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WEIGHTED BERGMAN KERNEL FUNCTIONS ASSOCIATED TO MEROMORPHIC FUNCTIONS

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ABSTRACT. We present a technique for computing explicit, concrete formulas for the weighted Bergman kernel on a planar domain with modulus squared weight of a meromorphic function in the case that the meromorphic function has a finite number of zeros on the domain and a concrete formula for the unweighted kernel is known. We apply this theory to the study of the Lu Qi-keng problem.

1. Introduction. The Bergman kernel function has been called a cornerstone of geometric function theory [**10**] and is the object of considerable study in complex analysis. The problems of computing explicit formulas for this function and determining its zero set are classical in complex analysis. Domains for which the associated Bergman kernel is zero-free are called Lu Qi-keng domains, and the problem of determining which domains are Lu Qi-keng is known as the Lu Qi-keng problem. This problem is of interest in the study of Bergman representative coordinates which require the kernel to be zero-free (see, [**7, 8**]). The Lu Qi-keng problem for smooth planar domains has been solved [**14**], but a solution for higher dimensions is not yet known [**2**]. The property of having a zero-free kernel is also a biholomorphic invariant and hence may be used to distinguish biholomorphic equivalence classes.

The main result of this paper is Theorem 3.1, which allows writing certain weighted Bergman kernels on the plane in terms of other weighted Bergman kernels with simpler weights. One consequence of this theorem is that, if one has an explicit, concrete formula for an unweighted kernel, then one can compute an explicit, concrete

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formula for the weighted kernel whenever the weight is the modulus squared of a meromorphic function with finitely many zeros on the associated domain. By the well-known technique of Theorem 2.1, weighted kernels for domains on the plane are related to unweighted kernels for domains in $\mathbb{C}^2$. Thus, the results presented here that are specific to complex dimension 1 have relevance to the classical problems of the first paragraph, in particular the Lu Qi-keng problem, in complex dimension 2. With the two-dimensional Lu Qi-keng problem in mind we study the zero sets of weighted kernels in Section 4.

The Bergman kernel for a domain $\Omega \subset \mathbb{C}$ is the unique skew-symmetric sesqui-holomorphic[1] function $K^\Omega\colon \Omega \times \Omega \to \mathbb{C}$ with the reproducing property

$$\begin{aligned} f(z) &= \langle f, K^\Omega(\cdot, z)\rangle \\ &= \int_\Omega f(w) K^\Omega(z,w)\, dV_w \qquad \text{for all } f \in A^2(\Omega), \end{aligned} \tag{1.1}$$

where dV_w is the real $2n$-dimensional Lebesgue volume (or area) measure, and $A^2(\Omega)$ is the Hilbert space of square-integrable holomorphic functions on Ω, called the Bergman space. (When the domain is clear, we will omit it from the superscript of K.) Equivalently, if $\{\phi_j\}_{j=0}^\infty$ is an orthonormal Hilbert space basis for $A^2(\Omega)$, then the Bergman kernel function $K^\Omega(z,w)$ is given by

$$K^\Omega(z,w) := \sum_{j=0}^\infty \phi_j(z)\overline{\phi_j(w)}. \tag{1.2}$$

Also of present interest is the weighted Bergman kernel with respect to a weight φ, which we denote $K^\Omega_\varphi(z,w)$. Here a *weight* φ is a measurable real-valued function $\varphi\colon \Omega \to [0,\infty]$. (Further assumptions on φ will be specified as needed.) Replacing the inner product in equation (1.1) with the weighted inner product

$$\langle f, g\rangle_\varphi := \int_\Omega f(w)\overline{g(w)}\varphi(w)\, dV_w,$$

$K^\Omega_\varphi(z,w)$ is the unique reproducing kernel for the weighted Bergman space

$$A^2_\varphi(\Omega) = \{f \mid \langle f, f\rangle_\varphi < \infty \text{ and } f \text{ holomorphic}\}.$$

The details of this classical theory may be found in many texts on complex analysis, for example, in [**1, 9**].

2. Preliminary theory. To study the Lu Qi-keng problem in higher dimensions, we would like concrete examples of kernels on domains in n-dimensional complex space, but, obtaining a closed-form formula for the kernel from (1.2) is possible only for domains with a high degree of symmetry. There are, however, several techniques for relating the kernel of one domain to the kernel of another domain of different complex dimension (see [**3**]). We shall make crucial use of the following known result.

Theorem 2.1. *Let D be a bounded domain in $\mathbb{C}$, let ψ be a weight function on D, and let Ω be defined by*

$$\Omega := \{(z, w) \in \mathbb{C}^2 \mid z \in D, |w| < \psi(z)\} \subset \mathbb{C}^2.$$

Then $K^D_{\pi\psi^2}(z, w) \equiv K^\Omega((z, 0), (w, 0))$.

The idea behind Theorem 2.1 appears in the literature in various forms. It is essentially [**11,** Corollary 2.1], in which Ligocka, generalizing an idea found in a proof due to Forelli and Rudin [**5**], calls the Forelli-Rudin construction. The term Forelli-Rudin construction appears elsewhere in subsequent literature in reference to similar techniques. Such techniques are surveyed in [**3**].

Our primary goal is to express a weighted kernel in terms of another weighted kernel that is in some sense simpler than the first. The next theorem is the simplest case of such a theorem and is fundamental to the rest of the theory.

