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ON THE SOBOLEV ORTHOGONALITY OF CLASSICAL ORTHOGONAL POLYNOMIALS FOR NON STANDARD PARAMETERS

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ABSTRACT. The discrete part of the discrete-continuous orthogonality

$$\mathscr{B}(f,g)=\mathscr{B}_d(f,g)+\mathscr{B}_c(f^{(N)},g^{(N)}),$$

is studied for families of classical orthogonal polynomials such that the associated three-term recurrence relation

$$xp_n=p_{n+1}+\beta_n p_n+\gamma_n p_{n-1},$$

presents one vanishing coefficient γ_n, as in the case of Laguerre polynomials $L_n^{(-N)}$, Jacobi polynomials $P_n^{(-N,\beta)}$ and Gegenbauer polynomials $C_n^{(-N+1/2)}$ with $N\in\mathbb{N}$. It is shown that the discrete bilinear functional $\mathscr{B}_d$ can be replaced by a linear functional, $\mathscr{L}$, or by another bilinear functional related with $\mathscr{L}$, which allows us to reformulate the orthogonality in a much simpler way in the case of Laguerre polynomials and in a totally explicit form in the case of Jacobi and Gegenbauer polynomials.

1. Introduction. Classical orthogonal polynomials play a distinguished role in many branches of applied mathematics. Probably the main reason for this use is that they are polynomial solutions of the second order differential equation

$$\sigma(x)y''(x)+\tau(x)y'(x)+\lambda_n y(x)=0,$$

with σ and τ polynomials of degree at most 2 and 1, respectively, and $\lambda_n\in\mathbb{R}$. One of the main tools in dealing with classical orthogonal polynomials, p_n, is precisely its orthogonality, which can be written as

$$\langle \mathscr{L}, p_n p_m\rangle = k_n\delta_{n,m},\quad \text{for all } n,m\geq 0, \tag{1.1}$$

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for some linear functionals $\mathscr{L}$ and $k_n > 0$. However, it is known that the existence of such a $\mathscr{L}$ does not hold for all classical orthogonal polynomials. In fact, it only occurs for Hermite polynomials H_n, Laguerre polynomials $L_n^{(\alpha)}$ with $\alpha > -1$ and Jacobi polynomials (Gegenbauer polynomials included) $P_n^{(\alpha,\beta)}$ with $\alpha, \beta \in (-1, +\infty)$. Hence, a natural problem is whether the orthogonality holds for the other choices of the parameters, the so-called non standard parameters. For almost all the rest of the parameters in $\mathbb{C}$, it is possible to find a linear functional $\mathscr{L}$ such that equation (1.1) holds but with $k_n \neq 0$ (the path of integration which is used as a contour in the complex plane, see for instance, **[9, 15]**), so that it is a non Hermitian orthogonality.

There are only a few cases in which it is not possible to find an orthogonality of type (1.1) which correspond with the existence of a vanishing coefficient γ_N in the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x). \tag{1.2}$$

Due to the Favard theorem, see **[6,** Theorem 4.4], a functional $\mathscr{L}$ satisfying

$$\langle \mathscr{L}, p_n p_m \rangle = 0, \quad \text{for all } n, m \geq 0,\ n \neq m,$$

is uniquely defined, but the condition $\langle \mathscr{L}, p_n^2 \rangle \neq 0$ does not hold for all $n \in \mathbb{N}$. Indeed, $\gamma_n \neq 0$ for $n < N$ if and only if $\langle \mathscr{L}, p_n^2 \rangle \neq 0$ for $n < N$. Thus, a different type of orthogonality is needed. The fact that, for the given $\mathscr{L}$, the condition (1.1) cannot be satisfied, also follows from the study of its sequence of moments

$$\mu_n = \langle \mathscr{L}, x^n \rangle.$$

The moments are known for classical orthogonal polynomials (a normalization is needed in some cases) and, for some families with non standard parameters, there exist some determinants of the Hankel matrices

$$\Delta_N = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_N \\ \mu_1 & \mu_2 & c\ldots & \mu_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_N & \mu_{N+1} & \cdots & \mu_{2N} \end{vmatrix},$$

which vanish; hence, a sequence of polynomials $\{p_n\}_{n\geq 0}$ satisfying equation (1.1) cannot exist, see for instance **[6,** Theorem 3.1]. Indeed, $\Delta_n \neq 0$ for $n < N$ if and only if $\langle \mathscr{L}, p_n^2 \rangle \neq 0$ for $n < N$. The problem arises because the family $\{p_n\}_{n\geq 0}$ is not uniquely determined by $\mathscr{L}$. Some properties of general polynomials $\{p_n\}$ satisfying

$$\langle \mathscr{L}, p_n x^m \rangle = 0, \quad \text{for } m = 0, 1, \ldots, n-1,$$

for a fixed general sequence of moments $\{\mu_n\}_{n\geq 0}$ such that $\Delta_n = 0$ can be seen in [**10,** Section 1].

The families of classical orthogonal polynomials satisfying equation (1.1), for which the existence of a linear functional $\mathscr{L}$ is not possible, are the Laguerre $L_n^{(-N)}$, Jacobi $P_n^{(-N,\beta)}$ or $P_n^{(\alpha,-N)}$ and Gegenbauer polynomials $C_n^{(-N+1/2)}$ with $N \in \mathbb{N}$ in all cases. These are the families which we study in this work (the orthogonality (1.1) also fails for $P_n^{(-N,-M)}$ and $C_n^{(-N)}$, but these polynomials are not well defined for all degrees in the sense of $\deg p_n = n$). These polynomials can be defined for all $n \in \mathbb{N}_0$ taking an adequate normalization (for instance, by considering monic polynomials) and all the main properties which appear in the literature (differential, three term recurrence relations, hypergeometric representation, etc.) remain true, but the orthogonality (see, for instance, [**4, 14, 24**] for properties of these polynomials with any parameters). These polynomials have been endowed with an orthogonality through a bilinear functional $\mathscr{B}$, see [**2, 3, 16**], with the property

$$\mathscr{B}(p_n, p_m) = k_n \delta_{n,m}, \tag{1.3}$$

with $k_n > 0$ so that the family $\{p_n : n \in \mathbb{N}_0\}$ is characterized by equation (1.3), i.e., $\{p_n : n \in \mathbb{N}_0\}$ is an orthogonal polynomial sequence with respect to $\mathscr{B}$.

