

MORITA EQUIVALENCES OF SPIN BLOCKS OF SYMMETRIC AND ALTERNATING GROUPS

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ABSTRACT. We complete the demonstration of source algebra equivalences between spin blocks of families of covering groups $\{\tilde{S}_n\}$ and $\{\tilde{A}_n\}$ of symmetric and alternating groups, for pairs of blocks at the ends of maximal strings. These equivalences remain within the family of groups if cores of the two blocks have the same parity and cross over from one family to the other if the cores are of opposite parity. This demonstrates Kessar and Schaps' crossover conjecture for the easier case of extremal points of maximal strings. We use this result to give an improved bound for the highest degree necessary in order to get representatives of all Morita equivalence classes of spin blocks for a given weight.

1. Introduction. Let G be a finite group, and let F be an algebraically closed field of characteristic p . We are interested here in the modular case when $p > 0$. Writing

$$FG = \bigoplus_{j=1}^r B_j$$

as a decomposition into blocks, we let D_j be the *defect group* of block B_j , the smallest subgroup over which B_j modules are relatively projective, which is determined up to conjugacy in G .

Donovan has conjectured that there are only a finite number of Morita equivalence classes although, as G runs through all finite groups, there are infinitely many blocks with given defect group D . Puig has generalized this to a conjecture that, for a given defect group, there are

2010 AMS *Mathematics subject classification.* Primary 20C05, 20C20.

Keywords and phrases. Donovan's conjecture, spin blocks, Scopes involution, Morita equivalence.

This work was partially supported by a grant from the Bar-Ilan University Research Authority, and formed part of the first author's Ph.D. dissertation at Bar-Ilan University.

Received by the editors on March 3, 2014, and in revised form on July 27, 2015.

only finitely many classes of blocks up to Puig equivalence, in which blocks are equivalent if they have the same source algebra.

Two blocks are derived-equivalent if there is an equivalence of categories between the corresponding bounded derived categories. If two blocks are derived-equivalent, then they share many important invariants, including the number of simple modules. Broué has conjectured [3, 4] that, if the defect group D is abelian, the block will be derived equivalent to its Brauer correspondent. For abelian D , this would reduce the number of possible derived equivalence classes to the number of possible blocks with normal defect group D , which is generally much smaller than the number of Morita equivalence classes.

The original proof of the Donovan conjecture for symmetric groups by Scopes [17] depended on certain operations on partitions known as Scopes involutions. These have since been shown to be artifacts of a much deeper theory of categorification developed in the symmetric group case by Chuang and Rouquier [8] in which blocks are organized into strings, on each of which a certain element of a Weyl group acts as a reflection. In the interior of the strings the image of the block under this reflection is only derived-equivalent, but, at the end of the strings, one obtains an actual Morita equivalence; these are the Scopes involutions.

In this paper, we are interested in faithful representations of a family of central extensions of the symmetric group. We assume, henceforward, that our ground field F is of positive, odd characteristic. For $n > 7$, there are only two possible central extensions; one in which involutions still lift to involutions and one in which they lift to elements of order 4. These two groups can also be defined for $n \leq 7$. The group algebras of the two groups are isomorphic; thus, from the point of view of representation theory it does not matter which we choose. We will denote by \tilde{S}_n the group in which the transpositions lift to elements whose squares are the non-trivial central element and refer to it as the *chosen covering group* or, more briefly, the *covering group*. Both of the possible choices have isomorphic alternating groups \tilde{A}_n , and the groups in this family will be called *covering groups of the alternating group*.

We consider the possible Morita equivalences among blocks in the two families $\{\tilde{S}_n\}$ and $\{\tilde{A}_n\}$, the chosen covering groups of the symmetric and alternating groups. Kessar [12] has already demonstrated

Donovan's conjecture for blocks of the covering groups of symmetric groups [12] using a version of the Scopes maps adapted for the covering groups, and she proved not just Morita equivalence but actual source algebra equivalence. In [13], Kessar and the second author demonstrated that, for this purpose, the two families, $\{\tilde{S}_n\}$ and $\{\tilde{A}_n\}$, should be treated together, in that, when the parity of the partitions corresponding under the Scopes map are different, the expected source algebra equivalences takes a block of $\{\tilde{S}_n\}$ to a block of $\{\tilde{A}_n\}$, and vice versa. The proof of a similar result, for the case where the parities are equal, is embedded in Kessar [12] but is difficult to extract because the primary aim of the author was to prove Donovan's conjecture.

In this paper, we redefine the Scopes map to be an involution, more compatible with the new understanding of the theory, and we give a new proof, using permutation modules, for the case of equal parity. This leads to a better bound for Donovan's conjecture.

The blocks are determined by a core ρ , a weight w and choice of family between $\{\tilde{S}_n\}$ and $\{\tilde{A}_n\}$. The crossover conjecture [2] asserts that, for any w , there are exactly two derived equivalence classes in the union of blocks from the two families $\{\tilde{S}_n\}$ and $\{\tilde{A}_n\}$. The correspondence of the blocks is determined by Scopes involutions on the cores. We show here that, for each such class under Scopes involution, there is a degree after which all of the Scopes involutions actually produce source algebra equivalences.

In Section 2, we review spin representations and give labels ρ^w to spin blocks based on the core and the weight.

In Section 3, we define i -strings of block labels in terms of the abacus notation and determine the combinatorics of these i -strings. Also, in Section 3, where we will be concerned with labels as combinatorial objects, we will work with the labels themselves, without discussion of whether they represent blocks of symmetric or alternating groups.

In Section 4, we demonstrate that the blocks corresponding to the labels at the two ends of an i -string, with crossovers where necessary, are source algebra equivalent, and thus, Morita equivalent. Also, in Section 4, where we consider the blocks over a complete discrete valuation ring \mathcal{O} and are concerned with the block as an algebra, we must distinguish between the blocks of the two groups.

In Section 5, we use the results of Sections 3 and 4 to give a sharp bound for Donovan's conjecture for spin blocks.

In order to study the source algebra equivalences we will work over a modular system (K, \mathcal{O}, F) , where \mathcal{O} is a discrete valuation ring, K is its quotient field, of characteristic 0 and F , of characteristic p , is the residue field after dividing out by the maximal ideal. Letting G be a covering group of either symmetric or alternating groups for some degree n , defined more precisely below, we associate to each block both a set of irreducibles from KG , which will be called *characteristic 0 irreducibles* or *ordinary irreducibles* and a set of irreducibles from FG , which will be called *modular irreducibles*. It is the spin blocks of $\mathcal{O}G$ which will form the bridge between them.

For any core ρ , let B_{ρ^w} be the block algebra of $\mathcal{O}\tilde{S}_n$ with core ρ and weight w , and let B'_{ρ^w} be the corresponding block of $\mathcal{O}\tilde{A}_n$ with core ρ and weight w . We will prove the following:

Theorem. *Suppose that the blocks with cores ν and μ and weight w lie at the ends of a maximal string. Then, if the parities are the same, B_{ν^w} is source algebra equivalent to B_{μ^w} , and B'_{ν^w} is source algebra equivalent to B'_{μ^w} . If the parities are different, B_{ν^w} is source algebra equivalent to B'_{μ^w} , and B'_{ν^w} is source algebra equivalent to B_{μ^w} .*

2. Definitions and notation.

2.1. Spin blocks of symmetric and alternating groups. In the notation fixed in Section 1 the symmetric group S_n and the alternating group A_n have central extensions \tilde{S}_n, \tilde{A}_n with kernel C_2 , the cyclic group of order 2, generated by an involution z . We assume henceforward that the characteristic of the ground field is finite but not 2. The group algebra of each of the covering groups can be decomposed into two subalgebras of equal dimension by the value of the characters on the central involution z . One of these subalgebras is isomorphic to the group algebra of the original group since, in every character, the value of z is 1. The characters of the second subalgebra, for which z takes the value -1 , will be called *spin representations*, and the corresponding blocks will be called *spin blocks*.

For every block of S_n , isomorphism classes of irreducible representations over a field of characteristic 0 are labeled by the partitions of n . For every spin block of \tilde{S}_n and \tilde{A}_n , isomorphism classes of characteristic 0 irreducible spin representations are labeled by the *strict* partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ where $\lambda_i \neq \lambda_j$ for i, j satisfying $1 \leq i, j \leq r$. For example, in \tilde{S}_6 , characteristic 0 irreducible spin representations are labeled by strict partitions

$$(6), (5, 1), (4, 2), (3, 2, 1).$$

Each strict partition λ has a parity $\epsilon(\lambda) \in \{0, 1\}$ that equals the parity of a permutation with cycle structure given by λ and which thus equals the parity of the number of even parts. If we denote by $|\lambda|$ the sum of the parts, which we call the *degree* of the partition, and the number of parts by $h(\lambda)$, then $\epsilon(\lambda) \equiv |\lambda| - h(\lambda) \pmod{2}$.

In addition to the characteristic 0 irreducibles we will also need to know something about characteristic p irreducibles. For the ordinary representations of the symmetric group characteristic p irreducibles can be labeled by a subset of the partitions. There are two dual choices, either p -regular partitions which do not have p or more copies of any one part, or p -restricted partitions, in which two adjacent parts do not differ by more than p . For spin representations, we need a different set of labels, determined by Brundan and Kleshchev [5] using supermodules.

Not only are the labels for characteristic p irreducibles not a subset of the labels of characteristic 0 irreducibles, but, even on the intersection of the two sets, the definition of parity may be different, as the formulae in the next definition will show. For a given p -block, either all of the characteristic p -irreducibles will be even and will come in pairs, or they will all be odd, and there will be a single characteristic p irreducible for each label.

For characteristic 0 irreducibles, if the block is not of defect 0, there will be some of each parity. In spin blocks of \tilde{S}_n odd irreducibles will be doubled and, for blocks of \tilde{A}_n , even irreducibles will be doubled. For example, for spin blocks of \tilde{S}_3 , two linear characteristic 0 irreducibles have the same odd label $(2, 1)$, and the unique characteristic 0 irreducible of degree 2 has even label (3) .

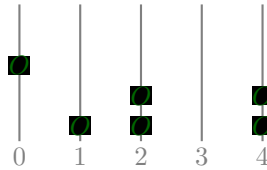


FIGURE 1. $\lambda = (10, 9, 7, 4, 2, 1)$.

Definition 2.1. A partition is *p-strict* if the only possible multiple parts are divisible by p . A p -strict partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is *restricted* if

- (i) $\lambda_{i+1} - \lambda_i \leq p$ for $i < r$,
- (ii) if $\lambda_{i+1} - \lambda_i = p$, then p does not divide λ_i .

For a restricted p -strict partition μ we define $h_{p'}(\mu)$ to be the number of parts not divisible by p , and then let

$$\epsilon'(\mu) \equiv |\mu| - h_{p'}(\mu) \pmod{2}$$

be the parity of μ as the label of a characteristic p irreducible. Obviously, $\epsilon'(\mu)$ will differ from the parity

$$\epsilon(\mu) \equiv |\mu| - h(\mu) \pmod{2}$$

defined above whenever the number of parts prime to p has a different parity than the total number of parts.

2.2. The abacus notation and p -bar cores. The strict partitions can be represented on an abacus with p runners labeled by residues $0, 1, \dots, (p - 1)$ where the parts of the strict partition are represented as beads, which can be in position $0, 1, 2, \dots$ [12, Section 4]. The part $\lambda_i = ap + b, 0 \leq b \leq p - 1$, corresponds to a bead on the runner b in position a , except that no bead is allowed on runner 0 at position 0 since this would correspond to a part of length 0. Since the positions are linearly ordered by $ap + b$, we will say that a bead is smaller or larger, respectively, than another according to numerical ordering on the corresponding positions, while beads in the same position, which can occur only on runner 0 for p -strict partitions, will be incomparable.

In the example in Figure 1, the largest bead, corresponding to 10, is at the top of the 0-runner, and the smallest is part 1 at the bottom of the 1-runner.

Definition 2.2. *Removing a p -bar.* Removing a p -bar from a strict or p -strict partition λ consists either of

- (i) lowering the position of one bead one place down on its runner into an available place (usually empty, but for the 0-runner of a p -strict partition any space is available), which corresponds to reducing a single part λ_i by p when possible;
- (ii) removing a bottom bead in the 0-runner, which corresponds to removing a part p from the partition λ ; or
- (iii) removing two bottom beads on runners i and $p - i$ for $0 < i < p$.