Theorem 2.2. *Let $\Omega \subset \mathbb{C}^n$, let $K_\varphi(z, w)$ be the weighted Bergman kernel on Ω with respect to a weight function φ, and let g be holomorphic on Ω. Suppose that, after possibly removing singularities, $K_\varphi(z, w)/g(z)$ is holomorphic in z. Then*

$$K_{\varphi\cdot|g|^2}(z, w) = \frac{K_\varphi(z, w)}{g(z)\overline{g(w)}}.$$

Proof. The weight φ plays no role in the following argument, so, for simplicity of notation, we suppress the subscript φ in the calculation. We have that

$$\int_\Omega \left|\frac{K(z,w)}{g(z)}\right|^2 |g(z)|^2\, dV_z = \|K(\cdot,w)\|^2 < \infty,$$

so

$$\frac{K(z,w)}{g(z)} \in A^2_{|g|^2}(\Omega) \quad \text{(as a function of } z). \tag{2.1}$$

Also,

$$\int_\Omega |K_{|g|^2}(z,w)|^2 |g(z)|^2\, dV_z = \left\|K_{|g|^2}(\cdot,w)\right\|^2_{|g|^2} < \infty.$$

so

$$K_{|g|^2}(z,w)g(z) \in A^2(\Omega) \quad \text{(as a function of } z). \tag{2.2}$$

By equation (2.1) and the reproducing property of $K_{|g|^2}(z,w)$, we have

$$\begin{aligned}
\frac{K(z,w)}{g(z)} &= \int_\Omega \frac{K(\zeta,w)}{g(\zeta)} K_{|g|^2}(z,\zeta)|g(\zeta)|^2\, dV_\zeta \\
&= \int_\Omega K(\zeta,w)K_{|g|^2}(z,\zeta)\overline{g(\zeta)}\, dV_\zeta \\
&= \overline{\int_\Omega K(w,\zeta)K_{|g|^2}(\zeta,z)g(\zeta)\, dV_\zeta}.
\end{aligned}$$

By equation (2.2) and the reproducing property of the kernel $K(z,w)$, this last expression is

$$\overline{K_{|g|^2}(w,z)g(w)} = \overline{g(w)}K_{|g|^2}(z,w).$$

We have shown that

$$\frac{K(z,w)}{g(z)} = \overline{g(w)}K_{|g|^2}(z,w),$$

from which the theorem follows. □

Theorem 2.2 and the ancillary Theorem 4.9 are the only multidimensional theorems in this paper. The other results are specific to domains of dimension 1. As described above, the one-dimensional results, together with Theorem 2.1, can be used to study domains in higher dimensions. Indeed, Theorem 2.2 provides a recipe, illustrated by Example 2.3, for constructing non Lu Qi-keng domains in $\mathbb{C}^2$. The technique of this example, though elementary, appears to be absent from the literature.

Example 2.3. Let c be a point in the open unit disk $\mathbb{D}$, and define $\psi(z) := (\sqrt{\pi}|z-c|)^{-1}$ and

$$\Omega := \left\{(z,w) \in \mathbb{C}^2 \mid z \in \mathbb{D}, |w| < \psi(z)\right\} \subset \mathbb{C}^2.$$

Applying Theorem 2.1 to this Ω, we have

$$K^{\mathbb{D}}_{|z-c|^{-2}}(z,w) = K^{\Omega}((z,0),(w,0)).$$

On the other hand, applying Theorem 2.2 to $\Omega' := \mathbb{D}$, $\varphi(z) := |z-c|^{-2}$ and $g(z) = z-c$ gives

$$K^{\mathbb{D}}_{|z-c|^{-2}}(z,w) = (z-c)K^{\mathbb{D}}(z,w)(\overline{w}-\overline{c}).$$

Of course, all of the hypotheses of Theorem 2.2 need to be satisfied. The claim is that $K^{\mathbb{D}}_{|z-c|^{-2}}(z,w)$ must have a zero at $z=c$. Indeed, since

$$\|K^{\mathbb{D}}_{|z-c|^{-2}}(\cdot,w)\|_{\varphi} < \infty,$$

the function $(z-c)^{-1}K^{\mathbb{D}}_{|z-c|^{-2}}(z,w)$ has a removable singularity at $z=c$, cf., Theorem 3.3 and Remark 3.5. Since $(z-c)K^{\mathbb{D}}(z,w)(\overline{w}-\overline{c})$ clearly has zeros whenever $z=c$ or $w=c$, Ω is not Lu Qi-keng.

The domain in Example 2.3 is an unbounded domain, but a bounded non Lu Qi-keng domain can be obtained via Ramanadov's theorem together with Hurwitz's theorem.

Our goal is now to obtain a formula for a weighted kernel explicitly in terms of the unweighted kernel when the weight is the modulus squared of a meromorphic function. Theorem 2.2 allows us to handle the poles: the poles appear as zeros of the same order in the formula for the weighted kernel given by Theorem 2.2. On the other hand, any zeros of the meromorphic function associated to the weight clearly cannot appear as poles in the formula for the weighted kernel since the kernel is holomorphic.

