ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 46, Number 5, 2016

# POLYNOMIAL FIRST INTEGRALS FOR WEIGHT-HOMOGENEOUS PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS OF WEIGHT DEGREE 4

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ABSTRACT. We classify all of the weight-homogeneous planar polynomial differential systems of weight degree 4 having a polynomial first integral.

**1. Introduction and statement of the main result.** In this paper, we deal with polynomial differential systems of the form:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \mathbb{C}^2, \tag{1.1}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), P_2(\mathbf{x}))$ and $P_i \in \mathbb{C}[x, y]$ for $i = 1, 2$. As usual, $\mathbb{Q}^+$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of non-negative rational, real and complex numbers, respectively, and $\mathbb{C}[x, y]$ denotes the polynomial ring over $\mathbb{C}$ in the variables $x$, $y$. Here, $t$ is real or complex.

System (1.1) is *weight homogeneous* or *quasi-homogeneous* if there exist $\mathbf{s} = (s_1, s_2) \in \mathbb{N}^2$ and $d \in \mathbb{N}$ such that, for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$,

$$P_i(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_i-1+d}P_i(x, y), \tag{1.2}$$

for $i = 1, 2$. We call $\mathbf{s} = (s_1, s_2)$ the *weight exponent* of system (1.1) and $d$ the *weight degree* with respect to the weight exponent $\mathbf{s}$. In the particular case where $\mathbf{s} = (1, 1)$, system (1.1) is called a *homogeneous polynomial differential system* of degree $d$.

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2010 AMS *Mathematics subject classification.* Primary 34A05, 34A34, 34C14.

*Keywords and phrases.* Polynomial first integrals, weight-homogeneous polynomial differential systems.

The first author is partially supported by MINECO/FEDER, grant Nos. MTM2008-03437 and MTM2013-40998-P, ICREA Academia, grant Nos. FP7-PEOPLE-2012-IRSES 318999 and 316338, AGAUR, grant No. 2014SGR-568, and grant No. UNAB13-4E-1604. The second author is supported by Portuguese national funds through FCT, Fundação para a Ciência e a Tecnologia, grant No. PEst-OE/EEI/LA0009/2013 (CAMGSD).

Received by the editors on October 9, 2012, and in revised form on January 9, 2015.

DOI:10.1216/RMJ-2016-46-5-1619 

Recently, such systems have been investigated by several authors. Labrunie **[12]** and Ollagnier **[15]** characterized all polynomial first integrals of the three-dimensional $(a, b, c)$ Lotka-Volterra systems. Maciejewski and Strelcyn **[14]** proved that the so-called Halphen system has no algebraic first integrals. However, some of the best results for general weight homogeneous polynomial differential systems have been provided by Furta **[10]** and Goriely **[11]**. For quadratic homogeneous polynomial differential systems, we refer the reader to **[13, 16]**. Additionally, we refer the reader to [**1**–**5**].

A non-constant function $H(x, y)$ is a *first integral* of system (1.1) if it is constant on all solution curves $(x(t),\ y(t))$ of system (1.1), i.e., $H(x(t), y(t))$is constant for all values of $t$ for which the solution $(x(t), y(t))$ is defined. If $H$ is $C^1$, then $H$ is a first integral of system (1.1) if and only if

$$P_1 \frac{\partial H}{\partial x} + P_2 \frac{\partial H}{\partial y} = 0. \tag{1.3}$$

The function $H(x, y)$ is weight homogeneous of weight degree $m$ with respect to the weight exponent $\mathbf{s}$ if it satisfies $H(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^m H(x, y)$, for all $\alpha \in \mathbb{R}^+$.

Given $H \in \mathbb{C}[x, y]$, we can split it into the form

$$H = H_m + H_{m+1} + \cdots + H_{m+l},$$

where $H_{m+i}$ is a weight homogeneous polynomial of weight degree $m+i$ with respect to the weight exponent $\mathbf{s}$, i.e.,

$$H_{m+i}(\alpha^{s_1} x, \alpha^{s_2} y) = \alpha^{m+i} H_{m+i}(x, y).$$

The following well-known proposition (see **[13]** for proof) reduces the study of the existence of analytic first integrals of a weight-homogeneous polynomial differential system (1.1) to the study of the existence of a weight-homogeneous polynomial first integral.

**Proposition 1.1.** *Let $H$ be an analytic function, and let*

$$H = \sum_i H_i$$

*be its decomposition into weight-homogeneous polynomials of weight degree $i$ with respect to the weight exponent* $\mathbf{s}$. *Then, $H$ is an analytic first* *integral of the weight-homogeneous polynomial differential system* (1.1) *if and only if each weight-homogeneous part* $H_i$ *is a first integral of system* (1.1) *for all* $i$.

The main goal of this paper is to classify all analytic first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. In view of Proposition 1.1, we only need to classify all polynomial first integrals of the weight-homogenous planar polynomial differential systems of weight degree 4. The classification of all polynomial first integrals (and hence, of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 1 is straightforward and trivial. The classification of all polynomial first integrals (and hence, of all analytic first integrals) of the weight-homogenous planar polynomial differential systems of weight degree 2 was given in [**8, 13**] and for systems of weight degree 3 in [**6, 8**].

Kowalevskaya exponents are used in the classification of all polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degrees 2 and 3. However, it was shown in [**6,** Theorem 4] that these exponents are useless for classifying the polynomial first integrals for weight-homogenous planar polynomial differential systems of weight degrees larger than 3.