In each case, we reduce the sum of the parts of the strict partition by p . For labels of the modular irreducibles, which allow multiple copies of parts divisible by p , we denote the multiplicity by an integer next to the bead.

Definition 2.3. A p -bar core is a strict partition from which no p -bars can be removed and still leave a strict partition after reordering. When the maximal number w of p -bars is removed from a strict partition or a restricted p -strict partition λ , a strict partition ρ called the p -bar core of λ and denoted by $\rho(\lambda)$ is attained. The integer w is called the *weight of the partition*.

Remark 2.4. Amongst other properties, a p -bar core has no parts divisible by p and no pairs of parts which sum to p . The p -core is independent of the order in which the p -bars are removed. As an example, if $p = 5$ and we begin with a partition $(10, 9, 7, 4, 2, 1)$ as in Figure 1, we cannot reduce 9 to 4 until we have removed the pair 4 and 1. To eliminate 10, we first reduce it to 5 and then we remove part 5, which we can do because it is the bottom bead on the 0-runner. There are many different ways to remove four p -bars, but the final result, whichever order is used, will be the same p -bar core $(7, 4, 2)$.

2.3. Types of blocks. All spin representations of \tilde{S}_n (or \tilde{A}_n) that have the same p -bar core belong to the same spin block [5] and have

the same weight w . We use the notation ρ^w as a label for such a block. When $w = 0$, ρ^0 labels three blocks, for if ρ is an even partition, it labels two irreducibles in $F\tilde{A}_n$ and one in $F\tilde{S}_n$ and, if it is odd, it labels two irreducibles in $F\tilde{S}_n$ and one in $F\tilde{A}_n$.

For $w > 0$, the symbol ρ^w labels two blocks, one for the group algebra $F\tilde{S}_n$ and the other for the group algebra $F\tilde{A}_n$. One of the two blocks has one irreducible for each restricted p -strict label and is said to be of type M . The other has two irreducibles for each restricted p -strict label and is said to be of type Q . The block of type M is a block of $F\tilde{S}_n$ if $\epsilon(\rho) - w$ is even and is a block of $F\tilde{A}_n$ if $\epsilon(\rho) - w$ is odd. The names Q and M come from the theory of superalgebras, where each restricted p -strict label corresponds to a unique supermodule irreducible, and it is only by forgetting the supermodule structure that we get two irreducibles for each label in the modules for blocks of type Q .

For a given integer weight w , all spin blocks which are of type M have the same number ℓ of simple modules, and all blocks of type Q have 2ℓ simples. We cannot have a source algebra equivalence between a block of type Q and a block of type M for a given weight w because a source algebra equivalence, being a Morita equivalence, must preserve the number of simple modules, and the block of type Q will have twice as many modular irreducibles. For this reason, we consider both families together in order to allow for matching of the types.

For example, for $p = 5$, the block of $F\tilde{S}_{19}$ that is labeled by $(7, 2)^2$ is of type Q and has ten modular irreducibles labeled by the restricted p -strict partitions $(10, 7, 2)$, $(7, 5, 5, 2)$, $(7, 5, 4, 2, 1)$, $(9, 7, 2, 1)$, $(7, 6, 4, 2)$, all of which are odd as labels of modular irreducibles, even though they are not all odd as partitions, since 19 is odd and the number of parts prime to 5 is even in each case.

2.4. The combinatorics of block labels. Let $t = (p - 1)/2$. In [2], we define a directed graph with edges labeled by residues $\{0, 1, \dots, t\}$, called the *block-reduced crystal graph*, in which the vertices were labels ρ^ω . Using the theory of i -addable and i -removable nodes from [5], the graph contained an edge of residue i from ρ^ω to σ^v if and only if there was a restricted p -strict partition for ρ^ω with an i -addable node producing a p -strict partition for σ^v . The corresponding graph for $n \leq 12$ and $p = 5$ is given in Figure 2. The paper [2] considered the combinatorial properties of what were called maximal i -strings, which

were sequences of vertices of maximal length connected by edges of residue i . In this paper, in place of i -addable and i -removable nodes in a Young diagram, we will add or subtract 1 from a part congruent to i or $p - i$, which will be represented in Section 3 below by shifting a bead from the i or $p - i$ runner to an adjacent runner in the abacus notation described just before Definition 2.2.

For the usefulness of studying combinatorial relationships between the labels we cite the work of Kang, Kashiwara and Tsuchioka [11] in which they construct a family of quiver Hecke superalgebras in which there is one block for each label. They conjecture [11, page 3] that the blocks of the quiver Hecke superalgebras are all of type M . This was proven by Hill and Wang [9, subsection 6.5].

From paper [11], blocks are obtained which are Morita equivalent to the type M block of a given label. The Morita equivalence involves replacing a block of form $M_r(R)$ in the symmetric or alternating group, for r a power of 2, with the block algebra R , thus eliminating superfluous powers of two.

However, for the result of this paper, we do not need the full strength of these methods, or even the block-reduced crystal graph. We have produced Figure 2 here merely as an aid to understanding the meaning of i -strings and Scopes involutions. Instead of using crystals and affine Lie algebras as was done in the original thesis [15], for the purpose of this paper, in Definition 3.5, we define i -strings in terms of i -shifts in the abacus notation.

2.5. Scopes involutions.

Definition 2.5 ([2]). Let $t = (p - 1)/2$, where, as always, p is an odd prime, any p -core ρ can be represented by a *core t -tuple*

$$c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$$

where ℓ_i is the number of beads on the runner numbered i or $p - i$, and we set $\epsilon_i = 0$ if there are beads on runner i ; otherwise, we set $\epsilon_i = 1$. Note the rather counter-intuitive choice that, if there are no beads on either runner, so that $\ell_i = 0$, then $\epsilon_i = 1$. We will abbreviate $c(\rho(\lambda))$ by $c(\lambda)$.

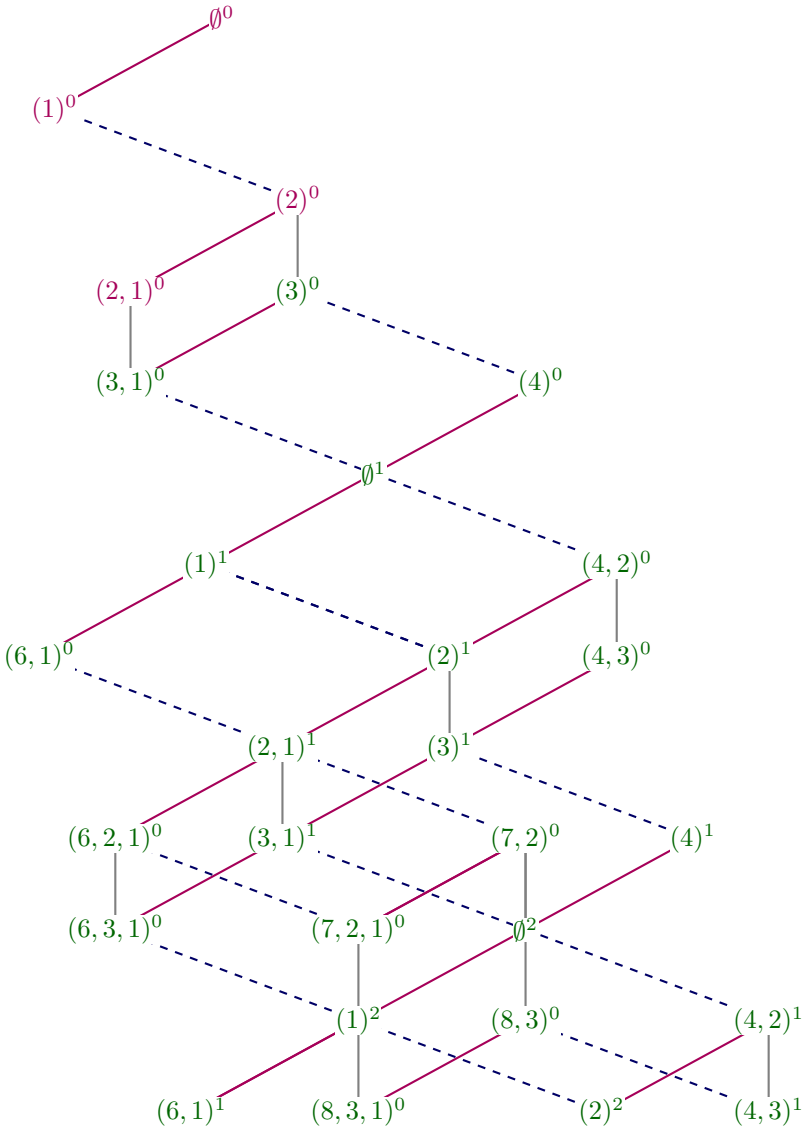


FIGURE 2. Block-reduced crystal graph for $p = 5$, $n \leq 12$.

In what follows, cores ρ will generally be represented by their core t -tuple $c(\rho)$, since the description of the actual partition is too bulky and does not exhibit special properties of the core. We give the core t -tuples for some of the cores in Figure 2:

- $\lambda = (6, 1), c(\lambda) = ((2, 0), (0, 1))$
- $\lambda = (4, 3), c(\lambda) = ((1, 1), (1, 1))$
- $\lambda = (6, 2, 1), c(\lambda) = ((2, 0), (1, 0))$
- $\lambda = (7, 2), c(\lambda) = ((0, 1), (2, 0))$.

Definition 2.6. Let D be the set of all p -strict partitions. For $0 \leq i \leq t$, we define *Scopes involutions* $K_i : D \rightarrow D$ by the following.

- For $0 < i < t$: the involution K_i interchanges beads on runner i and $i + 1$ as well as beads on runner $p - i$ and $p - i - 1$. For a core $\lambda \in D$, with core t -tuple

$$c(\lambda) = ((\ell_1, \epsilon_1), \dots, (\ell_i, \epsilon_i), (\ell_{i+1}, \epsilon_{i+1}), \dots, (\ell_t, \epsilon_t))$$

applying K_i gives a new core $\bar{\lambda}$ with core t -tuple

$$c(\bar{\lambda}) = ((\ell_1, \epsilon_1), \dots, (\ell_{i+1}, \epsilon_{i+1}), (\ell_i, \epsilon_i), \dots, (\ell_t, \epsilon_t)).$$

- For $i = t$: the involution K_t interchanges beads on runner $t + 1$ and runner t . In particular, for cores λ , we have $c(K_i(\lambda)) = c(\lambda)$ if $\ell_t = 0$, and otherwise, the core t -tuple becomes

$$c(\bar{\lambda}) = ((\ell_1, \epsilon_1), \dots, (\ell_t, 1 - \epsilon_t)).$$

- For $i = 0$: the involution K_0 moves beads in position $a \geq 1$ on runner 1 to the position $a - 1$ on runner $p - 1$ and, similarly, beads in position a' on runner $p - 1$ are moved to $a' + 1$ on runner 1. Furthermore, we add part 1 to $K_0(\lambda)$ if λ does not have part 1 and remove it if λ does have part 1, so that the total number of beads on runners 1 and $p - 1$ in λ differs by 1 from the number of beads on those runners in $K_0(\lambda)$. If λ is a core, we get that, if $\bar{\lambda} = K_0(\lambda)$, then

$$c(\bar{\lambda}) = ((\ell_1 - (-1)^{\epsilon_1}, 1 - \epsilon_1), (\ell_2, \epsilon_2), \dots, (\ell_t, \epsilon_t)).$$

Remark 2.7. For $i > 0$, the involution given here, restricted to strict partitions, is exactly that $\widetilde{S}_{c_{i+1}}$ defined in [12]. For $i = 0$, the map K_0 is actually an involution, unlike \widetilde{S}_{c_1} of [12].

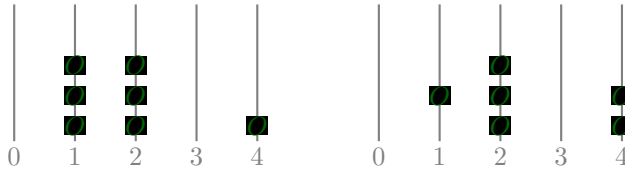


FIGURE 3. $\lambda = (12, 11, 7, 6, 4, 2, 1) \rightarrow K_0(\lambda) = (12, 9, 7, 6, 4, 2)$.