3. Decomposition theorems. Theorem 2.2 needs modification in the case where the meromorphic function in the weight vanishes. The goal of this section is to show how to accomplish this modification in dimension 1. For a general planar domain Ω and holomorphic function f, we are able to express $K^{\Omega}_{|f|^2}(z,w)$ in terms of the kernel associated to a "simpler" weight function and the basis functions for the orthogonal complement of $A^2_{|f|^2}(\Omega)$ in a larger space of functions; when φ is both bounded and bounded away from zero near c, the normalized function

$$\frac{K^{\Omega}_{\varphi}(z,c)}{(z-c)\sqrt{K^{\Omega}_{\varphi}(c,c)}}$$

turns out to span the orthogonal complement of

$$A^2_{|z-c|^2\varphi(z)}(\Omega)$$

in

$$A^2_{|z-c|^2\varphi(z)}(\Omega\setminus\{c\}).$$

Theorem 3.1. *Let $\Omega\subset\mathbb{C}$ be a domain, $c\in\Omega$, and let φ be a weight on Ω which is bounded in a neighborhood of c. Then*

$$K^{\Omega}_{|z-c|^2\varphi}(z,w)=\frac{K^{\Omega}_{\varphi}(z,w)}{(z-c)(\overline{w}-\overline{c})}-\frac{K^{\Omega}_{\varphi}(z,c)K^{\Omega}_{\varphi}(c,w)}{(z-c)(\overline{w}-\overline{c})K^{\Omega}_{\varphi}(c,c)}. \tag{3.1}$$

Remark 3.2. The requirement that φ be bounded in a neighborhood of c excludes degenerate cases such as $\varphi(z)=|z|^{-2}$ with $c=0$; indeed, this requirement makes the hypotheses of Theorem 3.1 mutually exclusive of the hypotheses of Theorem 2.2 as explained in the discussion after the proof. Apparently, the right hand side of equation (3.1) has singularities at $z=c$ and $w=c$ but these are removable.

Proof. Let

$$\psi(z):=\frac{K^{\Omega}_{\varphi}(z,c)}{z-c}.$$

Clearly,

$$\psi\in A^2_{|z-c|^2\varphi}(\Omega\setminus\{c\}).$$

Our strategy is as follows:

(1) $K_\varphi^\Omega(z,w)/[(z-c)(\overline{w}-\overline{c})]$ reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega\setminus\{c\})$.

(2) $\psi(z)$ is orthogonal to $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega\setminus\{c\})$; as a consequence,

(3) $\psi(z)$ is orthogonal to $K^\Omega_{|z-c|^2\varphi}(z,w)$ in $A^2_{|z-c|^2\varphi}(\Omega\setminus\{c\})$.

(4) From equations (1) and (2),

$$Q(z,w) := \frac{K_\varphi^\Omega(z,w)}{(z-c)(\overline{w}-\overline{c})} - c_0(w)\psi(z)$$

also reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega\setminus\{c\})$, where $c_0(w)$ is arbitrary.

(5) Setting

$$c_0(w) := \overline{\psi(w)}/K_\varphi^\Omega(c,c),$$

we have $Q\in A^2_{|z-c|^2\varphi}(\Omega)$; it follows from equation (4) and the uniqueness of the Bergman kernel that $Q(z,w)\equiv K^\Omega_{|z-c|^2\varphi}(z,w)$.

Once equations (1) and (2) are proven, equations (3) and (4) are obvious.

Proof of equation (1). Let $f\in A^2_{|z-c|^2\varphi}(\Omega)$. We have

$$\begin{aligned}
&\int_{\Omega\setminus\{c\}} f(w)\frac{K_\varphi^\Omega(z,w)}{(z-c)(\overline{w}-\overline{c})}|w-c|^2\varphi(w)\,dV_w\\
&\qquad = \frac{1}{z-c}\int_\Omega K_\varphi^\Omega(z,w)f(w)(w-c)\varphi(w)\,dV_w\\
&\qquad = \frac{1}{z-c}f(z)(z-c)\quad (\text{since } f(z)(z-c)\in A^2_\varphi(\Omega))\\
&\qquad = f(z).
\end{aligned}$$

This proves equation (1). □

Proof of equation (2). Let $f \in A^2_{|z-c|^2\varphi}(\Omega)$. We have

$$\begin{aligned}\int_{\Omega\setminus\{c\}} & f(w)\overline{\psi(w)}|w-c|^2\varphi(w)\,dV_w \\ &= \int_{\Omega\setminus\{c\}} f(w)\frac{\overline{K^\Omega_\varphi(w,c)}}{\overline{w}-\overline{c}}|w-c|^2\varphi(w)\,dV_w \\ &= \int_\Omega f(w)(w-c)K^\Omega_\varphi(c,w)\varphi(w)\,dV_w \\ &= 0 \quad (\text{since } f(z)(z-c)\in A^2_\varphi(\Omega)).\end{aligned}$$

This proves equation (2). □

To finish the proof, observe that, for $c_0(w) := \overline{\psi(w)}/K^\Omega_\varphi(c,c)$, we have that

$$Q(z,w) \equiv \frac{K^\Omega_\varphi(z,w)}{(z-c)(\overline{w}-\overline{c})} - \frac{K^\Omega_\varphi(z,c)K^\Omega_\varphi(c,w)}{(z-c)(\overline{w}-\overline{c})K^\Omega_\varphi(c,c)},$$

which has a removable singularity at $z = c$ and $w = c$. Thus, equation (5) holds, and the theorem is proven. □