**Proposition 1.2.** *The systems with weight degree* 4 *in* $\mathbb{C}^2$ *and their corresponding values of* $\mathbf{s}$ *can be written as follows*:

$$\begin{aligned}
\mathbf{s}=(1,1) &: \dot{x}=a_{40}x^4+a_{31}x^3y+a_{22}x^2y^2+a_{13}xy^3+a_{04}y^4,\\
&\quad\ \dot{y}=b_{40}x^4+b_{31}x^3y+b_{22}x^2y^2+b_{13}xy^3+b_{04}y^4;\\
\mathbf{s}=(1,2) &: \dot{x}=a_{40}x^4+a_{21}x^2y+a_{02}y^2,\\
&\quad\ \dot{y}=b_{50}x^5+b_{31}x^3y+b_{12}xy^2;\\
\mathbf{s}=(1,3) &: \dot{x}=a_{40}x^4+a_{11}xy, \qquad \dot{y}=b_{60}x^6+b_{31}x^3y+b_{02}y^2;\\
\mathbf{s}=(1,4) &: \dot{x}=a_{40}x^4+a_{01}y, \qquad \dot{y}=b_{70}x^7+b_{31}x^3y;\\
\mathbf{s}=(2,3) &: \dot{x}=a_{11}xy, \qquad \dot{y}=b_{30}x^3+b_{02}y^2;\\
\mathbf{s}=(2,5) &: \dot{x}=a_{01}y, \qquad \dot{y}=b_{40}x^4;\\
\mathbf{s}=(3,3) &: \dot{x}=a_{20}x^2+a_{11}xy+a_{02}y^2, \qquad \dot{y}=b_{20}x^2+b_{11}xy+b_{02}y^2;\\
\mathbf{s}=(6,9) &: \dot{x}=a_{01}y, \qquad \dot{y}=b_{20}x^2.
\end{aligned}$$

The proof of Proposition 1.2 is provided in Section 2.

In what follows, we state our main result, i.e., we classify when the systems of Proposition 1.2 exhibit a polynomial first integral. The systems with weight exponent $(1,1)$ having a polynomial first integral are given in Section 3. The systems with weight exponent $(3,3)$ having a polynomial first integral are studied inside the systems with weight exponent $(1,2)$. The polynomial first integrals for the other systems of Proposition 1.2 are provided in this introduction.

We introduce the change $(X,Y)=(x^2,y)$ in the planar weight homogeneous polynomial differential systems (1.1) of weight degree 4 with weight exponent $(1,2)$. With these new variables $(X,Y)$ the system with weight exponent $(1,2)$ becomes, after introducing the new independent variable $d\tau = x\,dt$,

$$\begin{aligned} X' &= 2a_{40}X^2 + 2a_{21}XY + 2a_{02}Y^2, \\ Y' &= b_{50}X^2 + b_{31}XY + b_{12}Y^2, \end{aligned} \tag{1.4}$$

where the prime denotes the derivative with respect to $\tau$.

System (1.4) is a homogeneous quadratic planar polynomial system with $\mathbf{s}=(1,1)$. It is well known (see **[9]**) that, for each quadratic homogeneous system, there exist some linear transformation and a rescaling of time which transform system (1.4) into systems in (1.5).

$$\begin{array}{ll} \dot{x} = -2xy + \frac{2}{3}x(p_1x+p_2y), & \dot{y} = -x^2+y^2+\frac{2}{3}y(p_1x+p_2y), \\ \dot{x} = -2xy + \frac{2}{3}x(p_1x+p_2y), & \dot{y} = x^2+y^2+\frac{2}{3}y(p_1x+p_2y), \\ \dot{x} = -x^2 + \frac{2}{3}x(p_1x+p_2y), & \dot{y} = 2xy+\frac{2}{3}y(p_1x+p_2y), \\ \dot{x} = \frac{2}{3}x(p_1x+p_2y), & \dot{y} = x^2+\frac{2}{3}y(p_1x+p_2y), \\ \dot{x} = \frac{2}{3}x(p_1x+p_2y), & \dot{y} = \frac{2}{3}y(p_1x+p_2y). \end{array} \tag{1.5}$$

We prove the following theorem which characterizes all polynomial first integrals for the systems in (1.5).

**Theorem 1.3.** *The homogeneous polynomial systems in* (1.5) *have a polynomial first integral $H$ if and only if one of the following conditions hold.*

(a) *The first system in* (1.5) *with $p_1=0$, $p_2=3(1-q)/(1+2q)$ with $q=n/m\in\mathbb{Q}^+$ and, in this case, $H=x^m(3y^2-x^2)^n$.*

(b) *The second system in* (1.5) *with* $p_1 = 0$, $p_2 = 3(1-q)/(1+2q)$ *with* $q = n/m \in \mathbb{Q}^+$ *and, in this case,* $H = x^m(3y^2 + x^2)^n$.
(c) *The third system, in* (1.5) *with* $p_1 = 0$ *and* $p_2 = 3(1-2q)/(2(1+q))$ *with* $q = n/m \in \mathbb{Q}^+$ *and, in this case,* $H = x^m y^n$.
(d) *The fourth system in* (1.5) *with* $p_1 = p_2 = 0$ *and, in this case,* $H = x$.

We note that systems with weight exponent $(3, 3)$ coincide with systems (1.4), and hence, it can be written into systems in (1.5). Therefore, Theorem 1.3 applies to those systems.

The proof of Theorem 1.3 is given in Section 4.

**Theorem 1.4.** *The weight homogeneous polynomial differential systems with weight exponent* $(1, 3)$ *and weight degree* 4 *have a polynomial first integral* $H$ *if and only if the following conditions hold*:

(a) $a_{11} = a_{40} = 0$ *with* $H = x$.
(b) $b_{60} = b_{31} = b_{02} = 0$ *with* $H = y$.
(c) $(3a_{11} - b_{02})(3a_{11} - 2b_{02}) \neq 0$, $a_{40} = -a_{11}b_{31}/(3a_{11} - 2b_{02})$, $3a_{11}/(6a_{11} - 2b_{02}) = m/n \in \mathbb{Q}^+$ *and* $m/n < 1$ *with*

$$\begin{aligned}H = x^{3(n-m)}&((3a_{11} - 2b_{02})b_{60}x^6 + 2(3a_{11} - b_{02})b_{31}x^3y\\&- (9a_{11}^2 - 9a_{11}b_{02} + 2b_{02}^2)y^2)^m.\end{aligned}$$