Example 2.8. For $p = 5$, let

$$\rho = (2), c(\rho) = ((0, 1), (1, 0))$$

be a partition and the core t -tuple of the p -core labeled by ρ^0 in Figure 1. Then,

$$\begin{aligned} K_0(\rho) &= (2, 1), c(K_0(\rho)) = ((1, 0), (1, 0)); \\ K_1(\rho) &= (1), c(K_1(\rho)) = ((1, 0), (0, 1)); \\ K_2(\rho) &= (3), c(K_2(\rho)) = ((0, 1), (1, 1)). \end{aligned}$$

Referring to Figure 2, one can see that Scopes involutions are reflecting strings in the diagram, where the heavy diagonal lines are 0-strings, the dotted diagonal lines are 1-strings and the vertical lines are 2-strings.

Example 2.9. For a more complicated example, for $p = 5$, let

$$\rho = (12, 7, 6, 2, 1), \lambda = (12, 11, 7, 6, 4, 2, 1)$$

be, respectively, the p -core and a partition in the block labeled by ρ^3 . Then,

$$\begin{aligned} K_0(\rho) &= (12, 7, 4, 2), K_0(\lambda) = (12, 9, 7, 6, 4, 2); \\ K_1(\rho) &= (11, 7, 6, 2, 1), K_1(\lambda) = (12, 11, 7, 6, 3, 2, 1); \\ K_2(\rho) &= (13, 8, 6, 3, 1), K_2(\lambda) = (13, 11, 8, 6, 4, 3, 1). \end{aligned}$$

A sample of Scopes involution K_0 in abacus notation is given in Figure 3.

In order for these involutions to be of use, we must show that they preserve block labels, which is equivalent to showing that they preserve cores.

Lemma 2.10. *If $\lambda \in D$ and, if $K_i(\lambda) = \chi$, then*

$$K_i(\rho(\lambda)) = \rho(\chi).$$

Proof. For $0 \leq i \leq p-1$, denote by n_i and n'_i the numbers of beads on runner i in the abacus representation of λ and χ , respectively. For $i > 0$, the lemma was proven in [12, Lemma 4.7], with changes in notation mentioned in Remark 2.7. For $i = 0$, we have $n'_1 = n_{p-1} + 1$ and $n'_{p-1} = n_1$ if 1 is not a part of λ and $n'_1 = n_{p-1}$ and $n'_{p-1} = n_1 - 1$, otherwise.

If $n_1 \leq n_{p-1}$, the core t -tuple of $\rho(\lambda)$ equals $((n_{p-1} - n_1, 1), \dots)$ and that of $\rho(\chi)$ equals $((n_{p-1} - n_1 + 1, 0), \dots)$, which is the image of the core $\rho(\lambda)$ under K_0 , as claimed in Lemma 2.10. If $n_1 > n_{p-1}$, then the core t -tuple of $\rho(\lambda)$ equals $((n_1 - n_{p-1}, 0), \dots)$ and that of $\rho(\chi)$ equals $((n_1 - n_{p-1} - 1, 1), \dots)$, again as claimed. \square

3. Equivalences of extremal spin blocks. The aim of this section is to determine combinatorial conditions under which a block label, in the language of [2], is extremal in a maximal i -string. Then, in the next section, we will show Scopes involution on such block labels ensures a source algebra equivalence of corresponding blocks. However, since we are avoiding the use of Lie algebra methods in this paper, we must first translate the concept of i -addable and i -removable nodes from [5] to the abacus notation. The degree of the Young diagram of a partition λ is $|\lambda|$, the sum of the parts. Since adding or removing a node from the Young diagram changes the degree by exactly 1, our basic move will be to shift a bead to an adjacent runner, which corresponds to increasing or decreasing the part corresponding to that bead by exactly 1.

3.1. The i -shifts.

Definition 3.1. Let $i = 0, \dots, t$, with, as usual, $t = (p-1)/2$, and let K_i be the corresponding Scopes involution.

(i) $i > 0$. An i -shift consists of moving a bead to an empty position at the same height on the adjacent runner with which its runner is interchanged by K_i and will be called an i -shift up or down according to whether the part represented by the bead is increased or decreased by the shift, and corresponding runners will be called *low* or *high*.

(ii) $i = 0$. A 0-shift down can be one of three possible types: removing part 1 on the 1-runner, moving a bead on the 1-runner to the 0 runner, or moving a bead on the 0-runner to an empty space on the $p - 1$ runner. A 0-shift up consists of moving a bead from the $p - 1$ runner to the 0 runner, moving a bead from the 0-runner to an empty space on the 1-runner, or adding a bead corresponding to part 1 if the 0 position on the 1-runner is open. In describing a 0-shift, the $p - 1$ runner will be called the *lowest runner*, the 0-runner will be called the *middle runner* and 1-runner will be called the *highest runner*.

Begin with a core ν of degree n such that $\mu = K_i(\nu)$ is of lower degree m , and generate all intermediate block labels in the i -string connecting the two cores. Then perform all possible i -shifts down on ν in random order to produce μ . As follows from the work of [5] the string of block labels of intermediate partitions would be independent of this choice of ordering. However, in order to extend the procedure to non-cores, Brundan and Kleshchev [5] introduced an ordering to the procedure of adding and removing nodes, which requires an ordering to the i -shifts as well. In the proofs involving i -shifts, we will make i -shifts up starting with the lowest part and i -shifts down starting with the highest part.

The basic order will be ascending for i -shifts up and descending for i -shifts down, but there is an additional subtlety which appears in their theory as the i -signature, and which we will translate into the abacus notation by recursively declaring certain beads to be inert while choosing one of the active beads for which an i -shift will be called “proper.” For those accustomed to the Brundan-Kleshchev signature theory, for $i > 0$, active beads on high runners correspond to $-$ and active beads on low runners correspond to $+$. The situation that we are trying to achieve, by eliminating “ $+ -$ ” pairs, is one in which all of the beads on high runners are numerically smaller than the beads on low runners. As usual, the situation for $i = 0$ is more complicated to describe.

Definition 3.2. (A proper i -shift.)

(i) $i \neq 0$. For $i > 0$, the original set of inert beads are those on runners not involved in an i -shift or those whose target position under K_i is already full. An active bead on a high runner can make itself inert as well as the first active bead on a low runner which is smaller than it in the ordering of parts, if such a bead exists. When all such pairs have been made inert, a proper i -shift up will move the smallest bead on a low runner to the adjacent high runner, and a proper i -shift down will move the largest bead on a high runner to the adjacent low runner.

(ii) $i = 0$. Initially, a bead on the middle runner will be *up*-inert if the adjacent position on the highest runner is full and otherwise it will be *up*-active. Similarly, it will be *down*-inert if the adjacent position on the lowest runner is full and otherwise it will be *down*-active. An *up*-active bead on the middle runner will make inert either the first active bead larger than it on a high runner or the first *down*-active bead on the middle runner, whichever comes first, if such a bead exists. An active bead on the lowest runner will make *down*-inert the first *down*-active bead on the middle runner larger than it or make inert the first active bead on the highest runner larger than it, whichever comes first, if it exists. When all such pairs have been made inert, a proper 0-shift up will move the smallest bead on a low runner to the adjacent high runner, and a proper 0-shift down will move the largest bead on a high runner to the adjacent low runner.

Remark 3.3. When we start with a core, the target is empty so there are no initially inert beads except those on irrelevant runners. If we make the i -shifts up from the bottom of low runners or from the top of the high runners, then we are always in the situation where the active beads on the high runners are smaller than the active beads on the low runners, and it is never necessary to invoke the recursion step. We have included the definition of a proper i -shift in order that i -strings will be well defined even when extremal points are not cores.

Example 3.4. Let $p = 5$, and consider the block with label $(3, 1)^{12}$. One of its restricted, p -strict partitions is $(18, 12, 10, 9, 6, 5, 3, 1)$. In Figure 4, we show the inert and active beads first for the 0-shift up to $(6, 3, 1)^{11}$ and then for the 2-shift down to $(2, 1)^{12}$. In the case of the 0-

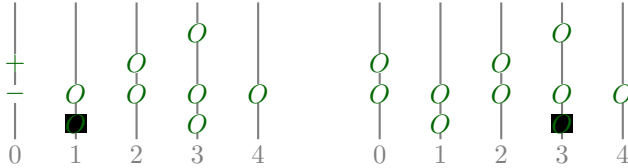


FIGURE 4. $\lambda = (18, 12, 10, 9, 8, 7, 6, 5, 3, 1)$, 0-shift up, 2-shift down.

shift up, the beads on runners 2 and 3 are initially inert. Bead 6 makes five *up*-inert, and bead 9 makes ten *down*-inert. The two active beads in the down direction are 1 and 5, which are both smaller than 10, the only active bead in the up direction. In order to make the 0-shift up, we increase 10 to 11, giving $(18, 12, 11, 9, 6, 5, 3, 1)$.

In the second case, 7 and 8 were initially inert, while 18 and 12 were eliminated by recursion because 12 was an active bead on a low runner which was below 18, an active bead on a high runner. This leaves 3, on a high runner, as the only active bead. In the diagram in Figure 4, we have indicated inert beads by circles, and on the 0-runner, a “-” marks an “up-inert, down-active” bead, while a “+” marks a “down-inert, up-active” bead.

Definition 3.5. If ρ^u and σ^u are two block labels with $K_i(\rho^u) = \sigma^u$ and $\deg(\rho) > \deg(\sigma)$, choose any restricted, p -strict partition λ for ρ^u . It defines an *i-string* of block labels obtained by making i -shifts down starting with the highest active bead until reaching σ and recording the block label after each i -shift. The same string could be obtained by starting with σ and taking i -shifts up, taking each time the lowest active bead. Such a string will be called a *maximal i-string* if it is not a substring of any other i -string. The block labels at the ends of a maximal i -string, together with their corresponding blocks, will be called *extremal* for that i -string, and those block labels in the interior will be called *internal* to the i -string.

Example 3.6. Consider the 0-string at the bottom of Figure 1, where, in this case, $p = 5$: $(4)^1 \rightarrow \emptyset^2 \rightarrow (1)^2 \rightarrow (6, 1)^1$. The block with label $(4)^1$ has two restricted p -strict labels of irreducibles, and for each, we

make a series of proper i -shifts up.

$$(5, 4) \longrightarrow (5, 4, 1) \longrightarrow (5, 5, 1) \longrightarrow (6, 5, 1),$$

$$(4, 3, 2) \longrightarrow (4, 3, 2, 1) \longrightarrow (5, 3, 2, 1) \longrightarrow (6, 3, 2, 1).$$

As the first step, part 1 was added to the 1-runner. At the second stage, part 4 on the 4-runner was moved to become 5 on 0-runner, and finally the 5 was pushed onto the 1-runner, where it became 6.

3.2. The w -allowed actions. Now we shall consider what restrictions are on w , in order that ρ^w be an extremal block label in its i -string, for $0 \leq i \leq t$. For the purposes of proving that the blocks corresponding to extremal block labels on an i -string are source algebra equivalent, we will formulate this condition in terms of ℓ_i , the number of parts in the core on runner i or $p - i$.

Later, in Section 5, we will formulate and prove a global condition, which will show that, for any w , there is an integer $N(w)$ such that any block label ρ^w occurring in degree $n \geq N(w)$ will be extremal.

Definition 3.7. When ρ^w is an extremal block label in its maximal i -string, we will say that the involution K_i is a w -allowed action for ρ^w .

The following technical lemma will be needed for the proof of the main theorem in Section 4. In this lemma, we fix a core ρ and a residue i and show that, for w larger than the bound established in the lemma, ρ^w is the label of a block which is internal in its i -string.

In Section 5, we will fix w while allowing i and ρ to vary, and show that, for all cores ρ such that ρ^w is the label of a block of degree n with $n \geq N(w)$, there exists an i for which ρ^w labels an extremal block in an i -string. In addition, for this i -string, the source algebra equivalent block corresponding to the label at the other end of the i -string will have lower degree, and this fact will allow us to establish an improved bound for Donovan's conjecture.

Lemma 3.8. *Let ρ be a p -core, with core t -tuple*

$$c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t)).$$

- (i) The involution K_0 is a w -allowed action for ρ^w if and only if $w \leq \ell_1 + \epsilon_1 - 1$.
- (ii) The involution K_t is a w -allowed action for ρ^w if and only if $w \leq 2\ell_t + 1$.
- (iii) The involution K_i , $1 < i < t$, is a w -allowed action for ρ^w if and only if:
 - $w \leq (\ell_{i+1} - \ell_i) \cdot (-1)^{\epsilon_i}$, for $\epsilon_i = \epsilon_{i+1}$,
 - $w \leq \ell_i + \ell_{i+1}$, for $\epsilon_i \neq \epsilon_{i+1}$.