This and related problems have been studied from several points of view in the last two decades, see [**1, 2, 3, 5, 7, 8, 12, 16, 18, 19, 20, 21, 23**]. We comment briefly on the state of the art.

The first results in this direction were given in 1995 by Kwon and Littlejohn [**16**], who established the Sobolev orthogonality for Laguerre polynomials $L_n^{(-N)}$ with $N \in \mathbb{N}$ (so $\gamma_N = 0$, or equivalently, $\Delta_N = 0$)

through the positive-definite inner product

$$\begin{aligned}\mathscr{B}(f,g) = &\sum_{k=0}^{N-1}\sum_{j=0}^{k} B_{k,j}(N)(f^{(k)}(0)g^{(j)}(0)+f^{(j)}(0)g^{(k)}(0)) \\ &+\int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}dx,\end{aligned} \tag{1.4}$$

where $f^{(j)}$ stands for the jth derivative, and

$$B_{k,j}(N) = \begin{cases} \displaystyle\sum_{p=0}^{j}(-1)^{k+j}\binom{N-1-p}{k-p}\binom{N-1-p}{j-p} & \\ & \text{for } 0 \le j < k \le N-1 \\ \displaystyle\frac{1}{2}\sum_{p=0}^{k}\binom{N-1-p}{k-p}^2 & \text{for } 0 \le j = k \le N-1. \end{cases}$$

The main idea of the previously obtained result **[16]** is to use the fact that the derivatives of classical orthogonal polynomials are also classical and, taking a derivative of adequate order, the parameters become standard. Then, the bilinear functional $\mathscr{B}$ is expressed as the sum of two bilinear forms: a discrete $\mathscr{B}_d$ (the fist term on the right hand side of equation (1.4)) and a continuous $\mathscr{B}_c$ evaluated at the derivatives of order N (the second term on the right hand side of equation (1.4), with the properties:

$$\mathscr{B}_d(p_n,p_m) = k_n\delta_{n,m}, \qquad \text{for } n,m \in \{0,1,\dots,N-1\}, \tag{1.5}$$

$$\mathscr{B}_d(p_n,p_m) = 0, \qquad \text{for } n \ge N \text{ or } m \ge N, \tag{1.6}$$

$$\mathscr{B}_c(p_n^{(N)},p_m^{(N)}) = k_n\delta_{n,m}, \qquad \text{for } n,m \ge N,$$

and $k_n > 0$. Thus, the role of $\mathscr{B}$ in dealing with the family $\{p_n\}_{n\ge 0}$ is to decompose it into two parts: for polynomials with degree $< N$, $\mathscr{B}$ acts only through $\mathscr{B}_d$; while, for polynomials of degree $\ge N$, $\mathscr{B}$ acts only through the part with $\mathscr{B}_c$. This idea was developed in 1998 by Álvarez de Morales, Pérez and Piñar **[3]**, who found a general approach which establishes the orthogonality for classical orthogonal polynomials when the parameters are such that the three-term recurrence relation (1.2) presents some $\gamma_N = 0$, or equivalently, $\Delta_N = 0$. The bilinear functional in [**3**] is defined by

$$\mathscr{B}(f,g)=\mathscr{B}_d(f,g)+\int f^{(N)}(x)g^{(N)}(x)\,d\nu(x), \tag{1.7}$$

where ν is the orthogonality measure associated with the Nth derivative of the corresponding classical orthogonal polynomial, always with classical parameters, and $\mathscr{B}_d$ satisfies properties (1.5)–(1.6). It is easy to check that, with these assumptions, $\{p_n\}_{n\in\mathbb{N}_0}$ is a sequence of orthogonal polynomials with respect to equation (1.7), so the main problem consists of finding a $\mathscr{B}_d$ satisfying equations (1.5)–(1.6).

The orthogonality of Gegenbauer polynomials $C_n^{(-N+1/2)}$ for $N\in\mathbb{N}$ (thus, $\gamma_{2N}=0$, or equivalently, $\Delta_{2N}=0$) was established in [**3**]

$$\mathscr{B}(f,g)=\mathbf{FAG}^t+\int_{-1}^{1} f^{(2N)}(x)g^{(2N)}(1-x^2)^N dx,$$

where

$$\mathbf{F}=(f(1),f'(1),\dots,f^{(N-1)}(1),f(-1),f'(-1),\dots,f^{(N-1)}(-1)),$$

and $\mathbf{G}$ is defined analogously. Since $C_{2N}^{(-N+1/2)}=\text{const}\,(x^2-1)^N$, property (1.6) is guaranteed and $\mathbf{A}$ is a symmetric positive definite matrix such that equation (1.5) holds. In the same way, the orthogonality of Jacobi polynomials $P_n^{(-N,\beta)}$ with β a non-negative integer (thus, $\gamma_N=0$ and $\Delta_N=0$) [**2**] is

$$\mathscr{B}(f,g)=\mathbf{FAG}^t+\int_{-1}^{1} f^{(N)}(x)g^{(N)}(x)(1+x)^{\beta+N}dx,$$

with

$$\mathbf{F}=(f(1),f'(1),\dots,f^{(N-1)}(1)),$$

since $P_N^{(-N,\beta)}=\text{const}\,(x-1)^N$.

