t u(x,0) = \varepsilon g(x), \end{cases} \tag{1.1}$$

where $V = V(x)$ is a potential, $\varepsilon > 0$ is a small parameter and (f, g) are compactly supported smooth functions. Here, n is the space dimension.

Problem (1.1) with $V \equiv 0$ has been intensively studied in the last several decades. Let $p_c(n)$ be the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

This is known as the critical exponent in the sense of the following: if $p \in (1, p_c(n)]$, then problem (1.1) with $V \equiv 0$ has no global solution for nonnegative initial data; while, if $p \in (p_c(n) + \infty)$, then the solution to (1.1) exists for all time. This is the well-known Strauss conjecture. We give only a brief summary here and refer the reader to **[12]** and the references therein for details. The case $n = 3$ was considered by John **[4]**. He proved that the solution blows up in finite time when $1 < p < p_c(3)$ and exists globally in time when $p > p_c(3)$. The same result was obtained by Glassey **[2, 3]** when $n = 2$. In high dimensions

2010 AMS *Mathematics subject classification.* Primary 35L70, 58J45.
Keywords and phrases. Blow up, wave equation, potential.
This work was supported by the NSFC, grant No. 11271322, the Fundamental Research Funds for the Central Universities and by the Scientific Research Foundation for Returned Overseas Chinese Scholars, State Education Ministry.
Received by the editors on December 15, 2015.
DOI:10.1216/RMJ-2017-47-5-1743

$n \geq 4$, the proof of the existence of global solutions when $p > p_c(n)$ can be found in the work of [**1, 6, 11**], while the corresponding blow up result when $1 < p < p_c(n)$ was established by Sideris [**8**]. The blow up result for the critical case $p = p_c(n)$ was shown by Schaeffer [**7**] when $n = 2, 3$, and independently by Yordanov and Zhang [**13**] and Zhou and Han [**15**] when $n \geq 4$.

When blow up happens, it is also interesting to obtain estimates for the lifespan $T(\varepsilon)$ of the solution, that is, the largest value T such that the solution exists for $x \in \mathbb{R}^n$, $0 \leq t < T$. It is conjectured that

$$T(\varepsilon) \approx \varepsilon^{[2p(p-1)]/[(n-1)p^2-(n+1)p-2]} \qquad \text{if } 1 < p \leq p_c(n),$$

and

$$T(\varepsilon) \approx e^{c\varepsilon^{-p(p-1)}} \qquad \text{if } p = p_c(n),$$

where, by $A \approx B$, we mean that there exist positive numbers C_1 and C_2 such that $A \leq C_1 B$ and $B \leq C_2 A$. This conjecture has been verified when $1 < p < p_c(n)$ and $n \leq 8$ in the work of [**5, 6, 10, 15**]. In [**10, 15**], the authors also obtained upper bound lifespan estimates for solutions in high dimensions, i.e.,

$$\begin{aligned} T(\varepsilon) &\leq c\varepsilon^{[2p(p-1)]/[(n-1)p^2-(n+1)p-2]} && \text{when } 1 < p < p_c(n); \\ T(\varepsilon) &\leq c' e^{c\varepsilon^{-p(p-1)}} && \text{when } p = p_c(n). \end{aligned}$$

The lower bound estimates for the lifespan in general when $n \geq 9$ is still an open question.

A natural question is to examine what happens if $V \not\equiv 0$. In [**9**], the authors showed that, in dimension 3, if the small potential $V(x)$ decays as $|x|$ goes to infinity and if $p > p_c(3)$, then the solution with small data exists globally in time; while, if the potential $V(x)$ is negative and decays in spatial variable, then the solution must blow up in finite time under some other assumptions. For a locally Hölder continuous potential $V(x)$ satisfying

$$0 \leq V(x) \leq \frac{C}{1 + |x|^{2+\delta}}$$

(for some $C, \delta > 0$), it was shown in [**13**] that the solution must blow up in finite time when $1 < p < p_c(n)$.

We are interested in what happens when the potential does not decay. It was shown in **[14]** that $T(\varepsilon) \geq c\varepsilon^{-3/2}$ in dimension $n \geq 2$ (slightly better in dimension 1) for $V(x) = |x|^2$ and $p = 2$.