One might reasonably wonder whether the hypotheses of Theorems 2.2 and 3.1 can be simultaneously satisfied, for, if so, the formally different conclusions would need to be reconciled. Consider the case when the domain $\Omega \subset \mathbb{C}^1$, $c \in \Omega$, and φ is a weight on Ω that is bounded in a neighborhood of c. If $K^\Omega_\varphi(z,w)/(z-c)$ is holomorphic in $z \in \Omega$, i.e. the singularity is removable, then it would appear that we have two formally different expressions for $K^\Omega_{|z-c|^2\varphi}(z,w)$. Observe that, if $K^\Omega_\varphi(z,w)/(z-c)$ is holomorphic, then $K^\Omega_\varphi(c,w) = 0$ for all $w \in \Omega$; in fact, a routine calculation shows that, in this case, $f(c) = 0$ for all $f \in A^2_\varphi(\Omega)$. Theorem 3.3 shows that this cannot happen if φ is bounded in a neighborhood of c, and hence, the hypotheses of Theorems 2.2 and 3.1 cannot be simultaneously satisfied.

Theorem 3.3. *Suppose $A^2_\varphi(\Omega)$ is a nontrivial weighted Bergman space for weight φ and domain $\Omega \subset \mathbb{C}^1$. If, for some $c \in \Omega$, $f(c) = 0$ for all $f \in A^2_\varphi(\Omega)$, then φ is not bounded in any neighborhood of c.*

Proof. Suppose the weight φ is bounded in a neighborhood of c. The goal is to construct a function in the weighted Bergman space that takes a nonzero value at c. By hypothesis, there is some nontrivial function f in the weighted Bergman space. Since f is not identically equal to zero, there is a least positive integer k such that the kth derivative of f at c is nonzero.

Let $g(z)$ denote $f(z)/(z-c)^k$. After the removable singularity is removed, the function g is holomorphic and has a nonzero value at c. The claim is that g lies in the weighted Bergman space.

Observe that g is weighted square-integrable in a small neighborhood of c, because the weight is bounded near c by hypothesis, and the holomorphic function g is locally bounded on its domain. On the other hand, outside a neighborhood of c, the factor $(z-c)^k$ is bounded away from zero, so $1/(z-c)^k$ is bounded above, whence the weighted norm of g is bounded above by a constant times the weighted norm of f, which is finite by hypothesis. □

Theorem 3.1 combined with Theorem 2.2 allows one to produce an explicit formula for $K^{\Omega}_{|f|^2}(z,w)$ in terms of $K^{\Omega}(z,w)$ in the case that f is a polynomial with zeros in Ω by just iterating the formula of equation (3.1). In fact, Theorem 3.1 is a special case of the following more general theorem.

Theorem 3.4. *Let Ω be a planar domain, $\{c_j\}_{j=1}^m$ a sequence of m distinct points in Ω, $\{\alpha_j\}_{j=1}^m$ a sequence of positive integers and φ a weight such that, for all j, φ is both bounded and bounded away from zero in a neighborhood of c_j. Define the following polynomials*:

$$\begin{aligned}
p(z) &:= (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_m)^{\alpha_m};\\
p_{j,k}(z) &:= (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_{j-1})^{\alpha_{j-1}}(z-c_j)^k,\\
&\qquad\qquad (1\le j\le m, 1\le k\le \alpha_j);\\
q_{j,k}(z) &:= p(z)/p_{j,k}(z)\\
&\quad = (z-c_j)^{\alpha_j-k}(z-c_{j+1})^{\alpha_{j+1}}(z-c_{j+2})^{\alpha_{j+2}}\cdots(z-c_m)^{\alpha_m}.
\end{aligned}$$

Then

$$K^{\Omega}_{|p(z)|^2\varphi}(z,w) = \frac{K^{\Omega}_{\varphi}(z,w)}{p(z)\overline{p(w)}} - \sum_{j=1}^{m}\sum_{k=1}^{\alpha_j}\frac{K^{\Omega}_{|q_{j,k}|^2\varphi}(z,c_j)K^{\Omega}_{|q_{j,k}|^2\varphi}(c_j,w)}{p_{j,k}(z)\overline{p_{j,k}(w)}K^{\Omega}_{|q_{j,k}|^2\varphi}(c_j,c_j)}.$$

Remark 3.5. By the L^2-version of the Riemann removable singularity theorem [**13,** E.3.2], when a weight ψ is both bounded and bounded away from zero in a neighborhood of c, then

$$K_{\psi}^{\Omega\setminus\{c\}}(z,w) \equiv K_{\psi}^{\Omega}(z,w).$$

Proof. We wish to show that the functions

$$\psi_{j,k}(z) := \frac{K^{\Omega}_{|q_{j,k}|^2\varphi}(z,c_j)}{p_{j,k}(z)}$$

form a basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega\setminus\{c_j\}_{j=1}^m)$. We only prove that the $\psi_{j,k}$ are mutually orthogonal, the rest of the proof being an easy exercise.