Proof. The core ρ is fixed. From [2], we know that the length of the string containing ρ^w is nondecreasing in w and that, whenever the string increases, there is one block label of weight 0 added at each end. Let $v_0(i)$ be the maximal weight w such that ρ^w is extremal in direction i , and let $v_1(i)$ be $v_0(i) + 1$, which is the minimal weight w such that ρ^w is internal in its i -string. By definition, only for extremal block labels do we get a w -allowed action, i.e., for block labels with weight w satisfying $w \leq v_0(i)$. For every $w \geq v_1(i)$, ρ^w will be internal because the string lengths are non-decreasing in w [2].

We assume i is fixed and write v_1 for $v_1(i)$. We investigate ρ^{v_1} . By the minimality of v_1 , we see that the block label which bounds it must be of weight 0, and we denote it by μ^0 . Thus, μ must be a core such that moving one bead to an adjacent runner produces the abacus representation of an element λ of D which reduces in v_1 moves to the core ρ . Since μ , which is a core, has no beads at all on the 0 runner, the resulting element λ is in fact a strict partition, since, for elements of D , all multiple parts must be divisible by p . For each i , there are two possible cases. The core μ can be at the high degree end of the string and require an i -shift down to reach the block label with core ρ , or it can be as the low-degree end of the string and require an i -shift up to reach the block label with core ρ .

(i) $i = 0$: First assume that a 0-shift down from block label μ^0 will give ρ^{v_1} , so the proper 0-shift down exists and must move a bead from runner 1 to the 0-runner. This requires that the core μ have $\ell'_1 > 0$ beads on runner 1, and thus, the core t -tuple has the form

$$c(\mu) = ((\ell'_1, 0), (\ell_2, \epsilon_2), \dots, (\ell_t, \epsilon_t)).$$

The change which produces a restricted partition λ in ρ^{v_1} corresponds in terms of the abacus to removing the upper bead on runner 1 and

placing it parallel in the 0 runner. When this is done, the bead goes down $\ell'_1 - 1$ times and disappears, and $v_1 = \ell'_1 - 1$, see Definition 2.2. Recalling that

$$c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t)),$$

we thus have $\ell_1 = \ell'_1 - 1$, so $v_1 = \ell_1$ and $v_0 = \ell_1 - 1 = \ell_1 + \epsilon_1 - 1$. The block label ρ^w is internal if and only if $w \geq \ell_1$.

Now assume that a 0-shift up from μ^0 will give ρ^{v_1} . This requires that μ have the form

$$c(\mu) = ((\ell'_1, 1), (\ell_2, \epsilon_2), \dots, (\ell_t, \epsilon_t)),$$

with $\ell'_1 > 0$ (since otherwise the resulting partition is also a core, with weight 0, and is not internal). The bead which is added is part 1. This cancels the lowest bead on runner $p - 1$, which gives one move, and the remaining $\ell'_1 - 1$ beads move down, so now we get $v_1 = \ell'_1 = \ell_1 + 1$, and $v_0 = \ell_1$. Since, in this case, $\epsilon_1 = 1$, this gives the formula in the statement of the lemma.

(ii) $i = t$: Every t -shift down from a core μ^0 for

$$c(\mu) = ((\ell_1, \epsilon_1), \dots, (\ell'_t, 1))$$

corresponds in terms of the abacus to removing the upper bead on runner $t + 1$ and putting it parallel in runner t . When this is done, the bead goes down $\ell'_t - 1$ times, and then the two bottom-most beads on runners t and $t + 1$ are removed, all of the beads on runner $t + 1$ go down, and the total number of moves is by $v_1 = 2\ell'_t - 2$. Let ρ^{v_1} be the block label obtained by the t -shift down. It will satisfy $\ell_t = \ell'_t - 2$, so $v_1 = 2\ell_t + 2$, and thus, $v_0 = 2\ell_t + 1$. The procedure making a t -shift up is similar, except that it is the bottom bead which moves from the t runner to the $t + 1$ runner. The total number of moves is the same.

(iii) $0 < i < t$: Every i -shift down from a block label μ^0 for $c(\mu) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$ corresponds in terms of the abacus to moving a bead from runner $i + 1$ to runner i or to moving a bead from runner $p - i$ to $p - i - 1$.

- First we examine the case that $\epsilon_i = \epsilon_{i+1}$:

Case 1. $\epsilon_i = \epsilon_{i+1} = 0$: In this case, an i -shift down corresponds to removing the upper bead in runner $i + 1$ and placing it parallel in runner i . This is only possible if $\ell'_{i+1} > \ell'_i$. When this is done, the bead goes down until there are no empty places, i.e., $\ell'_{i+1} - \ell'_i - 1$

moves. Let ρ^{v_1} be the block label after the shift down from the core μ . The core t -tuple of ρ satisfies $\ell_{i+1} = \ell'_{i+1} - 1$ and $\ell_i = \ell'_i + 1$; thus, $v_1 = (\ell_{i+1} - \ell_i + 1)$, and $v_0 = \ell_{i+1} - \ell_i$. The case of an i -shift up is similar.

Case 2. $\epsilon_i = \epsilon_{i+1} = 1$: In this case, an i -shift down corresponds to removing the upper bead in runner $p-i$ and placing it parallel in runner $p-i-1$. This is only possible if $\ell'_i > \ell'_{i+1}$. When this is done, the bead goes down until there is no empty place, i.e., $v_1 = \ell'_i - \ell'_{i+1} - 1$ times. Let ρ^{v_1} be the block label after removing an i -good node from the core μ . It satisfies $\ell_i = \ell'_i - 1$ and $\ell_{i+1} = \ell'_{i+1} + 1$; thus, $v_1 = \ell_i - \ell_{i+1} + 1$, and $v_0 = \ell_i - \ell_{i+1}$. The case of an i -shift up is similar.

- Finally, we examine the case where $\epsilon_i \neq \epsilon_{i+1}$. In order for an i -shift down to exist, the runners i and $p-i-1$ must be empty. Let $a_1p + (i+1)$ be the upper bead of runner $i+1$ and $a_2p + (p-i)$ the upper bead of runner $p-i$.

Case 1. $a_1p + (i+1) > a_2p + (p-i)$: In this case, an i -shift down corresponds to removing the upper bead from runner $i+1$ and placing it parallel in runner i . When this is done, the bead goes down $\ell'_{i+1} - 1$ times, then the bottom-most bead of runner $p-i$ and the bead in runner i are removed, and all of the beads on runner $p-i$ again go down ($\ell'_i - 1$ times).

Case 2. $a_1p + (i+1) \leq a_2p + (p-i)$: In this case, an i -shift down corresponds to removing the upper bead from runner $p-i$ and placing it parallel in runner $p-i-1$. When this is done, the bead goes down $\ell'_i - 1$ times, then the bottom-most bead of runner $i+1$ and the bead in runner $p-i-1$ are removed, and all of the beads on runner $i+1$ again go down ($\ell'_{i+1} - 1$ times).

In both cases, the weight is $v_1 = \ell'_i + \ell'_{i+1} - 1$. Let ρ^w be the block label after an i -shift down from the core μ . Its core t -tuple satisfies $\ell_i = \ell'_i - 1$ and $\ell_{i+1} = \ell'_{i+1} - 1$; thus, $v_1 = \ell_i + \ell_{i+1} + 1$ and $v_0 = \ell_i + \ell_{i+1}$.

The procedure for an i -shift up is similar, except that the bead to be transferred is taken from the first place which is empty on the runner with which we are making the exchange. □

4. Source algebra equivalences. Kessar [12] proved in certain cases that extremal blocks of maximal i -strings in the block-reduced crystal graph are source algebra equivalent. We now strengthen this

result so that it applies to every pair of extremal blocks in an i -string. In this section, we will demonstrate that a w -allowed action corresponds to a source algebra equivalence between appropriately chosen blocks labeled by the given cores.

Unlike in the previous section, we will now be concerned with char 0 representations, and thus, with strict partitions. Furthermore, we will no longer be concerned with making proper i -shifts and thus will not need all of the consideration of active or inert beads. Fix i and let λ and χ be two strict partitions such that $n = |\lambda| > |\chi| = m$. We now define $\mathcal{M}^{n-m}(\lambda, \chi)$ to be the set of possible paths leading from the abacus notation for λ to the abacus notations for χ by moving beads one at a time from a high runner for K_i to a low runner in such a way that, at every stage, we have a strict partition. The number will be 0 if there is no way to get from λ to χ by moving beads, in which case $K(\lambda) \neq \chi$.

The next definition is the key to proving the result we desire. Let J_n be the set of strict partitions with core ν and weight w , where $n = |\nu| + wp$, and let J_m be the set of strict partitions with core μ and weight w , where $m = |\mu| + wp$.

Definition 4.1 ([13]). A w -compatible pair (ν, μ) for i is defined to be a pair of cores such that:

- (i) $K_i : J_n \rightarrow J_m$ is one-to-one and onto, and $K_i(\nu) = \mu$.
- (ii) For any $\lambda \in J_n$ and $\chi \in J_m$,

$$|\mathcal{M}^{n-m}(\lambda, \chi)| = \begin{cases} 0 & \text{if } \chi \neq K_i(\lambda); \\ |\mathcal{M}^{n-m}(\nu, \mu)| & \text{if } \chi = K_i(\lambda). \end{cases}$$

- (iii) $\epsilon(\lambda) + \epsilon(K_i(\lambda)) = \epsilon(\mu) + \epsilon(\nu)$.

We shall now prove that, if K_i gives a w -allowed action for ν^w , then the pair of cores (ν, μ) , is a w -compatible pair. In the remainder of this section we will show that the w -compatible pair gives the desired source algebra equivalence.

We first prove some lemmas which will be needed to establish this result.

Lemma 4.2. *Let ν be a core, and let $i \in \{0, 1, \dots, t\}$ be such that $\mu = K_i(\nu)$ is different from ν and $|\mu| < |\nu|$. Then*

$$|\mathcal{M}^{n-m}(\nu, \mu)| = \begin{cases} (n - m)! & \text{if } i \neq 0; \\ (n - m)!/2^{(n-m-1)/2} & \text{if } i = 0. \end{cases}$$

Proof.

(i) $i > 0$: In this case, there are $n - m$ extra beads on the high runner, which can each be moved independently of the others, so we obtain $(n - m)!$ possible orders in which we can move these beads.

(ii) $i = 0$: We try to count the number of ways to get from the abacus for ν to the abacus for μ by moving one bead at a time. Except for part 1, which is destroyed in a single shift, all of the other i -shifts come in pairs: a shift to runner 0 and a further shift to runner $p - 1$. Thus, $n - m$ is odd, and we have $(n - m - 1)/2$ pairs of i -shifts. The number of possible orders in which we could make these moves if there were no restriction on the pairs would be $(n - m)!$. However, for each adjacent pair, we must first move the bead to the 0-runner and then later to the one $p - 1$ -runner, a consideration which divides the number of permissible orderings by 2. For each adjacent pair, we must divide by 2, so in total we must divide by $2^{(n-m-1)/2}$. This reduces the number of permissible orderings to $(n - m)!/2^{(n-m-1)/2}$. □

Remark 4.3. Let there be given a core t -tuple

$$c(\nu) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t)).$$

If $\mu = K_i(\nu)$ with $n \neq m$, then the number $|\mathcal{M}^{n-m}(\nu, \mu)|$ of possible paths by which to reduce beads is positive in the following cases:

(i) $K_i, 0 < i < t$:

- $\epsilon_i = 1, \epsilon_{i+1} = 0$, and we note that, by our convention, $\epsilon_{i+1} = 0$ implies that $\ell_{i+1} > 0$.

- $\epsilon_i = \epsilon_{i+1} = 0, \ell_{i+1} > \ell_i$.

- $\epsilon_i = \epsilon_{i+1} = 1, \ell_{i+1} < \ell_i$.

(ii) $K_0, \ell_1 > 0, \epsilon_1 = 0$.