In [**2, 3**], matrix $\mathbf{A}$ is proven to exist, and it can be constructed using the arguments of the proofs therein. We show a slightly different construction of $\mathbf{A}$, or equivalently, of $\mathscr{B}_d$, for the case of $\gamma_N=\Delta_N=0$ by using some ideas from [**17**]. This is valid for general orthogonal (not necessarily classical) polynomials, too. The first step is to define $\mathscr{B}_d$ via equation (1.5) in $\mathbb{P}_{N-1}$, the space of polynomials of degree less than or equal to $N-1$, with arbitrary positive constants k_n. This can be done because the polynomials $\{p_0,p_1,\dots,p_{N-1}\}$ form a basis of $\mathbb{P}_{N-1}$, and then, the Gram matrix of $\mathscr{B}_d$ in this basis is a diagonal matrix $\mathbf{K}$ with entries $k_0, \ldots, k_{N-1}$. The second step is, see **[17]**, to consider the basis of $\mathbb{P}_{N-1}$ composed of the basic Lagrange interpolation polynomials associated with the roots of p_N, taking into account their multiplicities. Thus, $\mathscr{B}_d$ in $\mathbb{P}_{N-1}$ can be written as

$$\mathscr{B}_d(f,g) = \mathbf{F}\mathbf{Q}\mathbf{K}\mathbf{Q}^t\mathbf{G}^t, \tag{1.8}$$

where $\mathbf{Q}$ is a nonsingular matrix and $\mathbf{F}$ and $\mathbf{G}$ denote the vectors whose entries are the evaluations of f and g as well as possibly its derivatives at the roots of p_N according to their multiplicities (and, hence, $\mathbf{A} = \mathbf{Q}\mathbf{K}\mathbf{Q}^t$). The entries of the matrix $\mathbf{Q}^{-1}$ are the polynomials $p_0, \ldots, p_{N-1}$ and possibly their derivatives evaluated at the roots of p_N, which are known. Thus, $\mathbf{Q}$ is computed as the inverse of a known matrix. Finally, $\mathscr{B}_d$ is considered as the extension of $\mathscr{B}_d$ to $\mathbb{P}$, the space of all polynomials (and even to the space of functions with bounded derivatives at the roots of p_N), via the representation (1.8). With this definition, $\mathscr{B}_d$ satisfies the conditions (1.5) by construction. Condition (1.6) holds since $\gamma_N = 0$ implies that p_N is a common factor of any p_n with $n \geq N$ (this fact could be obtained from the sequence of moments as well, see **[10]**). The approach described here has also been used for little q-Laguerre polynomials **[18]**, big and little q-Jacobi polynomials **[19]** and continuous q-Jacobi polynomials **[21]**.

This orthogonality was one of the reasons (among others as in **[13]**) that motivated the study of orthogonal polynomials with respect to a so called discrete-continuous Sobolev inner product, see **[2]**:

$$\mathscr{B}(f,g) = \mathbf{F}\mathbf{A}\mathbf{G}^t + \int f^{(k)}(x) g^{(k)}(x)\, d\nu(x),$$

with $\mathbf{A}$ any symmetric positive definite matrix, $\mathbf{F}$ and $\mathbf{G}$ row vectors whose entries consist of f and g, respectively, and possibly its derivatives evaluated at some nodes and ν a nontrivial probability measure (a nonatomic Borel positive measure) supported on the real line.

A different method for the discrete portion of orthogonality consists of the use of a linear functional $\mathscr{L}$, satisfying

$$\langle \mathscr{L}, p_n p_m \rangle = \delta_{n,m}, \qquad \text{for } n, m \in \{0, 1, \ldots, N-1\}, \tag{1.9}$$

$$\langle \mathscr{L}, p_n p_m \rangle = 0, \qquad \text{for } n \geq N \text{ or } m \geq N, \tag{1.10}$$

instead of the bilinear functional $\mathscr{B}_d$. This method has been applied to Racah, Hahn, dual Hahn and Krawtchouk polynomials **[7]**, where the functional $\mathscr{L}$ was previously known from the literature, and to Askey-Wilson, big q-Jacobi, dual q-Hahn, big q-Laguerre, q-Meixner and little q-Jacobi **[8]**, where $\mathscr{L}$ was also obtained from the literature and the relations between these families and their finite analogues (q-Racah, q-Hahn, continuous dual q-Hahn, affine q-Krawtchouk, quantum q-Krawtchouk and q-Krawtchouk, respectively). In all of these cases $\mathscr{L}$ was defined through a discrete measure supported on a finite amount of nodes and properties (1.9)–(1.10) were clearly satisfied.

Let us comment in regards to the manner of construction of $\mathscr{L}$. Functional $\mathscr{L}$ is a moment functional. The relations among a given sequence of moments, orthogonal polynomials with respect to the associated moment functional and the search of the largest space containing $\mathbb{P}$ in which the linear functional given by the moments can be defined, have been classically studied. In particular, moment functionals associated with classical orthogonal polynomials were studied in **[22]**, where they are expressed as

$$\mathscr{L} = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n}{n!} \delta^{(n)},$$

and, as usual,

$$\langle \delta^{(n)}, f \rangle = (-1)^n f^{(n)}(0),$$

which gives a representation of $\mathscr{L}$ using the moments as initial data for its construction.

Normalization of the moments would be necessary in our cases. For instance, for Laguerre polynomials, $L_n^{(\alpha)}$, the moments are usually considered to be $\mu_n^\alpha = \Gamma(\alpha + n + 1)$, and hence, μ_n^{-N} is not defined for $n \in \{0, \dots, N-1\}$, but normalizing

$$\widehat{\mu}_n^\alpha = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)} = (\alpha + 1)_n,$$

the moments $\widehat{\mu}_n^{-N}$ are well defined for $n \in \mathbb{N}_0$.

Now, we shall see a different manner of illustrating the nature of $\mathscr{L}$ when γ_N vanishes. Since we have the complete sequence of polynomials $\{p_n\}_{n\geq 0}$ with $\deg(p_n) = n$, functional $\mathscr{L}$ can be defined in $\mathbb{P}$ via the necessary condition

$$\langle \mathscr{L}, p_n \rangle = \delta_{n,0}, \tag{1.11}$$

and Favard's theorem [**6,** Theorem 4.4] guarantees the property

$$\langle \mathscr{L}, p_n p_m \rangle = 0, \quad \text{for all } n, m \in \mathbb{N}_0 \text{ and } n \neq m,$$

but $\langle \mathscr{L}, p_N^2 \rangle = 0$ (and this is exactly the reason why $\mathscr{L}$ is not substantial for an orthogonality which covers all the degrees; $\mathscr{L}$ cannot characterize the entire sequence of polynomials $\{p_n\}_{n\in\mathbb{N}_0}$). Furthermore, since $\gamma_N = 0$, using the three-term recurrence relation (1.2), it is easy to check that p_N is a factor of every p_n for $n \geq N$ and if, in addition, $\gamma_n \neq 0$ for $n < N$, then $\mathscr{L}$ can be represented as

$$\langle \mathscr{L}, p \rangle = C \int_\Gamma \frac{p(z)\, dz}{p_N(z) p_{N-1}(z)}, \quad \text{for all } p \in \mathbb{P}, \tag{1.12}$$

where Γ is a Jordan curve such that the roots of p_N lie inside Γ, the roots of p_{N-1} lie outside of Γ and $C \neq 0$ is a constant such that $\mathscr{L}(1) = 1$. The proof of equation (1.12) can easily be obtained using residues and setting

$$p_n = P p_N + Q p_{N-1} \quad \text{for } n < N$$

with polynomials P, Q such that $\deg(Q) = N - n - 1$ and $\deg(P) = N - n - 2$ using (1.2), and writing $p_n = P p_N$ for $n \geq N$.