For ψ_{j_0,k_0} and ψ_{j_1,k_1} distinct, we may assume $j_0 > j_1$ or else $j_0 = j_1$ and $k_0 > k_1$. Then

$$p_{j_0,j_1}(z) = p_{j_1,k_1}(z)(z-c_{j_1})^{\alpha_{j_1}-k_1}(z-c_{j_1+1})^{\alpha_{j_1+1}}\cdots(z-c_{j_0})^{k_0},$$

and

$$\begin{aligned}
&\langle\psi_{j_0,k_0}(z),\psi_{j_1,k_1}(z)\rangle_{|p|^2\varphi}\\
&\quad=\int_{\Omega\setminus\{c_j\}_{j=1}^m}\frac{K^{\Omega}_{|q_{j_0,k_0}|^2\varphi}(z,c_{j_0})}{p_{j_0,k_0}(z)}\,\overline{\frac{K^{\Omega}_{|q_{j_1,k_1}|^2\varphi}(c_{j_1},z)}{p_{j_1,k_1}(z)}}\,|p(z)|^2\varphi(z)\,dV_z\\
&\quad=\int_{\Omega}K^{\Omega}_{|q_{j_0,k_0}|^2\varphi}(z,c_{j_0})\\
&\qquad\times K^{\Omega}_{|q_{j_1,k_1}|^2\varphi}(c_{j_1},z)\overline{(z-c_{j_1})^{\alpha_{j_1}-k_1}(z-c_{j_1+1})^{\alpha_{j_1+1}}\cdots(z-c_{j_0})^{k_0}}\\
&\qquad\times|q_{j_0,k_0}(z)|^2\varphi(z)\,dV_z\\
&\quad=0.
\end{aligned}$$

$\square$

The proof does not depend on m being finite; we can still construct an orthonormal basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega\setminus\{c_j\}_{j=1}^m)$. However, this is of limited practical value since, in that case, Theorem 3.4 failed to give a closed form expression for the original weighted kernel. Moreover, in practice, the simpler Theorem 3.1 is sufficient.

4. Zeros of weighted kernels. We now study the relationship the zeros of these weighted kernels have to the zeros of the simpler kernels.

Theorem 4.1. *Let Ω be a domain in $\mathbb{C}$, let $c, z_0, w_0 \in \Omega$, and let φ be a weight on Ω that is bounded and bounded away from zero in a neighborhood of c. Suppose that $K_{|z-c|^2\varphi}(z_0, w_0) = 0$. Then $K_\varphi(z_0, w_0) = 0$ if and only if either $K_\varphi(z_0, c) = 0$ or $K_\varphi(c, w_0) = 0$.*

Proof. By the hypothesis and Theorem 3.1,

$$0 = \frac{K_\varphi(z_0, w_0)}{(z_0 - c)(\overline{w_0} - \overline{c})} - \frac{K_\varphi(z_0, c)K_\varphi(c, w_0)}{(z_0 - c)(\overline{w_0} - \overline{c})K_\varphi(c, c)},$$

from which the theorem is evident. □

Requiring that φ be bounded and bounded away from zero in a neighborhood of c determines the order of the zero of the weight $|z - c|^2\varphi(c)$ to be 2, a fact to which there are two significant consequences. First, as a consequence of the L^2-version of the Riemann removable singularity theorem,

$$K_\varphi^\Omega(z, w) \equiv K_\varphi^{\Omega\setminus\{c\}}(z, w) \quad \text{on } (\Omega \setminus \{c\}) \times (\Omega \setminus \{c\}).$$

We employ this fact in the several next theorems without comment. Second, for zeros of higher orders in the weight, we would need to use Theorem 3.4 rather than Theorem 3.1, which would not give the conclusion of Theorem 4.1.

Theorem 4.1 says the value of $K_\varphi^\Omega(z, w)$ at c affects the zero set of $K_{|z-c|^2\varphi}^\Omega(z, w)$. Compare this to the case where $c \notin \Omega$, in which case Theorem 2.2 says that the zero sets of both kernels coincide.

Theorem 4.1 assumes $K_{|z-c|^2\varphi}^\Omega(z, w)$ has a zero and then states when $K_\varphi^\Omega(z, w)$ has a zero. The next theorem assumes $K_\varphi(z, w)$ has a zero and then states when $K_{|z-c|^2\varphi}^\Omega(z, w)$ has a zero.

Theorem 4.2. *Let Ω be a domain in $\mathbb{C}$, let z_0, $c \in \Omega$ with $z_0 \neq c$, and let φ be a weight on Ω that is bounded and bounded away from zero in a neighborhood of c. Suppose $K_\varphi(z_0, c) = 0$. Then $K_{|z-c|^2\varphi}(z_0, w)$ has a zero of order $m - 1$ at $w = c$ if and only if $K_\varphi(z_0, w)$ has a zero of order m at $w = c$.*

Proof. By Theorem 3.1,

$$K_{|z-c|^2\varphi}(z_0,w) = \frac{K_\varphi(z_0,w)}{(z_0-c)(\overline{w}-\overline{c})} - \frac{K_\varphi(z_0,c)K_\varphi(c,w)}{(z_0-c)(\overline{w}-\overline{c})K_\varphi(c,c)}$$
$$= \frac{1}{z_0-c}\cdot\frac{K_\varphi(z_0,w)}{\overline{w}-\overline{c}}.$$

If m is the order of the zero of $K_\varphi(z_0,w)$ at $w=c$, then this last expression has a zero of order $m-1$ at $w=c$. □

Theorem 4.3. *Let Ω be a domain in $\mathbb{C}$, let $c_0, c_1, c_2 \in \Omega$ be distinct, and let φ be a weight on Ω that in some neighborhood of c_0 is bounded and bounded away from zero. Suppose either $K_\varphi(c_0,c_1)=0$ or $K_\varphi(c_0,c_2)=0$. Then $K_{|z-c_0|^2\varphi}(c_1,c_2)=0$ if and only if $K_\varphi(c_1,c_2)=0$.*