(iii) $K_t, \ell_t > 0, \epsilon_t = 1$. The complementary cases all increase the rank. For example, when $\epsilon_i = 0, \epsilon_{i+1} = 1$, we have $\ell_i > 0$. There are

no w -allowed actions which leave the rank fixed, except those which are trivial because there are identical configurations of beads on each pair of interchanged runners. The K_0 action is never trivial.

As in Definition 4.1, let J_n be the set of strict partitions with core ν and weight w , and let J_m be the set of strict partitions with core μ and weight w . Take $\lambda \in J_n$ and $\chi \in J_m$, where $n > m$ by our assumptions on ν and μ . To go back from ν to λ (or from μ to χ) we must perform w moves which correspond to adding p -bars. There are three kinds of moves:

- moving a bead up on its runner;
- inserting a pair of beads to runners i and $p - i$ where the bottom place in each runner is empty;
- creating a bead on runner 0.

Lemma 4.4. *Let ν be a core. The number of actions needed to insert n pairs of beads on runners $i, p - i$ for some $i > 0$ in the abacus representation of ν is:*

$$n^2 + \ell_i n.$$

Proof. By induction.

For $n = 1$: To insert one pair, we must lift the ℓ_i beads on runner i or on runner $p - i$ upwards, that is, ℓ_i moves and then insert the pair, that is, one more move. Altogether, we make $\ell_i + 1$ moves.

Assume that Lemma 4.4 is true for n , i.e., the number of actions needed to insert n pairs of beads on runners $i, p - i$ in the abacus representation of ν is $n^2 + \ell_i n$.

Now we prove that Lemma 4.4 is true for $n + 1$. To insert a pair after inserting n pairs: First we must lift the $n + \ell_i$ beads that are on runner i or runner $p - i$ upwards, that is, $n + \ell_i$ actions; second, we must lift the n beads that are on the other runner upwards, that is, another n moves. We can then insert the new pair, another move, doing $2n + \ell_i + 1$ new moves in total. Inserting n pairs, by the assumption $n^2 + n\ell_i$ moves, we have

$$n^2 + n\ell_i + 2n + \ell_i + 1 = (n + 1)^2 + \ell_i(n + 1). \quad \square$$

Corollary 4.5. *Suppose that K_i is a w -allowed action for ν^w .*

It is not possible to insert a pair of beads on runners j and $p - j$ in the following cases:

- (i) $j = i$ for $0 < i < t$, $\epsilon_i = \epsilon_{i+1} = 1$ and $\ell_i > \ell_{i+1}$.
- (ii) $j = i$ for $0 < i < t$, $\epsilon_i = \epsilon_{i+1} = 0$ and $2\ell_i + 1 \geq \ell_{i+1} > \ell_i$.
- (iii) $i = 0$ and $j = 1$.

Nor is it possible to insert more than one pair of beads on runners t and $t + 1$ when $\ell_t > 0$.

Proof. We divide the proof into cases, using Lemma 3.8 to bound w .

- $i \neq t$: (i) In this case, we know that $w \leq \ell_i - \ell_{i+1}$ and, inserting this pair is $\ell_i + 1$ moves, i.e., $\ell_i + 1 \leq w \leq \ell_i - \ell_{i+1}$, which implies that $\ell_{i+1} \leq -1$, a contradiction to the definition of the ℓ_i .

- In this case, we know that $w \leq \ell_{i+1} - \ell_i$ and, inserting this pair is $\ell_i + 1$ actions, i.e., $\ell_i + 1 \leq w \leq \ell_{i+1} - \ell_i \Rightarrow 2\ell_i + 1 \leq \ell_{i+1}$.

- In this case, we know that $w \leq \ell_1 - 1$ if $\epsilon_1 = 0$ and $w \leq \ell_1$ if $\epsilon_1 = 1$. Inserting this pair is $\ell_1 + 1$ actions, i.e., $\ell_1 + 1 \leq w \leq \ell_1 \Rightarrow 1 \leq 0$, a contradiction.

- $i = t$: In this case, we know that $w \leq 2\ell_t + 1$ and, inserting more than one pair, is at least $2\ell_t + 4$ actions, a contradiction. \square

Definition 4.6. We say that a bead can be *reduced* if and only if the bead can move from runner i to an empty place at the same height, in runner $i - 1$ for $2 < i < p - 1$, or from runner 1 to an empty place at one height less, in runner $p - 1$ for height bigger than 1, or from runner 1 at height 1 to disappearance of the bead. As mentioned in the proof of Lemma 4.4, the move from runner 1 to runner $p - 1$ is actually made in two steps: first to runner 0 and then to runner $p - 1$. This action will be called to *reduce a bead*. Such a reduction reduces the rank by 1 or 2, the latter only when moving the bead from runner 1 to runner $p - 1$.

We now have enough background to prove one of the main results of this section:

Proposition 4.7. *If K_i gives a w -allowed action with respect to ν^w , then (ν, μ) is a w -compatible pair via K_i .*

Proof.

(i) K_i is an involution. By Lemma 2.1, we see that, for every strict partition $\lambda \in J_n$, there is a suitable strict partition $\chi \in J_m$ obtained by K_i .

(ii) We need to compare $|\mathcal{M}^{n-m}(\lambda, \chi)|$ and $|\mathcal{M}^{n-m}(\nu, \mu)|$. The strict partition λ is obtained from the core ν by adding w p -bars. To the extent that these moves take place on runners not affected by K_i , the same moves will be involved in obtaining χ from μ . Thus, the only moves which affect the possibility of reducing beads are on runners affected by K_i , or on runner 0 in the case of $i = 0$. We consider the cases listed in Remark 4.3, the only relevant cases when K_i reduces the rank.

(a) $K_i, 0 < i < t$:

- *Case 1:* $\epsilon_i = 1, \epsilon_{i+1} = 0, l_i + l_{i+1} > 0$. Moving beads up does not block any bead reductions. By Corollary 4.5, we cannot add pairs unless $l_i > l_{i+1}$. If we add $d > 0$ pairs to ν , then we must use up $dl_i + d^2$ moves, and this is $\geq l_i + d$; thus,

$$l_i + d \leq w \leq l_i + l_{i+1}.$$

The number of moves remaining is then

$$\leq l_{i+1} - d.$$

Therefore, in the remaining moves on runner i , the topmost bead cannot rise above the topmost bead in runner $i + 1$. The number of bead on runner $i + 1$ which can be reduced decreases by d , but the number of beads which can be reduced on runner i increases by the same d , so the total number of beads which can be reduced is the same as in ν .

- *Case 2:* $\epsilon_i = \epsilon_{i+1} = 0, l_{i+1} > l_i$. Since w is the positive difference, we cannot move any beads on runner i up past the topmost bead of runner $i + 1$. We can add pairs only to runner i and only when

$$d \leq dl_i + d^2 \leq w \leq l_{i+1}.$$

The beads whose reduction is blocked on runner $i + 1$ will be compensated for by d new beads on runner i , thus again, the total number will be the same.

- *Case 3:* $\epsilon_i = \epsilon_{i+1} = 1, \ell_{i+1} < \ell_i$. This case is similar to Case 2, with i and $i + 1$ reversed.

(b) $K_0, \ell_1 > 0, \epsilon_1 = 0$. In this case, there are no pairs which can be added, and no beads on runner $p - 1$ to block the reduction of beads from runner 1. However, here, we must deal with the problem of beads on runner 0, in the manner described above.

(c) $K_t, \ell_t > 0, \epsilon_t = 1$. In this case, merely moving beads up will not block any reductions. If a pair is added, it is only one, and adding it uses up $\ell_t + 1$ moves, while $w \leq 2\ell_t + 1$; thus, the new bead on runner t cannot rise above the topmost bead of runner $t + 1$.

(iii) As stated before, λ belongs to the block with label ν^w , and χ to the block with label μ^w . For every core ρ , we define $\epsilon(\rho) \equiv |\rho| + h(\rho) \pmod{2}$. Adding a p -bar to the core ρ may be done in one of two ways: One is lifting up one bead on its runner so that the number of parts does not change but $|\rho|$ becomes $|\rho| + p$. The other is adding a pair of beads $(i, p - i)$, so $h(\rho)$ becomes $h(\rho) + 2$ and $|\rho|$ becomes $|\rho| + p$, recalling that p is odd. In both ways, the parity of the partition is changed from even to odd, or from odd to even. For w additions of p we can summarize the result as follows. If w is even, the parity changes w times, and therefore $\epsilon(\rho^w) = \epsilon(\rho)$; and, if w is odd, then the parity changes according to the scheme $\epsilon(\rho^w) = 1 - \epsilon(\rho)$. Therefore, for an even w , we obtain $\epsilon(\lambda) + \epsilon(\chi) = \epsilon(\nu) + \epsilon(\mu)$, and for odd w , we obtain $\epsilon(\lambda) = 1 - \epsilon(\nu)$ and $\epsilon(\chi) = 1 - \epsilon(\mu)$. Thus,

$$\epsilon(\lambda) + \epsilon(\chi) = 1 - \epsilon(\nu) + 1 - \epsilon(\mu) \equiv \epsilon(\nu) + \epsilon(\mu) \pmod{2}. \quad \square$$

We now arrive at the source algebra equivalence and the problem of crossovers. As mentioned in the introduction we will work over a modular system (K, \mathcal{O}, F) .

Let ν be a core of rank $n - wp$, and let μ be a core of rank $m - wp$, with $n > m$. A block idempotent is a centrally primitive idempotent in the center of the group algebra, and the center of $\mathcal{O}\tilde{A}_n$ is also the center of $\mathcal{O}\tilde{S}_n$ [12]. Let $b \in \mathcal{O}\tilde{A}_n$ represent the block idempotent of B_{ν^w} or B'_{ν^w} , and let $c \in \mathcal{O}\tilde{A}_m$ represent the block idempotent of B_{μ^w}

or B'_{μ^w} . If K_i is a w -allowed action for ν^w with $\mu = K_i(\nu)$, then we must try to prove that one of the following holds:

- (i) if the parities are the same, then B_{ν^w} is source algebra equivalent to B_{μ^w} , and B'_{ν^w} is source algebra equivalent to B'_{μ^w} .
- (ii) If the parities are different, then B_{μ^w} is source algebra equivalent to B'_{μ^w} , and B'_{ν^w} is source algebra equivalent to B_{μ^w} ,

The first set of equivalences was essentially proven in [12]. However, since the main thrust of that paper was the Donovan conjecture, it is rather hard to extract the particular result that we need. Therefore, we recast our results in a form which will allow us to apply [10, Theorem 2.5]. For the second set of equivalences, we will cite [13], which used permutation modules.

For any strict partition λ , when $\epsilon(\lambda) = 0$, we let θ_λ^+ and η_λ^\pm be the corresponding irreducible character or characters of \tilde{S}_n or \tilde{A}_n , and the characters η_λ^\pm are conjugate. When $\epsilon(\lambda) = 1$, we let θ_λ^\pm and η_λ^+ be corresponding irreducible characters or character of \tilde{S}_n or \tilde{A}_n , respectively. The characters θ_λ^\pm are called *associate* and are actually conjugate when \tilde{S}_n is embedded in \tilde{S}_{n+2} .

Definition 4.8 ([10]). Let λ be a strict partition of n and χ a strict partition of m , with $n > m$. Let θ_λ^\pm be an irreducible character of \tilde{S}_n corresponding to λ and θ_χ^\pm an irreducible character of \tilde{S}_m corresponding to χ . Let u be an idempotent of $(\mathcal{O}\tilde{S}_n)^{\tilde{S}_m}$. We define

$$r(\theta_\lambda^\pm, \theta_\chi^\pm, u) = \begin{cases} 0 & \text{if } \chi \neq K_i(\lambda); \\ \langle \text{Ind}_{\tilde{S}_m}^{\tilde{S}_n}(\theta_\chi^\pm) |_{K\tilde{S}_n u}, \theta_\lambda^\pm \rangle & \text{if } \chi = K_i(\lambda), \end{cases}$$

which is equal to the number of constituents of the irreducible module V_λ^\pm representing θ_λ^\pm in the projective module $K\tilde{S}_n u \otimes U_\chi^\pm$, where U_χ^\pm is the irreducible module representing the character θ_χ^\pm . The corresponding number for \tilde{A}_n will be denoted by r' .

We restrict ourselves henceforward to the case where the parities are the same; thus, either both characters belong to pairs or both are single.