Formula (1.12) shows that $\mathscr{L}$ is essentially an interpolatory quadrature formula associated with p_N: let $z_1, \dots, z_r$ be the roots of p_N and $\lambda_1, \dots, \lambda_r$ their multiplicities

$$\langle \mathscr{L}, p \rangle = \sum_{j=1}^{r} \sum_{k=0}^{\lambda_j - 1} A_{j,k} p^{(k)}(z_j), \tag{1.13}$$

with $A_{j,k}$ computed using equation (1.12). Indeed, this is a type of generalized Gaussian quadrature (it is a classical Gaussian quadrature when all the roots of p_N are simple and real and the Cotes numbers are $A_{j,0} > 0$). Hence, we have stated assertion (i) of the next lemma which is a key to the forthcoming results.

Lemma 1.1. *The following statements hold.*

(i) *Let a family of polynomials* $\{p_n\}_{n\in\mathbb{N}_0}$ *with* $\deg(p_n) = n$ *satisfy recurrence relation* (1.2) *with* $\gamma_n \neq 0$ *for* $1 \leq n \leq N-1$ *and* $\gamma_N = 0$*. Then* $\mathscr{L}$ *defined as* (1.11) *can be represented as the quadrature formula* (1.13) *associated to the orthogonal polynomial* p_N*.*

(ii) *Let a sequence* $\{\mu_n\}_{n\geq 0}$ *of real numbers be such that* $\Delta_n \neq 0$ *for* $0 \leq n \leq N-1$*,* $\Delta_N = 0$*. Consider* $\mathscr{L}$ *the associated moment functional. If there exists a family of polynomials* $\{p_n\}_{n\in\mathbb{N}_0}$ *with* $\deg p_n = n$ *such that*

$$\langle \mathscr{L}, p_n p_m\rangle = 0, \quad \text{for all } n, m \in \mathbb{N}_0,\ n \neq m,$$

then $\mathscr{L}$ *can be represented as the quadrature formula* (1.13) *associated to the orthogonal polynomial* p_N*.*

In particular, $\mathscr{L}$ *can be defined in any space of functions with bounded derivatives at the roots of* p_N*.*

Assertion (ii) can be easily proved by taking into account that a recurrence relation (1.2) also holds for $n < N$ and that p_N is a factor for all p_n with $n \geq N$ as well (expand the remainder of p_n divided by p_N in the basis $\{p_0, \dots, p_{N-1}\}$ and prove that all the coefficients vanish due to $\langle \mathscr{L}, p_N p\rangle = 0$ for any polynomial p and $\langle \mathscr{L}, p_n^2\rangle \neq 0$ for $n < N$). Then (ii) follows using the same arguments as those for (i).

Let us compare both methods:

• The use of $\mathscr{B}_d$ allows us to take the discrete part of the orthogonality positive semidefinite, since this property is controlled with the positivity of the coefficients k_n. However, $\mathscr{L}$ is positive semidefinite if and only if $\gamma_n > 0$ for $n = 1, \dots, N-1$.

• The bilinear functional $\mathscr{B}_d$ is not of Hankel type (i.e., $\mathscr{B}_d(xf, g) \neq \mathscr{B}_d(f, xg)$ so the matrix of moments is not Hankel) unless the coefficients k_n are chosen such that $k_n = k_{n-1}\gamma_n$. This means, for instance, that, with a different choice of the coefficients k_n, the bilinear functional does not directly reveal the structure given by the three-term recurrence relation. But, if some γ_n with $n \leq N-1$ is not positive, then the semidefinite positivity is incompatible with the Hankel property. Furthermore, $\mathscr{B}_d$ is chosen to be of Hankel type; therefore, it can be represented as

$$\mathscr{B}_d(\cdot, \cdot) = \text{const}\, \langle \mathscr{L}, \cdot \times \cdot\rangle.$$

• Since the matrix $\mathbf{Q}$ in equation (1.8) is computed as an inverse matrix, $\mathscr{B}_d$ is not totally explicit. The only exception is when all the roots of p_N are simple, which implies that, with suitable normalization, $\mathbf{Q}$ can be chosen as an orthogonal matrix. However, usually $\mathscr{L}$ is totally explicit.

• The method with $\mathscr{B}_d$ can be used even when some $\gamma_n = 0$ for $n < N$. However, the method with $\mathscr{L}$ is applicable when $N = \min\{n : \gamma_n = 0\}$ although an alternative using more than two linear functionals is common [**8**] when $\{n : \gamma_n = 0\}$ has more than one element.

Thus, if $\gamma_n > 0$ for $n = 1, \ldots, N-1$ or we are interested in the Hankel property for the discrete part, the method with the linear functional is adequate. However, if γ_n for $n < N$ is not positive, and we are interested in inner products, the method with the bilinear functional should be chosen although the matrix $\mathbf{Q}$ is not explicit.

In this paper, we obtain for the Laguerre $L_n^{(-N)}$ an explicit expression for the linear functional $\mathscr{L}$ (Section 2), Jacobi $P_n^{(-N,\beta)}$ (Section 3) and Gegenbauer polynomials $C_n^{(-N+1/2)}$ (Section 4) based on equation (1.12). Also, we consider an alternative form for the bilinear functional

$$\mathscr{B}_d(f,g) = \sum_{n=0}^{N-1} k_n \langle \mathscr{L}, p_n f\rangle \overline{\langle \mathscr{L}, p_n g\rangle},$$

which provides new expressions for orthogonalities of the polynomials under consideration. These expressions are simpler than those existent in the literature for the case of Laguerre polynomials, and they are totally explicit in the case of Jacobi and Gegenbauer polynomials.