(d) $b_{31}/a_{40} = -m/n$ *and* $m/n \in \mathbb{Q}^+$ *with* $H = x^{3m}(b_{60}x^3 + (b_{31} - 3a_{40})y)^{3n}$.
(e) $b_{02} = 0$, $a_{11} \neq 0$, $a_{40} = -b_{31}/3$ *and* $b_{06} = -(3a_{40} - b_{31})^2/(12a_{11})$ *with* $H = b_{31}x^3 - 3a_{11}y$.
(f) $(3a_{11} - b_{02})(3a_{11} - 2b_{02}) \neq 0$, $a_{40} = -a_{11}b_{31}/(3a_{11} - 2b_{02})$, $b_{06} = -(3a_{40} - b_{31})^2/(4(3a_{11} - b_{02}))$, $b_{02} \neq 0$ *and* $-3a_{11}/b_{02} = n/m \in \mathbb{Q}^+$, *with*

$$H = x^{3n}(b_{31}x^3 + (2b_{02} - 3a_{11})y)^m.$$

The proof of Theorem 1.4 is given in Section 5.

**Theorem 1.5.** *The weight homogeneous polynomial differential systems with weight exponent* $(1, 4)$ *and weight degree* 4 *have a polynomial first integral* $H$ *if and only if the following conditions hold*:

(a) $a_{40} = a_{01} = 0$ *and* $H = x$.
(b) $b_{70} = b_{31} = 0$ *and* $H = y$.
(c) $b_{31} = -4a_{40}$, *and* $4a_{40}^2 + a_{01}b_{70} \neq 0$ *with* $H = -b_{70}x^8 + 8a_{40}x^4y + 4a_{01}y^2$.
(d) $b_{31} = -4a_{40}$, $a_{40}b_{70} \neq 0$ *and* $4a_{40}^2 + a_{01}b_{70} = 0$ *with* $H = b_{70}x^4 - 4a_{40}y$.

The proof of Theorem 1.5 is given in Section 6.

**Theorem 1.6.** *The weight homogeneous polynomial differential systems with weight exponent* $(2,3)$ *and weight degree* $4$ *have a polynomial first integral* $H$ *if and only if* $a_{11} = 0$, *in which case* $H = x$, *or* $b_{30} = b_{02} = 0$, *in which case,* $H = y$, *or* $a_{11}(3a_{11} - 2b_{02}) \neq 0$ *and* $-2b_{02}/a_{11} = n/m \in \mathbb{Q}^+$, *in which case,*

$$H = x^n(2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2)^m.$$

The proof of Theorem 1.6 is given in Section 7.

**Theorem 1.7.** *The weight homogeneous polynomial differential systems with weight exponent* $(2,5)$ *and weight degree* $4$ *have the polynomial first integral* $H = 2b_{40}x^5 - 5a_{01}y^2$.

The proof of Theorem 1.7 is given in Section 8.

**Theorem 1.8.** *The weight homogeneous polynomial differential systems with weight exponent* $(6,9)$ *and weight degree* $4$ *have the polynomial first integral* $H = 2b_{20}x^3 - 3a_{01}y^2$.

The proof of Theorem 1.8 is given in Section 9.

**2. Proof of Proposition 1.2.** From the definition of weight homogeneous polynomial differential systems (1.1) with weight degree 4, the exponents $u_i$ and $v_i$ of any monomial $x^{u_i}y^{v_i}$ of $P_i$ for $i = 1, 2$, are such that they satisfy respectively the relations

$$s_1u_1 + s_2v_1 = s_1 + 3 \quad \text{and} \quad s_1u_2 + s_2v_2 = s_2 + 3, \tag{2.1}$$

respectively. Moreover, we can assume that $P_1$ and $P_2$ are coprime, and, without loss of generality, we can also assume that $s_1 \le s_2$. We consider different values of $s_1$.

*Case* $s_1 = 1$. If $s_2 = 1$, then in view of (2.1), we must have $u_1 + v_1 = 4$ and $u_2 + v_2 = 4$, that is, $(u_i, v_i) = (0, 4)$, $(u_i, v_i) = (1, 3)$, $(u_i, v_i) = (2, 2)$, $(u_i, v_i) = (3, 1)$ and $(u_i, v_i) = (4, 0)$, for $i = 1, 2$.

If $s_2 = 2$, then, in view of (2.1), we must have $u_1 + 2v_1 = 4$ and $u_2 + 2v_2 = 5$, that is, $(u_1, v_1) = (0, 2)$, $(u_1, v_1) = (2, 1)$ and $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (1, 2)$, $(u_2, v_2) = (3, 1)$, and finally, $(u_2, v_2) = (5, 0)$.

If $s_2 = 3$, then, in view of (2.1), we must have $u_1 + 3v_1 = 4$ and $u_2 + 3v_2 = 6$, that is, $(u_1, v_1) = (1, 1)$, $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (0, 2)$, $(u_2, v_2) = (3, 1)$, and finally, $(u_2, v_2) = (6, 0)$.

If $s_2 = 4$, then, in view of (2.1), we must have $u_1 + 4v_1 = 4$ and $u_2 + 4v_2 = 7$, that is, $(u_1, v_1) = (0, 1)$, $(u_1, v_1) = (4, 0)$, while $(u_2, v_2) = (3, 1)$, and finally, $(u_2, v_2) = (7, 0)$.

If $s_2 = 4 + l$ with $l \ge 1$, then equation (2.1) becomes

$$u_1 + (4 + l)v_1 = 4 \quad \text{and} \quad u_2 + (4 + l)v_2 = 7 + l. \tag{2.2}$$

From the first equation of (2.2), we obtain $v_1 = 0$ and $u_1 = 4$. By the second equation of (2.2), it follows that $v_2 \in \{0, 1\}$. If $v_2 = 0$, then $u_2 = 7 + l$; if $v_2 = 1$, then $u_2 = 3$. In both cases, $P_1$ and $P_2$ are not coprime. Thus, this case is not considered. $\square$

*Case* $s_1 = 2$. Now, we have $s_2 \ge 2$. If $s_2 = 2$, then in view of (2.1), we must have $2u_1 + 2v_1 = 5$ and $2u_2 + 2v_2 = 5$, which is not possible because 5 is not an even number.