Proof. By Theorem 3.1,

$$K_{|z-c_0|^2\varphi}(c_1,c_2) = \frac{K_\varphi(c_1,c_2)}{(c_1-c_0)(\overline{c_2}-\overline{c_0})} - \frac{K_\varphi(c_1,c_0)K_\varphi(c_0,c_2)}{(c_1-c_0)(\overline{c_2}-\overline{c_0})K_\varphi(c_0,c_0)}$$
$$= \frac{1}{(c_1-c_0)(\overline{c_2}-\overline{c_0})}\cdot K_\varphi(c_1,c_2),$$

from which the theorem is evident. □

Theorem 4.4. *Let Ω be a domain in $\mathbb{C}$, and let φ be a weight on Ω. Suppose that, for some $c_0 \in \partial\Omega$ and some sequence $\{c_j\}_{j=1}^\infty$ in Ω converging to c_0, we have*

$$\frac{K_\varphi(z,\, c_j)}{K_\varphi(c_j,\, c_j)} \longrightarrow 0 \quad \textit{as } j\to\infty$$

for all fixed $z \in \Omega$. Suppose also that there exist $z_0, w_0 \in \Omega$ such that $K_\varphi(z_0,w_0)=0$ and that $K_\varphi(z,c_j)$ is bounded away from 0 when j is large enough and z is in a compact subset of Ω. Then, for sufficiently large j, i.e., for c_j sufficiently close to $c_0 \in \partial\Omega$, there exists a $z_1 = z_1(c_j) \in \Omega$ near z_0 such that $K_{|z-c_j|^2\varphi}(z_1,w_0)=0$.

Proof. Define the following for all $\zeta, \omega, z \in \Omega$ and $\varepsilon > 0$:

$$g_{\zeta,\omega}(z) := \frac{K_\varphi(z,\omega)}{K_\varphi(z,\zeta)}; \qquad \alpha(\zeta) := |g_{\zeta,w_0}(\zeta)| = \left|\frac{K_\varphi(\zeta,w_0)}{K_\varphi(\zeta,\zeta)}\right|;$$

and

$$B(z,\varepsilon) := \{w \in \Omega \mid |z-w| < \varepsilon\} \quad \text{(the usual open } \varepsilon\text{-ball about } z).$$

Observe that, by hypothesis, $\alpha(c_j) \to 0$ as $j \to \infty$. Let $d := \frac{1}{2}\operatorname{dist}(z_0, \partial\Omega)$. Choose a $j_0 \in \mathbb{N}$ so that the following hold:

(6) $1/j_0 < d$, and
(7) $|c_j - c_0| < 1/j_0$ for all $j > j_0$.

By (6) and the definition of d,

(8) the closed ball $\overline{B(z_0, 1/j_0)}$ is contained in Ω.

By hypothesis, for j large enough, $K_\varphi(z, c_j)$ is bounded away from zero for $z \in \overline{B(z_0, 1/j_0)}$. Thus, for j large enough,

$$g_{c_j,w_0}(z) := \frac{K_\varphi(z,\, w_0)}{K_\varphi(z,\, c_j)}$$

and $K_\varphi(z, w_0)$ have the same zeroes on $\overline{B(z_0, 1/j_0)}$. So, by possibly increasing j_0, we can choose j_0 large enough so that we also have

(9) $\overline{B(z_0, 1/j_0)}$ contains a single zero of $g_{c_j,w_0}(z)$ when $j > j_0$, namely z_0.

Now, choose $j_1 \geq j_0$ such that

(10) $\alpha(c_j) < 1/j_0$ for all $j \geq j_1$, and
(11)
$$\alpha(c_j) < \inf\left\{|g_{c_{j_1},w_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right)\right\}$$
for all $j \geq j_1$.

Now, we argue that $C_0 := g_{c_{j_1},w_0}(\partial B(z_0, 1/j_0))$ is a closed curve about the origin and the point $g_{c_{j_1},w_0}(c_{j_1})$. Since z_0 is a zero of the holomorphic function $g_{c_{j_1},w_0}(z)$ and $\partial B(z_0, 1/j_0)$ is a closed curve about z_0, it follows from the argument principle of the elementary theory of holomorphic functions that C_0 is a closed curve about the origin. Moreover,

$$\alpha(c_{j_1}) < \inf\left\{|g_{c_{j_1},w_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right)\right\}$$

by equation (6), and so, C_0 also encloses a region containing $g_{c_{j_1},w_0}(c_{j_1})$, that is,

$$|g_{c_{j_1},w_0}(c_{j_1})| < |g_{c_{j_1},w_0}(z)|$$

on $\partial B\,(z_0, 1/j_0)$. By Rouché's theorem [**4,** page 110], it follows that the function $g_{c_{j_1},w_0}(z) - g_{c_{j_1},w_0}(c_{j_1})$ has a zero in $B\,(z_0, 1/j_0)$. Hence, for some $z_1 \in B\,(z_0, 1/j_0)$, we have $g_{c_{j_1},w_0}(z_1) = g_{c_{j_1},w_0}(c_{j_1})$, which is equivalent to

$$\frac{K_\varphi(z_1,\, w_0)}{K_\varphi(z_1,\, c_{j_1})} = \frac{K_\varphi(c_{j_1},\, w_0)}{K_\varphi(c_{j_1},\, c_{j_1})}.$$