2. Orthogonality for Laguerre $L_n^{(-N)}$. The Laguerre polynomials $L_n^{(-N)}$ can be defined as

$$L_n^{(-N)}(x) = \frac{(-N+1)_n}{n!}\, {}_1F_1\left(\begin{array}{c} -n \\ -N+1 \end{array} \,\middle|\, x \right),$$

for all $n \in \mathbb{N}$. Those which are monic,

$$\widehat{L}_n^{(-N)}(x) = n!(-1)^n L_n^{(-N)}(x),$$

satisfy the recurrence relation

$$x\widehat{L}_n^{(-N)}(x) = \widehat{L}_{n+1}^{(-N)}(x) + (2n-N+1)\widehat{L}_n^{(-N)}(x) + n(n-N)\widehat{L}_{n-1}^{-N}(x),$$

so $\{n : \gamma_n = 0\} = \{N\}$ and the functional $\mathscr{L}$ is defined in $\mathbb{P}$ as the quadrature rule associated with $\widehat{L}_N^{(-N)}(x) = x^N$, i.e.,

$$\langle \mathscr{L}, f \rangle = \sum_{j=0}^{N-1} A_j f^{(j)}(0),$$

for coefficients A_j. The following lemma shows that the orthogonality conditions of the family

$$\{L_n^{(-N)} : n = 0, \dots, N-1\}$$

reside in the series coefficients of $L_n^{(-N)} e^{-z}$ at $z = 0$, which is the essence of all the possible representations of $\mathscr{L}$.

Lemma 2.1.

$$L_n^{(-N)}(z)\, z^{-N} e^{-z} = \sum_{j=n+1}^{N} \frac{(-1)^{N-j}(-j+1)_n}{n!(N-j)!} \frac{1}{z^j} + \textit{an entire function.}$$

Proof. Multiplying the hypergeometric representation for $L_n^{(-N)}$ and the Laurent series for $z^{-N}e^{-z}$ at $z = 0$, one obtains

$$L_n^{(-N)}(z)\ z^{-N} e^{-z} = \sum_{j=1}^{N} B_{n,j} \frac{1}{z^j} + \text{entire function},$$

with

$$B_{n,j} = \frac{1}{n!} \sum_{k=0}^{\min\{N-j,n\}} \frac{(-n)_k(-N+1+k)_{n-k}}{k!} \frac{(-1)^{N-k-j}}{(N-k-j)!}.$$

Standard computations yield

$$\begin{aligned} B_{n,j} &= \frac{(-1)^{N-j}}{n!} \frac{(-N+1)_n}{(N-j)!} {}_2F_1\left(\begin{matrix} -n, & -N+j \\ & -N+1 \end{matrix} \;\middle|\; 1 \right) \\ &= \frac{(-1)^{N-j}}{n!} \frac{(-j+1)_n}{(N-j)!}. \end{aligned}$$

Finally, the lemma is proved taking into account that $B_{n,j} = 0$ for $j = 1, \dots, n$. $\square$

The first consequence of Lemma 2.1 are the representations for the linear functional $\mathscr{L}$.

Corollary 2.2. *The following representations for $\mathscr{L}$ hold:*

$$\begin{aligned}\langle \mathscr{L}, f\rangle &= \frac{(-1)^{N-1}(N-1)!}{2\pi i}\int_\Gamma f(z)\ z^{-N}e^{-z}dz\\ &= (-1)^{N-1}\ \frac{d^{N-1}}{dz^{N-1}}\bigg|_{z=0}(f(z)e^{-z})\\ &= \sum_{j=0}^{N-1}(-1)^j\binom{N-1}{j}f^{(j)}(0),\end{aligned}$$

where Γ is any (simple and closed) Jordan curve surrounding the origin, in particular, the unit circle.

The main result of this section is the next theorem.

Theorem 2.3. *Laguerre polynomials $L_n^{(-N)}$ for all degrees $n \in \mathbb{N}_0$ are orthogonal with respect to:*

(i) *The bilinear functional*

$$\mathscr{B}(f,g) = \sum_{j=0}^{N-1}(-1)^j\binom{N-1}{j}(fg)^{(j)}(0) + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}\,dx.$$

(ii) *The inner product*

$$\mathscr{B}(f,g) = \mathbf{FBKB}^t\mathbf{G}^t + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}dx,$$

where $\mathbf{F} = (f(0), f'(0), \ldots, f^{(N-1)}(0))$ and analogously for $\mathbf{G}$, $\mathbf{K}$ is an arbitrary diagonal positive definite matrix and

$$\mathbf{B} = (b_{j,n})_{j,n=0}^{N-1}, \qquad b_{j,n} = (-1)^j\binom{N-1}{j}\frac{(-j)_n}{n!}.$$

In particular, if $\mathbf{K} = I$, then

$$\mathbf{BKB}^t = (a_{i,j})_{i,j=0}^{N-1}, \quad a_{i,j} = (-1)^{i+j}\binom{N-1}{i}\binom{N-1}{j}\binom{i+j}{i},$$

Proof. The orthogonality given in (i) is a direct consequence of Corollary 2.2 and the results in Section 1.

Now consider the inner product given by

$$\mathscr{B}(f,g) = \sum_{n=0}^{N-1} k_n \mathscr{L}(fL_n^{(-N)})\mathscr{L}(gL_n^{(-N)}) + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}\,dx, \tag{2.1}$$

with $k_n > 0$. Using, for instance, the integral representation for $\mathscr{L}$ and Lemma 2.1, it is not difficult to obtain

$$\langle \mathscr{L}, L_n^{(-N)} f\rangle = \sum_{j=n}^{N-1} (-1)^j \binom{N-1}{j} \frac{(-j)_n}{n!} f^{(j)}(0).$$

Hence, $\mathbf{FBKB}^t\mathbf{G}^t$ is the matrix representation of the first term on the right hand side of equation (2.1) with $\mathbf{K}$ the diagonal matrix with entries $k_0, \dots, k_{N-1}$.