If $s_2 = 3$, then, in view of (2.1), we must have $2u_1 + 3v_1 = 5$ and $2u_2 + 3v_2 = 6$, that is, $(u_1, v_1) = (1, 1)$, while $(u_2, v_2) = (0, 2)$ and $(u_2, v_2) = (3, 0)$.

If $s_2 = 4$, then, in view of (2.1), we must have $2u_1 + 4v_1 = 5$ and $2u_2 + 4v_2 = 7$, which is not possible because 5 is not even.

If $s_2 = 5$, then, in view of (2.1), we must have $2u_1 + 5v_1 = 5$ and $2u_2 + 5v_2 = 8$, that is, $(u_1, v_1) = (0, 1)$ and $(u_2, v_2) = (4, 0)$.

If $s_2 = 5 + l$ with $l \ge 1$, then equation (2.1) becomes

$$2u_1 + (5 + l)v_1 = 5 \quad \text{and} \quad 2u_2 + (5 + l)v_2 = 8 + l. \tag{2.3}$$

The first equation of (2.3) is not possible because 5 is not an even number, $5+l \geq 6$ and $u_1$ and $v_1$ are non-negative integers. $\square$

*Case* $s_1 = 3$. Now, we have $s_2 \geq 3$. If $s_2 = 3$, then in view of (2.1), we must have $3u_1 + 3v_1 = 6$ and $3u_2 + 3v_2 = 6$, that is, $(u_i, v_i) = (0, 2)$, $(u_i, v_i) = (1, 1)$ and $(u_i, v_i) = (2, 0)$, for $i = 1, 2$.

If $s_2 = 3 + l$ with $l \geq 1$, then in view of (2.1), we must have

$$3u_1 + (3+l)v_1 = 6 \quad \text{and} \quad 3u_2 + (3+l)v_2 = 6 + l. \tag{2.4}$$

From the first equation of (2.4) we have that

$$v_1 = \frac{6 - 3u_1}{3+l} \leq \frac{6}{3+l},$$

and, using $l \geq 1$, then $v_1 \in \{0, 1\}$.

When $v_1 = 0$, then $3u_1 = 6$; thus, $u_1 = 2$. Then, from the second equation of (2.4) we obtain that $v_2 \in \{0, 1\}$. If $v_2 = 0$, then $u_2 \geq 1$, and, if $v_2 = 1$, then $u_2 = 1$. In both cases, we have that $P_1$ and $P_2$ are not coprime.

When $v_1 = 1$, then $3u_1 = 3 - l$, which is not possible since $u_1$ is an integer and $l \geq 1$. $\square$

*Case* $s_1 = 3 + l$ with $l \geq 1$. Now, we have $s_2 \geq 3 + l$ with $l \geq 1$, and equation (2.1) becomes

$$(3+l)u_1 + s_2 v_1 = 6 + l = (3+l) + 3 \tag{2.5}$$

and

$$(3+l)u_2 + s_2 v_2 = 3 + s_2.$$

From the first equation of (2.5), and taking into account that $l \geq 1$, we obtain that $u_1 \in \{0, 1\}$.

When $u_1 = 0$, we must have $s_2 v_1 = 6 + l$, and since $s_2 \geq 3 + l$, we obtain

$$v_1 = \frac{6+l}{s_2} \leq \frac{(3+l)+3}{3+l} = 1 + \frac{3}{3+l}.$$

Since $v_1 \neq 0$ and $l \geq 1$, we must have $v_1 = 1$. Then $s_2 = 6 + l$. Now, the second equation of (2.5) becomes

$$(3+l)u_2 + (6+l)v_2 = (6+l) + 3. \tag{2.6}$$

Then $v_2 = 0$. From equation (2.6), we have $u_2 = 1 + (6/l + 3)$. Thus, $l = 3$ and $u_2 = 2$, and we obtain the systems with weight exponent $(6, 9)$.

When $u_1 = 1$, we must have $s_2 v_1 = 3$, and, since $s_2 \geq 3+l$, we obtain $(3 + l)v_2 \leq 3$, which is not possible because $l \geq 1$. This concludes the proof of the proposition. $\square$

**3. Weight exponent s** $= (1, 1)$**.** A weight homogeneous polynomial system,

$$\dot{x} = P_1(x, y); \qquad \dot{y} = P_2(x, y),$$

with weight exponent $(1, 1)$ and weight degree $d$ is integrable, and its inverse integrating factor is $V(x, y) = xP_2(x, y) - yP_1(x, y)$. See **[7]** for more details.

As $P_1(x, y)$, $P_2(x, y)$ and $V(x, y)$ are homogeneous polynomials, if the degree of $P_1(x, y)$ and $P_2(x, y)$ is $d$, then, of course, the degree of $V(x, y)$ is $d + 1$. Thus, for $d = 4$, we can write the homogeneous polynomials as follows:

$$\begin{aligned} P_1(x, y) &= (p_1 - a_1)x^4 + (p_2 - 4a_2)x^3 y + (p_3 - 6a_3)x^2 y^2 \\ &\quad + (p_4 - 4a_4)xy^3 - a_5 y^4, \\ P_2(x, y) &= a_0 x^4 + (4a_1 + p_1)x^3 y + (6a_2 + p_2)x^2 y^2 \\ &\quad + (4a_3 + p_3)xy^3 + (a_4 + p_4)y^4, \end{aligned} \tag{3.1}$$

and

$$V(x, y) = a_0 x^5 + 5a_1 x^4 y + 10a_2 x^3 y^2 + 10a_3 x^2 y^3 + 5a_4 xy^4 + a_5 y^5.$$