In the next section, we shall prove that problem (1.1) with $V(x) = |x|^2 - n$ must blow up in finite time no matter how small the data and obtain an upper bound for the lifespan of the solution.

2. Main result and the proof.

Theorem 2.1. *Consider Cauchy problem* (1.1) *with* $V(x) = |x|^2 - n$, $(f, g) \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, $g \geq 0$, $\varepsilon > 0$. *Assume that* $p > 1$. *Then, the solution to* (1.1) *must blow up in finite time. Moreover, we have the lifespan estimate*:

$$T(\varepsilon) \leq c\varepsilon^{-(p-1)/(p+1)},$$

where c *is a positive constant independent of* ε.

Remark 2.2. We see that the solution to problem (1.1) with $V(x) = |x|^2 - n$ must blow up in finite time for any $p > 1$ and any n. No critical exponent appears. This is quite different from the case $V(x) \equiv 0$.

Remark 2.3. The initial data need not be smooth. We assume that $(f, g) \in C_0^\infty(\mathbb{R}^n)$ for the convenience of exposition.

In order to prove this theorem, we shall need the following lemma.

Lemma 2.4. *Let* $p > 1$, $a \geq 1$ *and* $(p-1)a > q - 2$. *If* $F \in C^2([0, T))$ *satisfies*

(i) $F(t) \geq \delta(t-1)^a$;
(ii) $d^2F(t)/dt^2 \geq k(t+1)^{-q}[F(t)]^p$,

with some positive constants δ *and* k, *then* $F(t)$ *will blow up in finite time,* $T < \infty$. *Furthermore, we have the following estimate for the lifespan* $T(\delta)$ *of* $F(t)$:

$$T(\delta) \leq c\delta^{-(p-1)/[(p-1)a-q+2]},$$

where c *is a positive constant independent of* δ.

We omit the proof since this is [**16,** Lemma 2.1].

Proof of Theorem 2.1. Denote by

$$H = -\Delta + |x|^2$$

the harmonic oscillator on $L^2(\mathbb{R}^n)$. We know that it is self-adjoint and its first eigenvalue is $\lambda = n$. Let

$$\phi(x) = \pi^{-n/4} e^{-|x|^2/2}$$

be the eigenfunction associated to the eigenvalue $\lambda = n$. Thus,

$$H\phi = n\phi. \tag{2.1}$$

The first equation of (1.1) may be written as

$$\partial_t^2 u = -Hu + nu + |u|^p.$$

Consequently, if we define

$$F(t) = \int_{\mathbb{R}^n} u(x,t)\phi(x)\,dx = \langle u(\cdot,t), \phi(\cdot)\rangle, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 pair with respect to the spatial variable, we get

$$\begin{aligned} \frac{d^2}{dt^2} F(t) &= \int_{\mathbb{R}^n} \partial_t^2 u(x,t)\phi(x)\,dx \\ &= -\langle Hu, \phi\rangle + n\langle u, \phi\rangle + \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx. \end{aligned} \tag{2.3}$$

Due to the fact that H is is self-adjoint, and by (2.1), we know

$$-\langle Hu, \phi\rangle + n\langle u, \phi\rangle = -\langle u, H\phi\rangle + n\langle u, \phi\rangle = 0. \tag{2.4}$$

Thus, (2.3) becomes

$$\frac{d^2}{dt^2} F(t) = \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx. \tag{2.5}$$

Since by Hölders inequality and the fact that $\phi > 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u(x,t)\phi(x)\,dx \right|^p &\le \left| \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx \right| \left| \int_{\mathbb{R}^n} \phi(x)\,dx \right|^{p/p'} \\ &= c \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx \end{aligned}$$

for some $c > 0$, where $p' = p/(p-1)$, from (2.5) we obtain

$$\frac{d^2}{dt^2} F(t)| \geq k|F(t)|^p \tag{2.6}$$

for some other positive constant k. This means that Lemma 2.2 (ii) holds with $q = 0$.