In order to quote Theorem 2.5 from [10] we must give a few general definitions. Let G be a group and H a subgroup containing a p -group D . Set $A = \mathcal{O}G$, and let

$$Br_D : A^D \longrightarrow FC_G(D)$$

be the Brauer homomorphism. For any idempotent u of $(\mathcal{O}G)^H$, we let $m_{H,D}(u\mathcal{O}Gu)$ be the number of idempotents i in a primitive idempotent decomposition of u in $(u\mathcal{O}Gu)^H$ for which $Br_D(i)$ is non-zero.

As for the case to which we wish to apply the theorem, we let b be the idempotent of a block of $\mathcal{O}G$ and let c be the idempotent of a block of $\mathcal{O}H$, assuming that the two blocks have a common defect group D . The product bc commutes with H because it is an idempotent, since b is central in $\mathcal{O}G$, and c is central in $\mathcal{O}H$. Thus, bc is an idempotent in $(\mathcal{O}G)^H$. We denote by $\text{Irr}(G, b)$ the set of K -irreducibles for the group G which lie in the block with idempotent b .

In what follows, we will take $u = bc$. Then $bc\mathcal{O}Gbc$ is a subalgebra of the block algebra $b\mathcal{O}Gb$ and, because b is central, can be written in the form $c\mathcal{O}Gbc$. The source algebra idempotent, which is found in [10, Proof of Theorem 2.5] and which gives the source algebra equivalence, will be an idempotent of $(c\mathcal{O}Gbc)^H$.

We set $G = \tilde{S}_n$ and $H = \tilde{S}_m$. For any core ρ , let B_{ρ^w} be the block algebra of $\mathcal{O}\tilde{S}_n$ with core ρ and weight w , and let $B_{\rho^{w'}}$ be the corresponding block of $\mathcal{O}\tilde{A}_n$ with core ρ and weight w .

Lemma 4.9. *Let $\alpha := n - m$, and let $\beta := |\mathcal{M}^\alpha(\nu, \mu)|$. Suppose that ν and μ form a w -compatible pair and that $\epsilon(\nu) = \epsilon(\mu)$. Write $\epsilon(\alpha) = 0$ if α is even, and $\epsilon(\alpha) = 1$ otherwise. Then $|\text{Irr}(\tilde{S}_n, b)| = |\text{Irr}(\tilde{S}_m, c)|$ and, for each $\theta_\lambda^\pm \in \text{Irr}(\tilde{S}_n, b)$,*

$$\sum_{\theta_\tau^\pm \in \text{Irr}(\tilde{S}_m, c)} r(\theta_\lambda^\pm, \theta_\tau^\pm, bc) = 2^{(\alpha - \epsilon(\alpha))/2} \beta,$$

and, for each $\theta_\chi^\pm \in \text{Irr}(\tilde{S}_m, c)$,

$$\sum_{\theta_\tau^\pm \in \text{Irr}(\tilde{S}_n, b)} r(\theta_\tau^\pm, \theta_\chi^\pm, bc) = 2^{(\alpha - \epsilon(\alpha))/2} \beta.$$

Proof. Let λ be a strict partition of n and χ a strict partition of m . Let θ_λ^\pm be an irreducible character of \tilde{S}_n corresponding to λ and θ_χ^\pm an irreducible character of \tilde{S}_m corresponding to χ . It follows from the branching rules for spin representations, see, for example, [18], that, if θ_λ^\pm is a constituent of $\text{Ind}_{\tilde{S}_m}^{\tilde{S}_n}(\theta_\chi^\pm)$, then there is at least one path from λ to χ so $\mathcal{M}^\alpha(\lambda, \chi)$ is non-empty, which implies by Definition 4.1 that $\mathcal{M}^\alpha(\nu, \mu)$ is also non-empty.

(i) $\alpha \neq 1$. Since in our situation ν and μ have the same parities, α is odd if $i = 0$ and even if $i > 0$, for i such that $K_i(\mu) = \nu$. We can calculate the coefficients of induced characters using the branching rules in [18]. Stembridge shows that, if we can build λ up step-by-step from χ , then there will be a character for λ occurring in the module induced from χ . In Proposition 4.1, we precisely showed that, in the case of a w -allowed action, λ could be built up by i shifts. Furthermore, since the module was induced from the block with idempotent c , it is not destroyed by multiplication by c , so $r(\theta_\lambda^\pm, \theta_\chi^\pm, bc)$ would be expected to be non-zero. The difficulty lies in counting exactly how large it is.

If α is even, then $i \neq 0$, so that λ and χ have the same number of parts, and the multiplicity of θ_λ^\pm as a constituent of $\text{Ind}_{\tilde{S}_m}^{\tilde{S}_n}(\theta_\chi^\pm)$ is $2^{\alpha/2}\beta$ if $\epsilon(\chi) = 0$ and is $2^{\alpha/2-1}\beta$ if $\epsilon(\chi) = 1$.

If α is odd, then $i = 0$ and λ has one more part than χ . Since $\alpha > 1$, the multiplicity of θ_λ^\pm as a constituent of $\text{Ind}_{\tilde{S}_m}^{\tilde{S}_n}(\theta_\chi^\pm)$ is $2^{\alpha-1/2}\beta$ if $\epsilon(\chi) = 0$ and $2^{\alpha-3/2}\beta$ if $\epsilon(\chi) = 1$.

These branching rules may be summarized in a single formula, recalling that $\epsilon(\alpha)$ is the parity of α . Recall that $\beta = |\mathcal{M}^\alpha(\nu, \mu)|$. Then, if $\alpha \neq 1$,

$$r(\theta_\lambda^\pm, \theta_\chi^\pm, bc) = 2^{(\alpha - \epsilon(\alpha) - \epsilon(\lambda) - \epsilon(\chi))/2} \beta.$$

We now verify that the sums in the statement of Lemma 4.9 are correct. When $\epsilon(\chi) = 0$, there is only one character in each of the sums in the statement of the lemma with non-zero coefficients, and that coefficient has the correct value required. When $\epsilon(\chi) = 1$, there are exactly two with non-zero coefficients, and the sum of those two coefficients has the correct value.

(ii) $\alpha = 1$. We have $\beta = 1$ since the only way to have the parities preserved is for ν to be obtained from μ by adding part 1. In this case, θ_χ^+ lifts to a unique character, either θ_λ^+ or θ_λ^- , with multiplicity 1 (and similarly for θ_χ^- , when $\epsilon(\chi) = 1$).

In the special case where $\alpha = 1$ and $\epsilon(\chi) = 0$,

$$r(\theta_\lambda^+, \theta_\chi^+, bc) = 1,$$

giving both sums in the lemma (recalling that $\beta = 1$ and $\alpha - \epsilon(\alpha) = 0$).

In the special case that $\alpha = 1$ and $\epsilon(\chi) = 1$, there is a specific correspondence between paired characters; thus,

$$r(\theta_\lambda^+, \theta_\chi^+, bc) + r(\theta_\lambda^+, \theta_\chi^-, bc) = 1,$$

and

$$r(\theta_\lambda^-, \theta_\chi^+, bc) + r(\theta_\lambda^-, \theta_\chi^-, bc) = 1,$$

which give the first sum in the statement of Lemma 4.9.

It is equally true that

$$r(\theta_\lambda^+, \theta_\chi^+, bc) + r(\theta_\lambda^-, \theta_\chi^+, bc) = 1,$$

and

$$r(\theta_\lambda^+, \theta_\chi^-, bc) + r(\theta_\lambda^-, \theta_\chi^-, bc) = 1,$$

which gives the second sum in the statement of Lemma 4.9.

Similar results for \tilde{A}_n and \tilde{A}_m can be proven in an almost identical fashion, by standard Clifford theory. It is done in full in [15]. \square

Theorem ([10, Theorem 2.5]). *Let u be an idempotent of $(c\mathcal{O}Gbc)$. Then, for any character $\phi \in \text{Irr}(G, b)$,*

$$\sum_{\psi \in \text{Irr}(H, c)} r(\phi, \psi, u) \geq m_{H, D}(u\mathcal{O}Gu),$$

and, for any character $\psi \in \text{Irr}(H, c)$ we have

$$\sum_{\phi \in \text{Irr}(G, b)} r(\phi, \psi, u) \geq m_{H, D}(u\mathcal{O}Gu).$$

Further, if $|\text{Irr}(H, c)| = |\text{Irr}(G, b)|$, and if

$$\sum_{\substack{\psi \in \text{Irr}(H, c) \\ \phi \in \text{Irr}(G, b)}} r(\phi, \psi, u) \leq m_{H, D}(u\mathcal{O}Gu)|\text{Irr}(H, c)|,$$

then, for any primitive idempotent i of $(u\mathcal{O}Gu)$, the image $\text{Br}_D(i)$ is non-zero and the map $\mathcal{O}Hc \rightarrow i\mathcal{O}Gi$ given by $x \mapsto ix$ is an isomorphism of interior H -algebras. In particular, if the above holds, then $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are source algebra equivalent.

Since it is assumed that (ν, μ) is a w -compatible pair, the situation of the second part of the theorem is now addressed. We have already calculated the sum in Lemma 4.9. In order to complete the proof, we need the next lemma, inspired by an unpublished result by Kessar from an early version of [13].

Lemma 4.10. *With the notation above,*

- if α is odd, $m_{H, D}(\mathcal{O}Gbc) = 2^{\alpha-1/2}\beta$.
- If α is even, $m_{H, D}(\mathcal{O}Gbc) = 2^{\alpha/2}\beta$.

Proof. By Brauer’s main theorem, see, for example, [1], the image $\text{Br}_D(b)$ of the block idempotent b in $FC_G(D)$ is the $N_G(D)$ conjugacy class sum of block idempotents of blocks whose image under the projection $\pi : FC_G(D) \rightarrow FC_G(D)/Z(D)$ is of defect 0. Let \bar{b} be $\pi \circ \text{Br}_D$, and define \bar{c} in a similar fashion for H .

In order to apply [10, Theorem 2.5] in the proof of our main theorem, we must first calculate the quantity $m_{H, D}(c\mathcal{O}Gbc)$, which is the number of primitive idempotents in an idempotent decomposition of $(c\mathcal{O}Gbc)^H$, non-zero under the action of $\pi \circ \text{Br}_D$. By [10, Proposition 2.6], this is the same as the number of idempotents in a primitive idempotent decomposition of the algebra

$$(\bar{c}F(C_G(D)/Z(D)\bar{b}\bar{c})^{N_H(D)},$$

where $N_H(D)$ acts on $FC_G(D)$ through the inclusion of H in G . This number is the product of the degree of the defect 0 block, times the length of the orbit under the action of $N_H(D)$.

According to the local structure of blocks of the double covers of the symmetric groups given in Cabanes [6], the defect group is isomorphic to the defect group of $\tilde{S}_{|pw|}$. There is a subgroup, $\tilde{S}_{|pw|}\tilde{S}_{|\nu|}$ of \tilde{S}_n lifting a Young subgroup of S_n isomorphic to $S_{|pw|} \times S_{|\nu|}$ such that $C_G(D)/Z(D) \cong \tilde{S}_{|\nu|}$. The inclusion of H in G and the defect group D can be chosen such that $C_H(D)/Z(D) \cong \tilde{S}_{|\mu|}$, where the induced embedding of $\tilde{S}_{|\mu|}$ in $\tilde{S}_{|\nu|}$ is the standard embedding, and $N_H(D) \cong \tilde{S}_{|\mu|}N_{\tilde{S}_{|pw|}}(D)$. Let $\Lambda(b)$ be the set of ordinary irreducible characters of $\tilde{S}_{|\nu|}$ corresponding to the partition ν which is either $\{\theta_\nu^+\}$ or $\{\theta_\nu^+, \theta_\nu^-\}$, depending on parity. Similarly, let $\Lambda(c)$ be the set of ordinary irreducible characters of $\tilde{S}_{|\mu|}$ corresponding to the partition μ .

Groups $\tilde{S}_{|pw|}$ and $\tilde{S}_{|\nu|}$ do not centralize each other, even though the sets on which their images in \tilde{S}_n act are disjoint, because commuting transpositions multiplies the product by the central involution z (which acts in spin representations as -1). However, group $\tilde{A}_{|pw|}$ centralizes $\tilde{S}_{|\nu|}$ since all of its elements are even products of transpositions; hence,

$$(\bar{c}F(C_G(D)/Z(D))\bar{b}\bar{c})^{N_H(D)} = (\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle \tilde{S}_{|\mu|}, \sigma \rangle},$$

where σ in $\tilde{S}_{|pw|} - \tilde{A}_{|pw|}$ is the lifting of a transposition, so that σ^2 is central. Also, since σ normalizes $\tilde{S}_{|\mu|}$, σ acts on $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$, and thus, we can compute

$$(\bar{c}F(C_G(D)/Z(D))\bar{b}\bar{c})^{N_H(D)}$$

by first computing the algebra $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$ and then taking fixed points under σ .