Thus, the first integral is

$$H(x, y) = \int \frac{P_1(x, y)}{V(x, y)}\, dy + g(x),$$

satisfying $\partial H/\partial x = -P_2/V$. The canonical forms appear in the factorization of $V$. Assume that $V(x, y)$ factorizes as:

(i) five simple real roots: $a_0(x - r_1 y)(x - r_2 y)(x - r_3 y)(x - r_4 y)(x - r_5 y)$,

(ii) one double and three simple real roots: $a_0(x - r_1 y)^2(x - r_2 y)(x - r_3 y)(x - r_4 y)$,

(iii) two double roots and one simple real root: $a_0(x-r_1y)^2(x-r_2y)^2(x-r_3y)$,
(iv) one triple and two simple real roots: $a_0(x-r_1y)^3(x-r_2y)(x-r_3y)$,
(v) one triple and one double real roots: $a_0(x-r_1y)^3(x-r_2y)^2$,
(vi) one quadruple and one simple real roots: $a_0(x-r_1y)^4(x-r_2y)$,
(vii) one quintuple real root: $a_0(x-ry)^5$,
(viii) three real and one couple of conjugate complex roots: $a_0(x-r_1y)(x-r_2y)(x-r_3y)(x^2+bxy+cy^2)$ with $b^2-4c<0$,
(ix) one double, one simple real and one couple of conjugate complex roots: $a_0(x-r_1y)^2(x-r_2y)(x^2+bxy+cy^2)$ with $b^2-4c<0$,
(x) one triple real and one couple of conjugate complex roots: $a_0(x-r_1y)^3(x^2+bxy+cy^2)$ with $b^2-4c<0$,
(xi) one simple real and two couples of conjugate complex roots: $a_0(x-ry)(x^2+b_1xy+c_1y^2)(x^2+b_2xy+c_2y^2)$ with $b_1^2-4c_1<0, b_2^2-4c_2<0$,
(xii) one simple real and one double couple of conjugate complex roots: $a_0(x-ry)(x^2+bxy+cy^2)^2$ with $b^2-4c<0$.

Now, we shall compute the first integral for each case and obtain the conditions in order to show that it is a polynomial.

We define the function:

$$f(r)=5(p_4+p_3r+p_2r^2+p_1r^3).$$

*Case* (i). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x-r_4y)^{\gamma_4}(x-r_5y)^{\gamma_5},$$

where

$$\gamma_i=\frac{f(r_i)+a_0\prod_{\substack{j=1\\ j\neq i}}^{5}(r_i-r_j)}{\prod_{\substack{j=1\\ j\neq i}}^{5}(r_i-r_j)}.$$

We note that an integer power of $H$ is a polynomial if and only if $\gamma_i\in\mathbb{Q}$ for $i=1$, 2, 3, 4, 5, if they all have the same sign.

*Case* (ii). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x-r_4y)^{\gamma_4}$$
$$\exp\left(\frac{f(r_1)x}{r_1(r_1-r_2)(r_1-r_3)(r_1-r_4)(x-r_1y)}\right),$$

with $\gamma_1 = A_1/B_1 = A_1/[(r_1-r_2)^2(r_1-r_3)^2(r_1-r_4)^2]$,

$$\begin{aligned}A_1 = {}& 5p_1((r_2+r_3+r_4)r_1^2 - 2(r_3r_4+r_2(r_3+r_4))r_1 + 3r_2r_3r_4)r_1^2 \\ &+ 5p_2(r_1^3 - (r_3r_4+r_2(r_3+r_4))r_1 + 2r_2r_3r_4)r_1 \\ &+ 5p_3(2r_1^3 - (r_2+r_3+r_4)r_1^2 + r_2r_3r_4) \\ &+ 5p_4(3r_1^2 - 2(r_2+r_3+r_4)r_1 + r_3r_4 + r_2(r_3+r_4)) - 2a_0B_1,\end{aligned}$$

for $i = 2, \ldots, 4$ and

$$\gamma_i = \frac{A_i}{B_i} = \frac{-(f(r_i)+a_0B_i)}{(r_1-r_i)^2\displaystyle\prod_{\substack{j=2\\ j\neq i}}^{4}(r_i-r_j)}.$$

We note that an integer power of $H$ is a polynomial if and only if $f(r_1) = 0$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, 3, 4$, if they all have the same sign.

*Case* (iii). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}\exp\left(-\sum_{i=1}^{2}\frac{f(r_i)x}{r_i(r_1-r_2)^2(r_i-r_3)(r_iy-x)}\right),$$

with $\gamma_i = A_i/B_i = A_i/[(r_1-r_2)^3(r_i-r_3)^2]$ for $i = 1, 2$, $\gamma_3 = A_3/B_3 = -(f(r_3)+a_0B_3)/[(r_1-r_3)^2(r_2-r_3)^2]$,

$$\begin{aligned}A_i = {}& -2a_0B_i + (-1)^{i+1}(5p_4(3r_i - r_j - 2r_3) - 5p_3((r_1+r_2)r_3 - 2r_i^2) \\ &+ 5p_2r_i(r_i(r_1+r_2) - 2r_jr_3) + 5p_1r_i^2(-3r_jr_3 + r_i(2r_j+r_3))),\end{aligned}$$

for $i$, $j = 1$, $2$ and $i \neq j$. We note that an integer power of $H$ is a polynomial if and only if $f(r_i) = 0$ for $i = 1$, $2$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, 3$, if they all have the same sign.