We multiply by $\phi(x)$ on both sides of the first equation of (1.1) and integrate over $\mathbb{R}^n \times [0, t]$. Then we get

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} \partial_t^2 u(x,t)\phi(x)\,dx\,dt + \int_0^t \langle (H-n)u, \phi\rangle\,dt \\ = \int_0^t \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx\,dt. \end{aligned} \tag{2.7}$$

By (2.4), the second term disappears. Therefore, (2.7) becomes

$$\int_{\mathbb{R}^n} \partial_t u(x,t)\phi(x)\,dx - \int_{\mathbb{R}^n} \partial_t u(x,0)\phi(x)\,dx = \int_0^t \int_{\mathbb{R}^n} |u(x,t)|^p \phi(x)\,dx\,dt.$$

Since $\phi > 0$, we obtain

$$\int_{\mathbb{R}^n} \partial_t u(x,t)\phi(x)\,dx \geq \int_{\mathbb{R}^n} \partial_t u(x,0)\phi(x)\,dx,$$

that is,

$$\frac{d}{dt} F(t) \geq \varepsilon \int_{\mathbb{R}^n} g(x)\phi(x)\,dx.$$

Integrating over $[0, t]$ yields

$$F(t) \geq \varepsilon t \int_{\mathbb{R}^n} g(x)\phi(x)\,dx + \varepsilon \int_{\mathbb{R}^n} f(x)\phi(x)\,dx \geq c\varepsilon(t+1) \tag{2.8}$$

for some constant $c > 0$ and large enough t since $f \geq 0$, $g \geq 0$ and $g \not\equiv 0$. This means that Lemma 2.2 (i) holds with $a = 1$ and $\delta = c\varepsilon$, where $c > 0$ is independent of ε.

Now we apply Lemma 2.2 with $q = 0$, $a = 1$, $\delta = c\varepsilon$. It is obvious that $(p-1)a > q - 2$ if $p > 1$. Thus, from Lemma 2.2, we know that $F(t)$ must blow up in finite time, and we also have

$$T(\varepsilon) \leq c\varepsilon^{-(p-1)/(p+1)}.$$

This concludes the proof. $\square$

REFERENCES

1. V. Georgiev, H. Lindblad and C. Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119** (1997), 1291–1319.

2. R.T. Glassey, *Finite-time blow-up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323–340.

3. ———, *Existence in the large for $\Box u = F(u)$ in two space dimensions*, Math. Z. **178** (1981), 233–261.

4. F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscr. Math. **28** (1979), 235–268.

5. H. Lindblad, *Blow up for solutions of $\Box u = |u|^p$ with small initial data*, Comm. Part. Diff. Eq. **15** (1990), 757–821.

6. H. Lindblad and C. Sogge, *Long-time existence for small amplitude semilinear wave equations*, Amer. J. Math. **118** (1996), 1047–1135.

7. J. Schaeffer, *The equation $\Box u = |u|^p$ for the critical value of p*, Proc. Roy. Soc. Edinburgh **101** (1985), 31–44.

8. T.C. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Diff. Eq. **52** (1984), 378–406.

9. A. Walter Strauss and K. Tsutaya, *Existence and blow up of small amplitude nonlinear wave with a negative potential*, Discr. Cont. Dynam. Syst. **3** (1997), 175–188.

10. H. Takamura and K. Wakasa, *The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions*, J. Diff. Eq. **251** (2011), 1157–1171.

11. D. Tataru, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), 795–807.

12. C.B. Wang and X. Yu, *Recent works on the Strauss conjecture*, Contemp. Math. **581**, American Mathematical Society, Providence, RI, 2012.

13. B. Yordanov and Q.S. Zhang, *Finite time blowup for wave equations with a potential*, SIAM J. Math. Anal. **36** (2005), 1426–1433.

14. Q.D. Zhang, *Long time existence for the quadratic wave equation associated to the harmonic oscillator*, Nonlin. Anal. **130** (2016), 202–213.

15. Y. Zhou and W. Han, *Life-span of solutions to critical semilinear wave equations*, Comm. Part. Diff. Eq. **39** (2014), 439–451.

16. ———, *Blow-up of solutions to semilinear wave equations with variable coefficients and boundary*, J. Math. Anal. Appl. **374** (2011), 585–601.

East China University of Science and Technology, Department of Mathematics, Meilong Road 130, Shanghai, 200237 China

Email address: qidizhang@ecust.edu.cn