The bimodule $W = (\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$ is isomorphic to

$$\text{End}_{F(\tilde{S}_{|\nu|} \times \tilde{S}_{|\mu|}^{\text{op}})}(F\tilde{S}_{|\nu|}\bar{b}\bar{c}),$$

the algebra of $F(\tilde{S}_{|\nu|} \times \tilde{S}_{|\mu|}^{\text{op}})$ invariant endomorphisms of the $F(\tilde{S}_{|\nu|} \times \tilde{S}_{|\mu|}^{\text{op}})$ -module $F\tilde{S}_{|\nu|}\bar{b}\bar{c}$. The map taking $\bar{b}\bar{c}$ to $v \in W$ is obviously a homomorphism of left modules, and it is a homomorphism for the right action as well because v is fixed under conjugation by elements of $\tilde{S}_{|\mu|}$.

Since \bar{b} and \bar{c} are of defect 0, $F(\widetilde{S}_{|\nu|} \times \widetilde{S}_{|\mu|}^{\text{op}})$ -modules are sums of tensor products of the form $V \otimes U$, where V is an irreducible left module for $F\widetilde{S}_{|\nu|}$, and U is an irreducible right module for $F\widetilde{S}_{|\mu|}$. Furthermore, for the bimodule $F\widetilde{S}_{|\nu|}\bar{b}\bar{c}$, which is actually an $(F\widetilde{S}_{|\nu|}\bar{b} \times F\widetilde{S}_{|\mu|}^{\text{op}}\bar{c})$ -module, the factors must be such that V is a composition factor in the module induced up from U .

In general, the bimodules over a pair of semisimple algebras are direct sums of tensor products $V \otimes W$ of irreducibles. Thus, in fact, $F\widetilde{S}_{|\nu|}\bar{b}\bar{c}$ is isomorphic to

$$\bigoplus_{\substack{\phi \in \Lambda(b) \\ \psi \in \Lambda(c)}} (V_\phi \otimes U_\psi)^{\oplus d_{\phi, \psi}},$$

where V_ϕ and U_ψ are simple projective modules for $F\widetilde{S}_{|\nu|}$ and $F\widetilde{S}_{|\mu|}$ corresponding to the ordinary irreducible characters ϕ and ψ , respectively, and where $d_{\phi, \psi}$ is the multiplicity of ϕ in $\text{Ind}_{\widetilde{S}_{|\mu|}}^{\widetilde{S}_{|\nu|}}(\psi)$. Thus, $(\bar{c}F\widetilde{S}_{|\nu|}\bar{b}\bar{c})^{\widetilde{S}_{|\mu|}}$ is isomorphic to the semi-simple algebra

$$\prod_{\substack{\phi \in \Lambda(b) \\ \psi \in \Lambda(c)}} \text{Mat}_{d_{\phi, \psi}}(F).$$

(i) α is odd. Let us consider the case $\epsilon(\nu) = 0, \epsilon(\mu) = 0$. Here, $\Lambda(b)$ consists of the unique irreducible character θ_ν^+ of $\widetilde{S}_{|\nu|}$ corresponding to ν , and $\Lambda(c)$ consists of the unique irreducible character θ_μ^+ , of $\widetilde{S}_{|\mu|}$ corresponding to μ . The multiplicity of θ_ν^+ in $\text{Ind}_{\widetilde{S}_{|\mu|}}^{\widetilde{S}_{|\nu|}}(\theta_\mu^+)$ is $2^{\alpha-1/2}\beta$; thus, $(\bar{c}F\widetilde{S}_{|\nu|}\bar{b}\bar{c})^{\widetilde{S}_{|\mu|}}$ is a matrix algebra of size $2^{\alpha-1/2}\beta$.

In particular, σ acts as an inner automorphism on $(\bar{c}F\widetilde{S}_{|\nu|}\bar{b}\bar{c})^{\widetilde{S}_{|\mu|}}$. Since σ^2 is central and thus acts as the identity and, since p is of odd characteristic, we may assume that the action of σ is through a diagonal matrix with 1s and -1 s on the diagonal. Thus, the fixed points of this action are block diagonal matrices corresponding to the decomposition into eigenspaces of σ . The total number of primitive idempotents remains the same, equal to the total degree of the block diagonal matrix. It follows that the number of idempotents in any primitive idempotent decomposition of $(\bar{c}F\widetilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle \widetilde{S}_{|\mu|}, \sigma \rangle}$ is $2^{\alpha-1/2}\beta$.

Now let us consider the case $\epsilon(\nu) = 1, \epsilon(\mu) = 1$. Each of $\Lambda(b)$ and $\Lambda(c)$ consists of two characters.

(a) $\alpha > 1$: The multiplicity of any irreducible character in $\Lambda(b)$ in the induced character of any irreducible character in $\Lambda(c)$ is $2^{\alpha-3/2}\beta$. Thus, $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$ is a direct product of four matrix algebras, each of size $2^{\alpha-3/2}\beta$. Then σ permutes these matrix factors in pairs so that $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle\tilde{S}_{|\mu|},\sigma\rangle}$ is isomorphic to a direct product of two matrix algebras, each of size $2^{\alpha-3/2}\beta$. Thus, the number of idempotents in a primitive idempotent decomposition of $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle\tilde{S}_{|\mu|},\sigma\rangle}$ is $2^{\alpha-1/2}\beta$.

(b) $\alpha = 1$: In this case, $\beta = 1$, and there is a pairing between the elements of $\Lambda(b)$ and $\Lambda(c)$ such that the number of constituents is either 0 or 1. We may assume that, in this special case, θ_μ^+ lifts to θ_ν^+ and θ_μ^- lifts to θ_ν^- . The total number of idempotents lifting one of the elements of $\Lambda(c)$ is 1, but this is exactly equal to $2^{\alpha-1/2}\beta$, as in the case $\alpha > 1$.

Thus, the number of idempotents in a primitive idempotent decomposition of $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle\tilde{S}_{|\mu|},\sigma\rangle}$ is $2^{\alpha-1/2}\beta$.

(ii) α is even. Let us consider the case $\epsilon(\nu) = 0, \epsilon(\mu) = 0$. Here, $\Lambda(b)$ consists of the unique irreducible character θ_ν^+ of $\tilde{S}_{|\nu|}$ corresponding to ν , and $\Lambda(c)$ consists of the unique irreducible character θ_μ^+ of $\tilde{S}_{|\mu|}$ corresponding to μ . The multiplicity of θ_ν^+ in $\text{Ind}_{\tilde{S}_{|\mu|}}^{\tilde{S}_{|\nu|}}(\theta_\mu^+)$ is $2^{\alpha/2}\beta$; thus, $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$ is a matrix algebra of size $2^{\alpha/2}\beta$.

Now let us consider the case $\epsilon(\nu) = 1, \epsilon(\mu) = 1$. Then each of $\Lambda(b)$ and $\Lambda(c)$ consists of two characters, and the multiplicity of any irreducible character in $\Lambda(b)$ in the induced character of any irreducible character in $\Lambda(c)$ is $2^{\alpha/2-1}\beta$. Thus, $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\tilde{S}_{|\mu|}}$ is a direct product of four matrix algebras, each of size $2^{\alpha/2-1}\beta$. Then σ permutes these matrix factors in pairs; thus, $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle\tilde{S}_{|\mu|},\sigma\rangle}$ is isomorphic to a direct product of two matrix algebras, each of size $2^{(\alpha/2)-1}\beta$. Therefore, the number of idempotents in a primitive idempotent decomposition of $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle\tilde{S}_{|\mu|},\sigma\rangle}$ is $2^{\alpha/2}\beta$. □

Example 4.11. We now illustrate with the case in Example 3.4, where $i = 0, w = 1, \nu = (6, 1), n = 12$, while $\mu = (4), m = 9$. The difference

$\alpha = n - m = 3$ is odd, and, by Lemma 4.4,

$$\beta = |\mathcal{M}^\alpha(\nu, \mu)| = 3!/2 = 3.$$

The block with label ν^1 has two characteristic 0 irreducibles, labeled by $(6, 3, 2, 1)$ and $(6, 5, 2, 1)$, while the block algebra of c has two characteristic 0 irreducibles, labeled by $(4, 3, 2)$ and $(5, 4)$. Both cores are odd; thus, there are two irreducible characters

$$\Lambda(b) = \{\theta_\lambda^+, \theta_\nu^-\}, \Lambda(c) = \{\theta_\chi^+, \theta_\mu^-\}.$$

The decomposition number in this case, by Lemma 4.4, is 6; therefore, we obtain

$$\bigoplus_{\substack{\theta_\nu^\pm \in \Lambda(b) \\ \theta_\mu^\pm \in \Lambda(c)}} (V_{\theta_\nu^\pm} \otimes U_{\theta_\mu^\pm})^{\oplus 6}$$

for the bimodule. Then, $(\bar{c}F\tilde{S}_{|\nu|}\bar{b}\bar{c})^{\langle \tilde{S}_{|\mu|}, \sigma \rangle}$ is isomorphic to a direct product of two matrix algebras, each of size 6.

Theorem 4.12. *Suppose that ν^w and μ^w are extremal block labels of an i -string.*

(i) *If the parities are the same, B_{ν^w} is source algebra equivalent to B_{μ^w} , and B'_{ν^w} is source algebra equivalent to B'_{μ^w} .*

(ii) *If the parities are different, B_{ν^w} is source algebra equivalent to B'_{μ^w} , and B'_{ν^w} is source algebra equivalent to B_{μ^w} .*

Proof. We have shown in Section 3 that, if ν and μ are extremal points of an i -string, then K_i is a w -allowed action, and thus, (ν, μ) is a w -compatible pair.

(i) Suppose that the parities of ν and μ are the same. We have shown that $m_{H,D}(\mathcal{O}Gbc)$ is exactly the number calculated in Lemma 4.9. Then we sum over $\text{Irr}(\tilde{S}_n, b)$ or $\text{Irr}(\tilde{S}_m, c)$, which have the same number of elements, and obtain $m_{H,D}(\mathcal{O}Gbc)|\text{Irr}(\tilde{S}_n, b)|$. Thus, [10, Theorem 2.5] applies, and the block algebras $\mathcal{O}\tilde{S}_n b$ and $\mathcal{O}\tilde{S}_m c$ are source algebra isomorphic.

(ii) Suppose the parities are different. Then, this source algebra equivalence is obtained from [13, Lemma 5.1, Theorem 6.3]. \square

5. A sharp bound for Donovan’s conjecture.

Definition 5.1. We say that two blocks with labels ρ^w and σ^w of the same weight are *allowed equivalent* if one can be obtained from the other by a sequence of w -allowed actions.

Now we wish to find properties which will indicate that a block is allowed-equivalent to a block of lower rank. This will allow us to find a rank N_0 such that every block is allowed equivalent to a block of rank N , $N \leq N_0$. This was accomplished in [12]. However, by using crossovers and tighter analysis of possible actions, we can make the bound in [12] sharp and exhibit a block ρ^w which attains the bound.

Lemma 5.2. *Let ρ^w be a block with the p -core*

$$c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$$

such that, for each $i \in I = \{1, \dots, t\}$, either $\ell_i \geq w$, or $\ell_i = 0$. Then, there is a block μ^w , of a lower or equal rank, with the p -core $c(\mu) = ((\ell'_1, 0), \dots, (\ell'_r, 0), (0, 1), \dots, (0, 1))$, that is allowed equivalent, by w -allowed actions, to block ρ^w , and such that the values of ℓ'_j form a permutation of those values of ℓ_i with $\ell_i \geq w$.

Proof. Let ρ^w be a block with the p -core $c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$, satisfying, for each $i \in I$, either $\ell_i \geq w$ or $\ell_i = 0$. (Note that, for $\ell_i = 0$, by definition, $\epsilon_i = 1$.)