*Case* (iv). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}\exp\left(\frac{5\beta x}{2r_1^2(r_1-r_2)^2(r_1-r_3)^2(r_1y-x)^2}\right),$$

with

$$\begin{aligned}
\beta &= (p_1(r_1^2-3r_2r_1-3r_3r_1+5r_2r_3)r_1^3-p_2(r_1^2+r_2r_1+r_3r_1-3r_2r_3)r_1^2\\
&\quad -p_3(3r_1^2-r_2r_1-r_3r_1-r_2r_3)r_1+p_4(-5r_1^2+3r_2r_1+3r_3r_1-r_2r_3))x\\
&\quad + 2r_1(p_1(r_1r_2-2r_3r_2+r_1r_3)r_1^3+p_3(2r_1-r_2-r_3)r_1^2\\
&\quad + p_2(r_1^2-r_2r_3)r_1^2+p_4(3r_1^2-2r_2r_1-2r_3r_1+r_2r_3))y,\\
\gamma_1 &= A_1/B_1 = A_1/[(r_1-r_2)^3(r_1-r_3)^3],\\
A_1 &= -3a_0B_1 - 5(r_2(p_2+p_1r_2)r_1^3+(r_1-3r_2)(p_2+p_1r_2)r_3r_1^2\\
&\quad + (p_2r_2^2+p_1r_1(r_1^2-3r_2r_1+3r_2^2))r_3^2)+5p_3(r_1^3-3r_2r_3r_1\\
&\quad + r_2r_3(r_2+r_3))+5p_4(3r_1^2-3(r_2+r_3)r_1+r_2^2+r_3^2+r_2r_3),\\
\gamma_i &= A_i/B_i = (-1)^i(f(r_i)+a_0B_i)/[(r_1-r_i)^3(r_2-r_3)]
\end{aligned}$$

for $i=2,3$. We note that an integer power of $H$ is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3$, if they all have the same sign.

*Case* (v). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}\exp(\beta),$$

where

$$\begin{aligned}
\beta &= \frac{2(r_1-r_2)xf(r_2)r_1^2}{r_2(r_2y-x)}+\frac{(r_1-r_2)^2x^2f(r_1)}{(x-r_1y)^2}\\
&\quad + \frac{10(r_1-r_2)((2p_3+2p_1r_1r_2+p_2(r_1+r_2))r_1^2+p_4(3r_1-r_2))x}{r_1y-x},
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 &= -2r_1^2\big(3a_0(r_1-r_2)^4+15p_4+5p_3(r_1+2r_2)\\
&\quad + 5r_2(3p_1r_1r_2+p_2(2r_1+r_2))\big),\\
\gamma_2 &= -2r_1^2\big(2a_0(r_1-r_2)^4-15p_4-5p_3(r_1+2r_2)\\
&\quad - 5r_2(3p_1r_1r_2+p_2(2r_1+r_2))\big).
\end{aligned}$$

We note that an integer power of $H$ is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2$, if they all have the same sign.

*Case* (vi). A first integral $H$ is

$$(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}\exp(\beta),$$

where

$$\begin{aligned}\beta = {} & \frac{2(r_1-r_2)^3 f(r_1)x^3}{r_1^3(r_1y-x)^3} \\ & + \frac{30(r_1-r_2)((p_3+r_2(p_2+p_1r_2))r_1^3+p_4(3r_1^2-3r_2r_1+r_2^2))x}{r_1^3(r_1y-x)} \\ & + \frac{15(r_1-r_2)^2(p_4(3r_1-2r_2)+r_1((p_2+p_1r_2)r_1^2+p_3(2r_1-r_2)))x^2}{r_1^3(x-r_1y)^2},\end{aligned}$$

and

$$\begin{aligned}\gamma_1 &= -6(-4a_0(r_1-r_2)^4+5p_4+5r_2(p_3+r_2(p_2+p_1r_2))), \\ \gamma_2 &= 6(a_0(r_1-r_2)^4+5p_4+5r_2(p_3+r_2(p_2+p_1r_2))).\end{aligned}$$

We note that an integer power of $H$ is a polynomial if and only if $\beta = 0$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2$, if they all have the same sign.

*Case* (vii). A first integral $H$ is

$$(x-ry)^{\gamma_1} \exp\left(\frac{\beta x}{(x-ry)^4}\right),$$

where

$$\begin{aligned}\beta = {} & rx(rx(-p_2x+3p_1rx+4p_2ry)+p_3(x^2-4ryx+6r^2y^2)) \\ & -3p_4(x-2ry)(x^2-2ryx+2r^2y^2),\end{aligned}$$

and $\gamma_1 = -12a_0r^4$. We note that $x - ry$ is a polynomial first integral if and only if $\beta = 0$.

*Case* (viii). A first integral $H$ is

$$\begin{gathered}(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x-r_3y)^{\gamma_3}(x^2+bxy+cy^2)^{\gamma_4} \\ \exp\left(\frac{\beta x}{\prod_{i=1}^3(c+b\,r_i+r_i^2)\sqrt{(4c-b^2)x^2}}\arctan\left(\frac{bx+2cy}{\sqrt{(4c-b^2)x^2}}\right)\right),\end{gathered}$$

where

$$\begin{aligned}\beta = {} & 5(2p_1c^3-(b(p_2-p_1(r_1+r_2+r_3))+2(p_3+p_2(r_1+r_2+r_3) \\ & +p_1(r_2r_3+r_1(r_2+r_3)))))c^2+((p_3+p_1r_1r_2+p_1(r_1+r_2)r_3)b^2 \\ & +(3p_4-3p_1r_1r_2r_3+p_3(r_1+r_2+r_3)-p_2(r_2r_3+r_1(r_2+r_3)))b \\ & +2(p_2r_1r_2r_3+p_4(r_1+r_2+r_3)+p_3(r_2r_3+r_1(r_2+r_3))))c\end{aligned}$$

$$- bp_4(b+r_1)(b+r_2)+((p_1r_1b^3-p_2r_1b^2-p_4b+p_3r_1b-2p_4r_1)r_2 - bp_4(b+r_1))r_3),$$

and $\gamma_i = A_i/B_i = -(f(r_i)+a_0B_i)/[(c+br_i+r_i^2)\prod_{j=1,\, j\neq i}^3(r_i-r_j)]$ for $i=1,\,2,\,3$, $\gamma_4 = A_4/B_4 = A_4/[2\prod_{i=1}^3(c+b\,r_i+r_i^2)]$

$$\begin{aligned} A_4 = -a_0B_4 + 5((&p_2+p_1(r_1+r_2+r_3))c^2-(p_4+p_1r_1r_2r_3\\ &+p_3(r_1+r_2+r_3)+p_2(r_2r_3+r_1(r_2+r_3))\\ &+b(p_3-p_1(r_2r_3+r_1(r_2+r_3))))c+p_4(b+r_1)(b+r_2)\\ &+(p_4(b+r_1)+(p_4+(p_1b^2-p_2b+p_3)r_1)r_2)r_3). \end{aligned}$$

We note that an integer power of $H$ is a polynomial if and only if $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3,4$, if they all have the same sign.