If the coefficients k_n are chosen such that $k_n = 1$, then

$$a_{i,j} = \sum_{n=0}^{N-1} b_{i,n} b_{j,n} = (-1)^{i+j} \binom{N-1}{i}\binom{N-1}{j}\binom{i+j}{i}. \qquad \square$$

3. Orthogonality for Jacobi $P_n^{(-N,\beta)}$. In this section, we consider the Jacobi polynomials $P_n^{(-N,\beta)}$ with $\beta \notin \mathbb{Z}^-$ (this condition ensures that $\deg(p_n) = n$ and that these polynomials are not Gegenbauer polynomials). These polynomials can be defined for all degrees through the hypergeometric representation

$$P_n^{(-N,\beta)}(x) = \frac{(-N+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n,\ n-N+\beta+1 \\ -N+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right),$$

and the monic $\widehat{P}_n^{(-N,\beta)}$ to satisfy the recurrence relation

$$\begin{aligned} x\widehat{P}_n^{(-N,\beta)} &= \widehat{P}_{n+1}^{(-N,\beta)} + \frac{\beta^2 - N^2}{(2n-N+\beta)(2n-N+\beta+2)} \widehat{P}_n^{(-N,\beta)} \\ &\quad + \frac{4n(n-N)(n+\beta)(n-N+\beta)}{(2n-N+\beta-1)(2n-N+\beta)^2(2n-N+\beta+1)} \widehat{P}_{n-1}^{(-N,\beta)}; \end{aligned}$$

thus, γ_N is the unique vanishing coefficient γ_n, and $\mathscr{L}$ is defined in $\mathbb{P}$ as

$$\langle \mathscr{L}, p\rangle = \sum_{j=0}^{N-1} A_j p^{(j)}(1),$$

for some A_j, since $\widehat{P}_N^{(-N,\beta)}(x) = (x-1)^N$.

Lemma 3.1.

$$P_n^{(-N,\beta)}(1-z)^{-N}(1+z)^{\beta} = \sum_{j=n+1}^{N} \frac{2^{\beta}(-1)^j(-j+1)_n(-\beta-n)_{N-j}}{2^{N-j}n!(N-j)!}\,\frac{1}{(z-1)^j} + \textit{an analytic function at } 1.$$

Proof. Multiplying the hypergeometric representation for $P_n^{(-N,\beta)}$ and the Laurent series for $(1-z)^{-N}(1+z)^{\beta}$ at $z=1$, one obtains

$$P_n^{(-N,\beta)}(z)(1-z)^{-N}(1+z)^{\beta} = \sum_{j=1}^{N} B_{n,j}\frac{1}{(z-1)^j} + \text{ analytic function at } 1,$$

with

$$B_{n,j} = \sum_{k=0}^{\min\{N-j,n\}} \frac{(-N+1)_n}{n!}\,\frac{(-n)_k(n-N+\beta+1)_k(-1)^k}{(-N+1)_k k! 2^k} \times \frac{2^{\beta}(-1)^N(-1)^{N-j-k}(-\beta)_{N-j-k}}{(N-j-k)!2^{N-j-k}}.$$

Standard computations yield, see, for instance, [**11,** page 66] for the summation of the ${}_3F_2$ series,

$$\begin{aligned} B_{n,j} &= \frac{(-1)^j 2^{\beta}(-N+1)_n(-\beta)_{N-j}}{2^{N-j}n!(N-j)!} \\ &\quad \times {}_3F_2\left(\begin{matrix} -n,\ n-N+\beta+1,\ -N+j \\ -N+1,\ \beta-N+j+1 \end{matrix}\,\middle|\, 1\right) \\ &= \frac{2^{\beta}(-1)^j(-j+1)_n(-\beta-n)_{N-j}}{2^{N-j}n!(N-j)!}, \end{aligned}$$

and $B_{n,j}$ vanishes for $j \le n$. $\square$

Corollary 3.2. *The following representations for $\mathscr{L}$ hold*:

$$\begin{aligned}\langle \mathscr{L}, f\rangle &= \frac{-2^{N-1-\beta}(N-1)!}{(-\beta)_{N-1}2\pi i}\int_\Gamma \frac{f(z)(1+z)^\beta}{(1-z)^N}\,dz\\ &= \frac{(-1)^{N-1}2^{N-1-\beta}}{(-\beta)_{N-1}}\;\frac{d^{N-1}}{dz^{N-1}}\bigg|_{z=1} f(z)(1+z)^\beta\\ &= \sum_{j=0}^{N-1}\binom{N-1}{j}\frac{2^j}{(\beta-N+2)_j}f^{(j)}(1),\end{aligned}$$

where Γ is any Jordan curve in $\mathbb{C}\backslash(-\infty,-1]$ surrounding 1 (for instance a circle around 1 with radius lower than 2).

Theorem 3.3. *Jacobi polynomials $P_n^{(-N,\beta)}$ with $\beta\notin\mathbb{Z}^-$ are orthogonal with respect to*:

(i) *the bilinear functional*

$$\mathscr{B}(f,g) = \sum_{j=0}^{N-1}\binom{N-1}{j}\frac{2^j}{(\beta-N+2)_j}(fg)^{(j)}(1) + \int_{-1}^1 f^{(N)}(x)g^{(N)}(x)(1+x)^{\beta+N}dx.$$

(ii) *The inner product*

$$\mathscr{B}(f,g) = \mathbf{FBKB}^t\mathbf{G}^t + \int_{-1}^1 f^{(N)}(x)g^{(N)}(x)(1+x)^{\beta+N}dx,$$

with $\mathbf{F}=(f(1),f'(1),\ldots,f^{(N-1)}(1))$ and analogously for $\mathbf{G}$, $\mathbf{K}$ is an arbitrary diagonal positive definite matrix and

$$\mathbf{B}=(b_{j,n})_{j,n=0}^{N-1},\quad b_{j,n}=\binom{N-1}{j}\frac{(-2)^j(-j)_n(-\beta-n)_{N-1-j}}{(-\beta)_{N-1}n!}.$$

In particular, if $\mathbf{K}=I$ then $\mathbf{BKB}^t=(a_{i,j})_{i,j=0}^{N-1}$, with

$$a_{i,j} = \frac{2^{i+j}}{(\beta-N+2)_i(\beta-N+2)_j}\binom{N-1}{i}\binom{N-1}{j} \times {}_4F_3\left(\left.\begin{matrix}-i,\ -j,\ \beta+1,\ \beta+1\\ 1,\ \beta-N+2+i,\beta-N+2+j\end{matrix}\ \right|\ 1\right).$$