Since both $|z_0 - z_1| < d$ and $|c_0 - c_{j_1}| < d$, it must be that $z_1 \neq c_{j_1}$. Therefore, $K_{|z-c_{j_1}|^2\varphi}(z_1, w_0) = 0$. □

When $c \notin \Omega$, then

$$K_{|z-c|^2\varphi}(z, w) = \frac{K_\varphi(z,\, w)}{(z-c)(\overline{w}-\overline{c})}$$

by Theorem 2.2, so the zero set of $K_{|z-c|^2\varphi}(z, w)$ corresponds to the zero set of $K_\varphi(z, w)$ in that case. An interpretation of Theorem 4.4 is that, for $c \in \Omega$ as c approaches the boundary of Ω, the zero set of $K_{|z-c|^2\varphi}(z, w)$ approaches the zero set of $K_\varphi(z, w)$. The next corollary to Theorem 4.2 does not assume that c is near the boundary of Ω, although unlike in Theorem 4.4, we assume c is adapted to a zero of the kernel.

Corollary 4.5 (Corollary to Theorem 4.2)**.** *Let Ω be a domain in $\mathbb{C}$, and let φ be a weight on Ω. Suppose $c, w_0 \in \Omega$ is such that $K_\varphi(z, w_0)$ has a zero of order $m > 1$ at $z = c$. Then, there exist $z_1, z_2, \ldots, z_{m-1}, w_1 \in \Omega$ with the z_j near z_0 and w_1 near w_0 such that $K_{|z-c|^2\varphi}(z_j, w_1) = 0$ for $j = 1, \ldots, m-1$.*

Proof. Apply Hurwitz's theorem to the conclusion of Theorem 4.2. □

Theorem 4.6.

(A) *Suppose $\Omega \subset \mathbb{C}$ is a domain, φ a weight, and $\{c_j\}_{j=1}^{\infty}$ is a sequence in Ω converging to a point $c_0 \in \partial\Omega$ such that, for fixed z,*

$$\frac{K_\varphi(z, c_j)}{\sqrt{K_\varphi(c_j, c_j)}} \longrightarrow 0 \quad \text{as } j \to \infty.$$

Suppose also that $K_{|z-c|^2\varphi}(z_0, w_0) = 0$ for all $c \in \Omega$. Then, either

(a) *both $K_\varphi(z_0, w) \equiv 0$ and $K_{|z-c|^2\varphi}(z_0, w) \equiv 0$ as functions of w for all c; or*

(b) *both $K_\varphi(z, w_0) \equiv 0$ and $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as functions of z for all c.*

(B) *For any domain Ω and weight φ, if $K_\varphi(z, w_0) \equiv 0$ as a function of z, then for all $c \in \mathbb{C}$, $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as well.*

Remark 4.7. Part (B) is similar to Theorem 4.1 and follows from Theorem 4.1, the hypothesis that $K_\varphi(z, w_0) \equiv 0$, and continuity.

Proof. We prove part (A) first. The proof of part (B) will be obvious from the proof of part (A) and is omitted.

Let $c \in \Omega$. Assume first that $z_0 \neq c$ and $w_0 \neq c$. Then, by Theorem 3.1, we must have

$$K_\varphi(z_0, w_0) = \frac{K_\varphi(z_0, c) K_\varphi(c, w_0)}{K_\varphi(c, c)}. \tag{4.1}$$

The right hand side of equation (4.1), vanishes when we replace c with c_j and let $j \to \infty$. Hence $K_\varphi(z_0, w_0) = 0$, and therefore either

$$K_\varphi(z_0, c) = 0$$

or

$$K_\varphi(c, w_0) = 0.$$

One of these two conditions must hold for a set of values of c having an accumulation point, hence for all c. Assume, without loss of generality, that $K_\varphi(c, w_0) = 0$ for all c. Thus, $K_\varphi(z, w_0) \equiv 0$ as a function of z.

But, then

$$K_\varphi(z,w_0)=\frac{K_\varphi(z,c)K_\varphi(c,w_0)}{K_\varphi(c,c)}=0 \quad \text{for all } z,$$

and hence, (by Theorem 3.1) $K_{|z-c|^2\varphi}(z,w_0)\equiv 0$ as a function of z. $\square$

Since Theorems 4.4 and 4.6 have a hypothesis requiring or implied by the condition

$$\frac{K_\varphi(z,c)}{\sqrt{K_\varphi(c,c)}}\longrightarrow 0 \quad \text{as } c\to c_0\in\partial\Omega,$$

we state sufficient conditions on a domain for this limit condition to be satisfied. Below is [**6,** Lemma 4.1 (2)] which is "implicit in work of Pflug (see [**8,** subsection 7.6]) and Ohsawa [**12**] on the completeness of the Bergman metric" according to Fu and Straube [**6**].