*Case* (ix). A first integral $H$ is

$$\begin{aligned} &(x-r_1y)^{\gamma_1}(x-r_2y)^{\gamma_2}(x^2+bxy+cy^2)^{\gamma_3}\\ &\exp\bigg(-\frac{f(r_1)x}{r_1(r_1-r_2)(c+r_1b+r_1^2)(r_1y-x)}\\ &\quad+\frac{\beta x}{\prod_{i=1}^2(c+b\,r_i+r_i^2)^{3-i}\sqrt{(4c-b^2)x^2}}\ \arctan\bigg(\frac{bx+2cy}{\sqrt{(4c-b^2)x^2}}\bigg)\bigg), \end{aligned}$$

where

$$\begin{aligned} \beta = 5(&2p_1c^3-(b(p_2-p_1(2r_1+r_2))+2(p_3+p_2(2r_1+r_2)\\ &+p_1r_1(r_1+2r_2)))c^2+((p_3+p_1r_1(r_1+2r_2))b^2\\ &+(3p_4+p_3(2r_1+r_2)-r_1(3p_1r_1r_2+p_2(r_1+2r_2)))b\\ &+2(p_4(2r_1+r_2)+r_1(p_2r_1r_2+p_3(r_1+2r_2))))c-bp_4(b+r_1)^2\\ &+(-p_4b^2-2p_4r_1b+(b(p_1b^2-p_2b+p_3)-2p_4)r_1^2)r_2),\\ \gamma_1 = -&\frac{A_1}{B_1} = -\frac{A_1}{(r_1-r_2)^2(c+b\,r_1+r_1^2)^2},\\ A_1 = 2&a_0c^2(r_1-r_2)^2+2a_0b^2r_1^2(r_1-r_2)^2\\ &+c\big(-5p_4+r_1(4a_0(b+r_1)(r_1-r_2)^2+5p_1r_1(2r_1-3r_2)\\ &+5p_2(r_1-2r_2))-5p_3r_2)+b((4a_0r_1(r_1-r_2)^2-5p_3\\ &+5p_1r_1(r_1-2r_2)-5p_2r_2)r_1^2+5p_4(r_2-2r_1))+r_1(5p_4(2r_2-3r_1)\\ &+r_1((2a_0(r_1-r_2)^2-5p_2-5p_1r_2)r_1^2+5p_3(r_2-2r_1))), \end{aligned}$$

$$
\begin{aligned}
\gamma_2 &= \frac{A_2}{B_2} = \frac{-f(r_2)+a_0B_2}{(r_1-r_2)^2(c+b\,r_2+r_2^2)},\\
\gamma_3 &= \frac{A_3}{B_3} = \frac{A_3}{2(c+b\,r_1+r_1^2)^2(c+b\,r_2+r_2^2)},\\
A_3 &= -a_0B_3+5((p_2+p_1(2r_1+r_2))c^2-(p_4+p_3(2r_1+r_2)\\
&\quad +r_1(p_1r_1r_2+p_2(r_1+2r_2))+b(p_3-p_1r_1(r_1+2r_2)))c\\
&\quad +p_4(b+r_1)^2+((p_1b^2-p_2b+p_3)r_1^2+2p_4r_1+bp_4)r_2).
\end{aligned}
$$

We note that an integer power of $H$ is a polynomial if and only if $f(r_1)=0$, $\beta=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1,2,3$, if they all have the same sign.

*Case* (x). A first integral $H$ is

$$
\begin{aligned}
&(x-r_1y)^{\gamma_1}(x^2+bxy+cy^2)^{\gamma_2}\exp\left(\frac{f(r_1)(c+b\,r_1+r_1^2)^2x^2}{r_1^2(x-r_1y)^2}\right.\\
&\quad\left.-\frac{\beta_1(c+b\,r_1+r_1^2)x}{r_1^2(x-r_1y)}+\frac{\beta_2x}{\sqrt{(4c-b^2)x^2}}\arctan\left(\frac{bx+2cy}{\sqrt{(4c-b^2)x^2}}\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_1 &= 10((p_2-bp_1)r_1^4+2(p_3-cp_1)r_1^3+(-cp_2+bp_3+3p_4)r_1^2\\
&\quad +2bp_4r_1+cp_4),\\
\beta_2 &= 10((p_1r_1^3-p_4)b^3-r_1(p_2r_1^2+3p_4)b^2+r_1^2(p_3r_1-3p_4)b\\
&\quad -2p_4r_1^3+2c^3p_1-c^2(2p_3+b(p_2-3p_1r_1)+6r_1(p_2+p_1r_1))\\
&\quad +c((3p_1r_1^2+p_3)b^2+3(p_4+r_1(p_3-r_1(p_2+p_1r_1)))b\\
&\quad +2r_1(3p_4+r_1(3p_3+p_2r_1)))),
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_1 &= -2(3a_0(c+r_1(b+r_1))^3+5((p_1b^2-p_2b+p_3)r_1^3+3p_4r_1^2\\
&\quad +3bp_4r_1+b^2p_4+c^2(p_2+3p_1r_1)-c(p_4+b(p_3-3p_1r_1^2)\\
&\quad +r_1(3p_3+r_1(3p_2+p_1r_1))))),\\
\gamma_2 &= 5((p_1b^2-p_2b+p_3)r_1^3+3p_4r_1^2+3bp_4r_1+b^2p_4+c^2(p_2+3p_1r_1)\\
&\quad -c(p_4+b(p_3-3p_1r_1^2)+r_1(3p_3+r_1(3p_2+p_1r_1))))\\
&\quad -2a_0(c+r_1(b+r_1))^3.
\end{aligned}
$$

We note that an integer power $H$ is a polynomial if and only if $f(r_1) = 0$, $\beta_1 = \beta_2 = 0$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2$, if they all have the same sign.