Proof. We focus our attention only on (ii). Consider the inner product

$$\begin{aligned}\mathscr{B}(f,g) &= \sum_{n=0}^{N-1} k_n \langle \mathscr{L}, fP_n^{(-N,\beta)}\rangle\langle \mathscr{L}, gP_n^{(-N,\beta)}\rangle \\ &\quad + \int_{-1}^{1} f^{(N)}(x)g^{(N)}(x)(1+x)^{\beta+N}\,dx,\end{aligned} \tag{3.1}$$

with $k_n > 0$. Using Lemma 3.1,

$$\begin{aligned}&\langle \mathscr{L}, P_n^{(-N,\beta)} f\rangle \\ &\quad = \frac{1}{(-\beta)_{N-1}n!}\sum_{j=n}^{N-1}\binom{N-1}{j}(-2)^j(-j)_n(-\beta-n)_{N-1-j}f^{(j)}(1).\end{aligned}$$

Hence, $\mathbf{FBKB}^t\mathbf{G}^t$ is the matrix representation for the first term of the right hand side of equation (3.1) with $\mathbf{K}$ the diagonal matrix with entries $k_0, k_1, \ldots, k_{N-1}$. Finally, the expression for coefficients $a_{i,j}$ is obtained using

$$a_{i,j} = \sum_{n=0}^{N-1} b_{i,n}b_{j,n},$$

and standard computation. $\square$

4. Orthogonality for Gegenbauer $C_n^{(-N+1/2)}$. The Gegenbauer polynomials $C_n^{(-N+1/2)}$ are defined for all $n \in \mathbb{N}$ as

$$C_n^{(-N+1/2)}(x) = \frac{(-2N+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n,\ n-2N+1 \\ -N+1 \end{array}\middle|\ \frac{1-x}{2}\right),$$

since the numerators $n-2N+1$ and $(-2N+1)_n$ compensate the denominator $-N+1$ for $N \le n \le 2N-1$ and $n \ge 2N$, respectively. Those which are monic satisfy the three-term recurrence relation

$$\begin{aligned}x\widehat{C}_n^{(-N+1/2)}(x) &= \widehat{C}_{n+1}^{(-N+1/2)}(x) \\ &\quad + \frac{n(n-2N)}{(2n-2N-1)(2n-2N+1)}\widehat{C}_{n-1}^{(-N+1/2)}(x).\end{aligned}$$

Thus, γ_{2N} is the unique coefficient γ_n which vanishes. Taking into account the symmetry and the relation $\widehat{C}_{2N}^{(-N+1/2)} = (x^2-1)^N$, $\mathscr{L}$ can be defined in $\mathbb{P}$ as

$$\langle \mathscr{L}, f\rangle = \sum_{j=0}^{N-1} A_j (f^{(j)}(1) + (-1)^j f^{(j)}(-1)),$$

with some coefficients A_j.

Lemma 4.1. *For $n \in \{0, 1, \dots, 2N-1\}$,*

$$\begin{aligned} &C_n^{(-N+1/2)}(z)(1-z^2)^{-N} \\ &= \sum_{j=m+1}^{N} \frac{(-2N+1)_m (N)_{N-j} (-j+1)_m}{(-2)^{2N-j} m! (N-j)! (-2N+1+j)_m} \left(\frac{\chi_n}{(z-1)^j} + \frac{(-1)^{m+j}}{(z+1)^j} \right) \\ &\qquad + \text{ an entire function}, \end{aligned}$$

where $m = \min\{n, 2N-1-n\}$ and

$$\chi_n = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ -1 & \text{for } N \le n \le 2N-1. \end{cases}$$

Proof. Multiplying the hypergeometric representation for $C_n^{(-N+1/2)}$ and the Laurent series of $(1-z^2)^{-N}$ at $z=1$, one obtains

$$C_n^{(-N+1/2)}(z)(1-z^2)^{-N} = \sum_{j=1}^{N} B_{n,j} \frac{1}{(z-1)^j} + \text{analytic function at } 1,$$

with

$$\begin{aligned} B_{n,j} = &\sum_{k=0}^{\min\{N-j,n\}} \frac{(-2N+1)_n}{n!} \frac{(-n)_k (n-2N+1)_k (-1)^k}{(-N+1)_k k! 2^k} \\ &\times \frac{(N)_{N-j-k}}{(-2)^N (N-j-k)! (-2)^{N-j-k}}. \end{aligned}$$

By using standard computations, we can give $B_{n,j}$ as the Saalschützian hypergeometric series (see, for instance, **[11,** page 66])

$$B_{n,j} = \frac{(-2N+1)_n (N)_{N-j}}{n! (-2)^{2N-j} (N-j)!} {}_3F_2 \left(\begin{matrix} -n, \ -N+j, \ n-2N+1 \\ -N+1, \ -2N+j+1 \end{matrix} \,\middle|\, 1 \right),$$

which is expressed for $n < N$ as

$$B_{n,j} = \frac{(-2N+1)_n (N)_{N-j} (-j+1)_n}{(-2)^{2N-j} n! (N-j)! (-2N+1+j)_n},$$

and, for $N \leq n \leq 2N-1$, as

$$B_{n,j} = \frac{-(-2N+1)_{2N-1-n} (N)_{N-j} (-j+1)_{2N-1-n}}{(-2)^{2N-j} (2N-1-n)! (N-j)! (-2N+1+j)_{2N-1-n}}.$$

Note that $B_{n,j}$ vanishes for $j \leq \min\{n, 2N-1-n\}$. Finally, Lemma 4.1 is proved by considering the singular part at $z = -1$ using the symmetry. $\square$

Corollary 4.2. *The following representations hold for $\mathscr{L}$:*

$$\begin{aligned}
\langle \mathscr{L}, f \rangle &= \frac{(N-1)!(-2)^{2N-1}}{(N)_{N-1} 4\pi i} \int_\Gamma f(z)(1-z^2)^{-N} dz \\
&= \frac{(-2)^{2N-1}}{(N)_{N-1} 2} \Bigg((-1)^N \left. \frac{d^{N-1}}{dz^{N-1}} \right|_{z=1} f(z)(1+z)^{-N} \\
&\qquad\qquad - \left. \frac{d^{N-1}}{dz^{N-1}} \right|_{z=-1} f(z)(1-z)^{-N} \Bigg) \\
&= \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{2^{j-1}}{(-2N+2)_j} (f^{(j)}(1) + (-1)^j f^{(j)}(-1)),
\end{aligned}$$

where Γ is a contour composed by two circles of radius lower than 2, one of which surrounds $z = 1$ in the counterclockwise direction and the other $z = -1$ in the clockwise direction.