Step 1: If all $\epsilon_i = 0$, then ρ is already in the desired form. If not, let k be the first place in the p -core ρ satisfying $\epsilon_k = 1$, and let j be the first place in the p -core ρ , after k , satisfying $\epsilon_j = 0$, if such exists (i.e., all runners from runner k to runner $j - 1$ are empty and $\ell_j \neq 0$, i.e., $\ell_j \geq w$). We shall run a recursion on k in order to show that we can transform ρ by w -allowed actions to the form

$$c(\nu) = ((\tilde{\ell}_1, 0), \dots, (\tilde{\ell}_r, 0)(\tilde{\ell}_{r+1}, 1), \dots, (\tilde{\ell}_t, 1)).$$

If no j exists, then we can take $\nu = \rho$ and proceed to the second step.

If j exists, we have $\ell_j + \ell_{j-1} \geq w$; thus, we can perform the w -allowed action K_{j-1} and obtain a block with the p -core such that the pair $(\ell_{j-1}, 0), (\ell_j, 1)$ has been swapped. Currently, the new ℓ_{j-1} is the old ℓ_j and $\ell_{j-1} + \ell_{j-2} \geq w$; thus, we can perform the w -allowed

action K_{j-2} , and so on. In summary, we perform the w -allowed action $K_k \circ K_{k+1} \circ \dots \circ K_{j-1}$ until we reach the situation where all runners from runner $k + 1$ to runner j are empty. If there is no $j > k + 1$ with $\epsilon_j = 0$, we have finished the first step. Otherwise, we replace k by $k + 1$ and continue.

Step 2: If, in the p -core

$$c(\nu) = ((\tilde{\ell}_1, 0), \dots, (\tilde{\ell}_r, 0)(\tilde{\ell}_{r+1}, 1), \dots, (\tilde{\ell}_t, 1)),$$

all of the $\tilde{\ell}_i$ for $i > r$ equal 0, then the lemma has been proved. If not, let s be the last place in the p -core ρ satisfying $\tilde{\ell}_s \neq 0$, meaning $\tilde{\ell}_s \geq w$ and $\epsilon_s = 1$. A backwards recursion is performed on $s - r$.

By $K_{t-1} \circ \dots \circ K_{s+1} \circ K_s$, we can bring the pair $(\tilde{\ell}_s, 1)$ to place t and perform the w -allowed action K_t to invert ϵ_t from 1 to 0. Step 1 is repeated to obtain a new ν with r replaced by $r + 1$. When Step 2 is again applied, the new s' will be no greater than the previous s because the actions of Step 1 will return all of the pairs which came after $(\tilde{\ell}_s, 1)$ to their previous places.

A block μ^w is obtained with the p -core,

$$c(\mu) = ((l'_1, 0), \dots, (l'_r, 0), (0, 1), \dots, (0, 1)),$$

which is allowed equivalent to the block ρ^w , and of a lower rank than ρ^w (because of w -allowed actions which reduced the rank of the p -strict partition). Since in Section 4 we showed that allowed equivalent blocks have equivalent source algebras, and a block is Morita equivalent to its source algebra, we have actually shown that the blocks are Morita equivalent. □

Lemma 5.3. *Let ρ^w be a block with the p -core*

$$\rho = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$$

satisfying, for each $i \in I = \{1, \dots, t\}$, either $\ell_i > w$ or $\ell_i = 0$. There is an allowed equivalent block σ^w of lower rank with the p -core

$$c(\sigma) = ((\ell'_1 - 1, 0), \dots, (\ell'_r - 1, 0), (0, 1), \dots, (0, 1)),$$

where $(\ell'_1, \dots, \ell'_r)$ is a permutation of the non-zero ℓ_i .

Proof. Let $(\ell_{i_1}, \dots, \ell_{i_r})$ be the set of all $\ell_i > 0$. First, by Lemma 5.3, the block ρ^w is allowed equivalent to a block μ^w with the p -core $c(\mu) = ((\ell'_1, 0), \dots, (\ell'_r, 0), (0, 1), \dots, (0, 1))$, where $(\ell'_1, \dots, \ell'_r)$ is a permutation of $(\ell_{i_1}, \dots, \ell_{i_r})$.

We next want to reduce each ℓ_i , $1 \leq i \leq r$, by 1, with $\epsilon_i = 1$. In terms of the abacus this will bring all of the beads onto runners $p - r, \dots, p - 1$. In order to accomplish this a recursion is run on i , for $1 \leq i \leq r$. For $i = 1$, we perform K_0 . In order to reduce some ℓ_i by 1, we must bring it to ℓ_1 , by performing $K_1 \circ K_2 \circ \dots \circ K_{i-1}$ and then K_0 . We know that $\ell'_i > w$ for $1 \leq i \leq r$; thus, the involution K_0 is a w -allowed action ($\ell'_1 - 1 \geq w$), and also K_j when $\epsilon_j \neq \epsilon_{j+1}$ is a w -allowed action since $(\ell'_j + \ell'_{j+1} \geq w)$.

Finally, we apply Lemma 5.3 again to change all ϵ_i to 0. This is possible because all $\ell_i - 1 \geq w$. □

Lemma 5.4. *Let ρ^w be a block label with the p -core t -tuple given by*

$$c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t)),$$

let $(\ell_{i_1}, \dots, \ell_{i_r})$ be the set of all $\ell_i > 0$, let the sequence (m_1, \dots, m_r) be a permutation which satisfies $0 < m_1 \leq m_2 \leq \dots \leq m_r$, and let m_{gap} be the maximal gap between all m_i . Choose a j such that $m_{\text{gap}} = m_{j+1} - m_j$ where $m_j > 0$, i.e., m_{j+1} is the smallest of the big m_i s and m_j is the biggest of the small m_i s. If $m_{\text{gap}} \geq w$, then all ℓ_i satisfying $\ell_i \geq m_{j+1}$ are reducible by 1 by w -allowed actions.

Proof. Note first that $m_{j+1} \geq w + m_j > w$. Every pair (ℓ_i, ϵ_i) for which $\ell_i \geq m_{j+1}$, hereafter called a *tall* pair, can be commuted with every pair (ℓ_k, ϵ_k) for which $\ell_k \leq m_j$, hereafter called a *short* pair, whether the ϵ are the same or not since we always have $\ell_i - \ell_k \leq w$. Thus, if we let i be the first of the tall pairs when runners are ordered from 1 to $p - 1$, we can move it toward the front by w -admissible actions of type $K_{i'}$ for $0 < i' < t$. If $\epsilon_i = 1$, then, when runner $t + 1$ is reached, the action K_t must be performed, but this is also admissible since $2\ell_i + 1 \geq w$. Performing these actions recursively, we reach a situation in which all of the tall pairs have $\epsilon = 0$ and are in 1 through s , for some $s < r$. Then they can all be lowered by 1 as in Lemma 5.3. □

Proposition 5.5. *Let ρ^w with $c(\rho) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$ be a block label that cannot be reduced by w -allowed actions, and let m_{gap} be as before. Then the following hold:*

- *the p -core ρ satisfies $\min\{\ell_i \mid 1 \leq i \leq r\} \leq w$.*
- *$m_{\text{gap}} \leq w - 1$ i.e., the maximal gap is $w - 1$. Among these blocks, that with the maximal rank has core t -tuple of the form $c(\rho_w) = ((w, 0), \dots, (w + (w - 1)(t - 1), 0))$.*

Proof. Let $I = \{1, \dots, t\}$. By Lemma 5.2, we obtain that every core satisfying either $\ell_i \geq w$ or $\ell_i = 0$, for each $i \in I$, can be reduced by w -allowed actions to a core of the form $((\ell_1, 0), \dots, (\ell_r, 0), (0, 1), \dots, (0, 1))$.

By Lemma 5.3, a core $((\ell_1, 0), \dots, (\ell_r, 0), (0, 1), \dots, (0, 1))$ satisfying $\ell_i > w$ for $1 \leq i \leq r$ can be reduced to a core $((\ell'_1 - 1, 0), \dots, (\ell'_r - 1, 0), (0, 1), \dots, (0, 1))$, i.e., in the core that cannot be reduced there is an i satisfying $0 \neq \ell_i \leq w$.

By Lemma 5.4, if there is a gap, $m_{\text{gap}} = m_{j+1} - m_j$ satisfying $m_{\text{gap}} \geq w$ then this gap can be reduced until it is less than w , i.e., in the core that cannot be reduced, the maximal gap is $w - 1$. The maximal rank is attained when the minimum is as large as possible, all gaps are as large as possible, and the ordering of runners gives the largest possible rank. This gives the core ρ_w of the statement of the lemma. □

Lemma 5.6. *Let $c(\nu) = ((\ell_1, \epsilon_1), \dots, (\ell_t, \epsilon_t))$ be a core. The rank of this block is*

$$N(\nu) = \sum_{i=1}^t \ell_i \cdot i^{1-\epsilon_i} \cdot (p - i)^{\epsilon_i} + \frac{\ell_i(\ell_i - 1)}{2} \cdot p.$$

Proof. For every pair (ℓ_i, ϵ_i) , we consider the addition made to the rank by all beads on the i th runner. If $\epsilon_i = 0$, then there are ℓ_i beads, corresponding to parts of the form $ap + i$ for $a = 0, \dots, \ell_i - 1$, and, if $\epsilon_i = 1$, then there are ℓ_i beads corresponding to parts of the form $ap + p - i$ for $a = 0, \dots, \ell_i - 1$.

If $\epsilon_i = 0$, then the rank will be

$$\ell_i \cdot i + \frac{\ell_i(\ell_i - 1)}{2} \cdot p,$$

and, if $\epsilon_i = 1$, then that rank will be

$$\ell_i \cdot (p - i) + \frac{\ell_i(\ell_i - 1)}{2} \cdot p.$$

Thus, in each case, we obtain rank

$$\ell_i \cdot i^{1-\epsilon_i} \cdot (p - i)^{\epsilon_i} + \frac{\ell_i(\ell_i - 1)}{2} \cdot p$$

for every pair (ℓ_i, ϵ_i) , and we are finished. □

Theorem 5.7. *Block ρ_w^w of maximal rank $N(w)$, which does not lie at the maximal rank end of any i -string, has rank*

$$(5.1) \quad N(w) = pw + \left(\frac{p(w-1)}{2} + 1 \right) \cdot \left(\sum_{i=1}^t i^2(w-1) + i \right).$$

Every block of weight w in \mathcal{OS}_n or \mathcal{OA}_n is source algebra equivalent to a block of rank $\leq N(w)$.

Proof. According to Lemma 5.5, the block of maximal rank has core

$$((w, 0), (2w - 1, 0), \dots, (tw - (t - 1), 0)).$$

Now we substitute these values into formula (5.1):

$$\begin{aligned} N(\rho_w) &= \sum_{i=1}^t (iw - (i - 1))i + \frac{(iw - (i - 1))(iw - i)p}{2} \\ &= \sum_{i=1}^t (i^2w - (i - 1)i) + \frac{p(w - 1)}{2} \sum_{i=1}^t (i^2w - (i - 1)i) \\ &= \left(\frac{p(w - 1)}{2} + 1 \right) \cdot \left(\sum_{i=1}^t i^2(w - 1) + i \right). \end{aligned}$$

Finally, adding pw for the weight of the block gives the desired formula.

Since every block of higher rank lies at the end of a maximal i -string, it is source algebra equivalent to a lower-rank block. □

Remark 5.8. The number calculated in Theorem 5.7 is actually an integer.

- If w is odd, then $w - 1$ is even; thus, $w - 1/2$ is an integer.
- If w is odd, then parity of the sum depends on parity of $\sum_{i=1}^t i - i^2$, and each term of this sum is even.

Remark 5.9. A bound N_w is given in [12] above which every block is source algebra equivalent to a block of lower degree. The aim there was to prove Donovan's conjecture by finding some bound, and no effort was made to find a tight bound. In addition, then the problem of crossovers was not understood, so the steps were all parity-preserving. Kessar first defined a number

$$a_w = 3p^2(w + 1) + 3(p - 1).$$

The formula in [12] then becomes

$$N_w = a_w(p - 1) + p \left(\frac{a_w(a_w + 1)}{2} \right).$$

For an example which is in the range of manual computation, if one uses the block-reduced crystal graph from [2], for $p = 5$ and $w = 2$, there are 10 source algebra equivalence classes, of which the ninth appears at degree 22 and the tenth at degree 38. Bound $N(2)$ given above in Theorem 5.7 is the exact bound 38, while the previous bound N_2 was 141,963. The sharp bound would be impossible if one did not take crossovers into account.

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