Theorem 4.8. *Let $\Omega\subset\mathbb{C}^n$ be a bounded pseudoconvex domain. Suppose p_0 is a point in the boundary of Ω satisfying the following outer cone condition*:

> *there exist $r\in(0,1]$, $a\geq 1$ and a sequence $\{w_\ell\}_{\ell=1}^\infty$ of points $w_\ell\notin\Omega$ with $\lim_{\ell\to\infty}w_\ell=p_0$ and $\Omega\cap B(w_\ell,r\|w_\ell-p_0\|^a)=\emptyset$.*

Then, for any sequence $\{p_j\}_{j=1}^\infty\subset\Omega$ converging to p_0,

$$\lim_{j\to\infty}\frac{K^\Omega(z,p_j)}{\sqrt{K^\Omega(p_j,p_j)}}=0.$$

The outer cone condition of Theorem 4.8 is satisfied when Ω has a C^1 boundary, for example. Pseudoconvexity is a central notion in several complex variables which reduces to a triviality for domains of a single complex dimension: every domain in the plane is pseudoconvex [**9**]. Because we also wish to have the conclusion of the above theorem for certain weighted kernels, we show that the property addressed by the theorem is preserved when the weight of a kernel is multiplied by the modulus squared of a linear factor.

Theorem 4.9. *Suppose $\Omega \subset \mathbb{C}$ is a domain, $p_0 \in \partial\Omega$, and $\{p_j\}_{j=1}^{\infty} \subset \Omega$ is a sequence with $p_j \to p_0$ as $j \to \infty$ such that*

$$\frac{K_\varphi(z, p_j)}{\sqrt{K_\varphi(p_j, p_j)}} \longrightarrow 0 \quad \text{as } j \to \infty$$

locally uniformly. Then, for any $c \in \Omega$ with $K_\varphi(c, c) \neq 0$,

$$\frac{K_{|z-c|^2\varphi}(z, p_j)}{\sqrt{K_{|z-c|^2\varphi}(p_j, p_j)}} \longrightarrow 0 \quad \text{as } j \to \infty$$

locally uniformly.

Proof. From Theorem 3.1, we obtain

$$\frac{K_{|z-c|^2\varphi}(z, p_j)}{\sqrt{K_{|z-c|^2\varphi}(p_j, p_j)}} = \frac{\frac{K_\varphi(z,p_j)K_\varphi(c,c) - K_\varphi(z,c)K_\varphi(c,p_j)}{(z-c)(\overline{p_j}-\overline{c})K_\varphi(c,c)}}{\left(\frac{K_\varphi(p_j,p_j)K_\varphi(c,c) - |K_\varphi(p_j,c)|^2}{|p_j-c|^2 K_\varphi(c,c)}\right)^{1/2}}$$

$$= \frac{|p_j - c|^2 K_\varphi(c,c)^{1/2}}{(z-c)(\overline{p_j} - \overline{c})K_\varphi(c,c)} \cdot \frac{K_\varphi(z,p_j)K_\varphi(c,c) - K_\varphi(z,c)K_\varphi(c,p_j)}{(K_\varphi(p_j,p_j)K_\varphi(c,c) - |K_\varphi(p_j,c)|^2)^{1/2}}$$

$$= \frac{(p_j - c)}{(z-c)K_\varphi(c,c)^{1/2}} \cdot \frac{\frac{K_\varphi(z,p_j)K_\varphi(c,c)}{K_\varphi(p_j,p_j)^{1/2}} - \frac{K_\varphi(z,c)K_\varphi(c,p_j)}{K_\varphi(p_j,p_j)^{1/2}}}{\left(K_\varphi(c,c) - \frac{|K_\varphi(p_j,c)|^2}{K_\varphi(p_j,p_j)}\right)^{1/2}}.$$

The first factor approaches a constant as $j \to \infty$. In the second factor, every fraction in the numerator and the denominator approaches zero as $j \to \infty$ locally uniformly by hypothesis, so the second factor approaches zero as $j \to \infty$ locally uniformly. This proves the theorem. □

5. Further questions. Consider the (unweighted) kernel $K(z, w)$ for the unit disk $\mathbb{D}$. By summing an appropriate orthonormal basis in equation (2), it can be shown that, for any real α greater than -2,

$$\begin{aligned} K_{|z|^\alpha}(z, w) &= K(z, w) + \frac{\alpha}{2\pi(1 - z\overline{w})} \\ &= \left(1 + \frac{\alpha}{2} - \frac{\alpha}{2} z\overline{w}\right) K(z, w). \end{aligned}$$

(The reader might verify that this formula agrees with Theorem 3.1 when $\alpha = 2p$, $p \in \mathbb{N}$.) Now let $c \in D$ and $p \in \mathbb{N}$. Using this formula, the classical change of variables theorem for Bergman kernels and Theorem 2.2, one obtains

$$
\begin{aligned}
K_{|z-c|^{2p}}(z,w) &= \frac{K_{|\mu_c|^{2p}}(z,w)}{(1-\overline{c}z)^p(1-c\overline{w})^p} \\
&= \left((p+1) - p\mu_c(z)\,\overline{\mu_c(w)}\right)\frac{K(z,w)}{(1-\overline{c}z)^p(1-c\overline{w})^p}.
\end{aligned}
$$

What is the formula if p is allowed to be real, that is, what is the formula for $K_{|z-c|^\alpha}(z,w)$, $\alpha \in \mathbb{R}$? In particular, what is the formula when $\alpha = 1$?

Generalizing the previous question, is there a technique for computing $K^\Omega_\varphi(z,w)$ explicitly in terms of $K^\Omega(z,w)$ in the case where φ is the modulus of a meromorphic function rather than the *square* of the modulus a meromorphic function? Is there such a technique when φ is harmonic?

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ENDNOTES

1. Sesqui-holomorphic means holomorphic in the first variable and conjugate holomorphic in the second variable.

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