Theorem 4.3. *The Gegenbauer polynomials $C_n^{(-N+1/2)}$ are orthogonal with respect to:*

(i) *The bilinear functional*

(4.1)

$$\begin{aligned}
\mathscr{B}(f,g) = \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{2^{j-1}}{(-2N+2)_j} ((fg)^{(j)}(1) + (-1)^j (fg)^{(j)}(-1)) \\
+ \int_{-1}^{1} f^{(2N)}(x) g^{(2N)}(x) (1-x^2)^N dx.
\end{aligned}$$

(ii) *The inner product*

$$\mathscr{B}(f,g) = \mathbf{FBKB}^t\mathbf{G}^t + \int_{-1}^{1} f^{(2N)}(x)g^{(2N)}(x)(1-x^2)^N dx, \tag{4.2}$$

with $\mathbf{F} = (f(1), f'(1), \dots, f^{(N-1)}(1), f(-1), f'(-1), \dots, f^{(N-1)}(-1))$ *and analogously for* $\mathbf{G}$, $\mathbf{K}$ *is an arbitrary diagonal positive definite matrix and* $\mathbf{B} = (b_{j,n})_{j,n=0}^{2N-1}$,

$$b_{j,n} = \begin{cases} \binom{N-1}{j} \frac{2^{j-1}(-j)_n(-2N+1)_n}{n!(-2N+2)_j(-2N+2+j)_n} & \text{if } 0 \le j \le N-1,\ 0 \le n \le N-1, \\ -b_{j,2N-1-n} & \text{if } 0 \le j \le N-1,\ N \le n \le 2N-1, \\ b_{j-N,n}(-1)^{n+j} & \text{if } N \le j \le 2N-1. \end{cases}$$

In particular, if $\mathbf{K} = I$, *then* $\mathbf{BKB}^t = (a_{i,j})_{i,j=0}^{2N-1}$ *with*

$$\begin{aligned} a_{i,j} = &\binom{N-1}{i}\binom{N-1}{j} \frac{2^{i+j-1}}{(-2N+2)_i(-2N+2)_j} \\ &\times {}_4F_3\left(\begin{matrix} -i, -j, -2N+1, -2N+1 \\ 1, -2N+2+i, -2N+2+j \end{matrix} \,\middle|\, 1 \right), \end{aligned}$$

for $i, j \in \{0, 1, \dots, N-1\}$ *and*

$$a_{i,j} = \begin{cases} 0 & \text{if } i \in \{N, N+1, \dots, 2N-1\}, \\ & \quad j \in \{0, 1, \dots, N-1\}, \\ 0 & \text{if } i \in \{0, 1, \dots, N-1\}, \\ & \quad j \in \{N, N+1, \dots, 2N-1\} \\ (-1)^{i+j} a_{i-N,j-N} & \text{if } i, j \in \{N, N+1, \dots, 2N-1\}. \end{cases}$$

Proof. We focus our attention only on (ii). Consider the inner product

$$\begin{aligned} \mathscr{B}(f,g) = &\sum_{n=0}^{N-1} k_n \langle \mathscr{L}, f C_n^{(-N+1/2)} \rangle \langle \mathscr{L}, g C_n^{(-N+1/2)} \rangle \\ &+ \int_{-1}^{1} f^{(2N)}(x) g^{(2N)}(x)(1-x^2)^N dx, \end{aligned} \tag{4.3}$$

with $k_n > 0$. Using Lemma 4.1,

$$\langle \mathscr{L}, C_n^{(-N+1/2)} f\rangle = \sum_{j=m}^{N-1} \binom{N-1}{j} \frac{2^{j-1}(-j)_m(-2N+1)_m}{m!(-2N+2)_j(-2N+2+j)_m} (\chi_n f^{(j)}(1) + (-1)^{m+j} f^{(j)}(-1)).$$

Hence, $\mathbf{FBKB}^t\mathbf{G}^t$ is the matrix representation for the first term of the right hand side of equation (4.3) with $\mathbf{K}$ the diagonal matrix with entries $k_0, k_1, \ldots, k_{N-1}$. Finally, the expression for coefficient $a_{i,j}$ follows from

$$a_{i,j} = \sum_{n=0}^{2N-1} b_{i,n} b_{j,n}$$

and standard computations. □

Remark 4.4. The continuous part of the inner product in equation (4.2) can be replaced by

$$\mathscr{B}_c^M(f^{(M)}, g^{(M)}) = \int_{-1}^{1} f^{(M)}(x) g^{(M)}(x)(1-x^2)^{-N+M} dx,$$

with M any integer in $\{N, \ldots, 2N\}$ since the Mth derivative of $C_n^{(-N+1/2)}$ is a Gegenbauer polynomial with standard parameter, and, with such a replacement, equation (1.3) is satisfied:

- if $n < M$, then $\mathscr{B}(p_n, p_m) = \mathscr{B}_d(p_n, p_m) = k_n \delta_{n,m}$.
- If $n \in \{M, \ldots, 2N-1\}$ and $m \neq n$, then

  $$\mathscr{B}(p_n, p_m) = \mathscr{B}_d(p_n, p_m) + \mathscr{B}_c^M(p_n^{(M)}, p_m^{(M)}) = 0,$$

  but

  $$\mathscr{B}(p_n, p_n) = \mathscr{B}_d(p_n, p_n) + \mathscr{B}_c^M(p_n^{(M)}, p_n^{(M)}) > 0.$$

- If $n \geq 2N$, then

  $$\mathscr{B}(p_n, p_m) = \mathscr{B}_c^M(p_n^{(M)}, p_m^{(M)}) = k_n \delta_{n,m}.$$

Note that a replacement cannot be made in equation (4.1) since the property $\mathscr{B}(p_n, p_n) \neq 0$ was not guaranteed for $n \in \{M, \ldots, 2N-1\}$.

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