*Case* (xi). A first integral $H$ is

$$(x^2 + b_1xy + c_1y^2)^{\gamma_1}(x^2 + b_2xy + c_2y^2)^{\gamma_2}(x - r_1y)^{\gamma_3}$$
$$\exp\left(\sum_{i=1}^{2} \frac{\beta_i x}{\sqrt{(4c_i - b_i^2)x^2}} \arctan\left(\frac{b_ix + 2c_iy}{\sqrt{(4c_i - b_i^2)x^2}}\right)\right),$$

with $\beta_i = \alpha_i/\delta_i$ for $i = 1, 2$, where

$$\begin{aligned}
\alpha_i = {}& 5(-b_i(c_i^2p_2 + c_ic_jp_2 + b_jc_i(c_ip_1 + p_3) - 3c_ip_4 + c_jp_4)\\
&+ b_i(c_i^2p_1 + c_jp_3 + c_i(-3c_jp_1 + b_jp_2 + p_3) + b_jp_4)r_1\\
&+ b_i^3(-p_4+c_jp_1r_1)+b_i^2(c_ic_jp_1+c_ip_3+b_jp_4 - (b_jc_ip_1+c_jp_2+p_4)r_1)\\
&+ 2(c_i^3p_1 - c_jp_4r_1 - c_i^2(c_jp_1 + p_3 + p_2r_1 - b_j(p_2 + p_1r_1))\\
&+ c_i(p_4r_1 + c_j(p_3 + p_2r_1) - b_j(p_4 + p_3r_1)))),\\
\delta_i = {}& ((b_2^2c_1 + (c_1 - c_2)^2 + b_1^2c_2 - b_1b_2(c_1 + c_2))(c_i + b_ir_1 + r_1^2),
\end{aligned}$$

for $i$, $j = 1, 2$ and $i \neq j$,

$$\begin{aligned}
\gamma_i = {}& -2a_0(b_j^2c_i + (c_i - c_j)^2 + b_i^2c_j - b_ib_j(c_i + c_j))(c_i + r_1(b_i + r_1))\\
&+ 5(b_i^2(p_4 + c_jp_1r_1) + b_i(c_ic_jp_1 - c_ip_3 - c_jp_2r_1 + p_4r_1)\\
&+ (c_i - c_j)(-p_4 - p_3r_1 + c_i(p_2 + p_1r_1)) - b_j(c_i^2p_1 + p_4(b_i + r_1)\\
&- c_i(p_3 + (-b_ip_1 + p_2)r_1))),
\end{aligned}$$

for $i, j = 1, 2$ and $i \neq j$. Finally,

$$\gamma_3 = \frac{A_3}{B_3} = \frac{-(f(r_1) + a_0B_3)}{\displaystyle\prod_{i=1}^{2}(c_i + b_i\,r_1 + r_1^2)}.$$

We note that an integer power of $H$ is a polynomial if and only if $\beta_1 = \beta_2 = 0$ and $\gamma_i \in \mathbb{Q}$ for $i = 1, 2, 3$, if they all have the same sign.

*Case* (xii). A first integral $H$ is

$$(x-ry)^{\gamma_1}(x^2+bxy+cy^2)^{\gamma_2}\exp\left(\frac{\beta_1x^3}{[(4c-b^2)x^2]^{3/2}}\arctan\left(\frac{bx + 2cy}{\sqrt{(4c-b^2)x^2}}\right)\right.$$

$$+\frac{10\beta_2(c+b\,r+r^2)x}{(b^2-4c)c(x^2+bxy+cy^2)}\bigg),$$

where

$$\begin{aligned}\beta_1 &= -10((p_4+r(p_3-r(p_2+p_1r)))b^3+4p_3r^2b^2+2r^2(p_3r-3p_4)b\\ &\quad -4p_4r^3+4c^3p_1+c^2(4(p_3+r(p_2+3p_1r))-2b(p_2-3p_1r))\\ &\quad -2c(2p_2rb^2+(3p_4-r(p_3+r(3p_1r-p_2)))b\\ &\quad +2r(3p_4+r(p_3+p_2r)))),\\ \beta_2 &= -p_4yb^3+(c(p_1rx+p_3y)-p_4(x+ry))b^2+((p_1(x+ry)-p_2y)c^2\\ &\quad +(-p_2rx+3p_4y+p_3(x+ry))c-p_4rx)b\\ &\quad +2c(p_1yc^2-(p_1rx+p_3y+p_2(x+ry))c+p_3rx+p_4(x+ry)),\\ \gamma_1 &= 2(a_0(c+br+r^2)^2+f(r)),\\ \gamma_2 &= 4a_0(c+br+r^2)^2-f(r).\end{aligned}$$

We note that an integer power of $H$ is a polynomial if and only if $\beta_1=\beta_2=0$ and $\gamma_i\in\mathbb{Q}$ for $i=1$, 2, if they all have the same sign.

**4. Weight exponent $\mathbf{s}=(1,2)$.** In this section, we prove Theorem 1.3. Since systems in equation (1.5) are homogeneous, we know that they are integrable because they have the inverse integrating factor $V=x\dot{y}-y\dot{x}$. The strategy will be to obtain such first integrals and to determine which of them are polynomials. Denoting systems in equation (1.5) by $\dot{x}=P(x,y)$ and $\dot{y}=Q(x,y)$, the first integral is

$$H(x,y)=\int\frac{P(x,y)}{V(x,y)}\,dy+g(x),$$

satisfying $\partial H/\partial x=-Q(x,y)/V(x,y)$.

The first system in equation (1.5) has the first integral

$$H=x^{-3-2p_2}(3y^2-x^2)^{-3+p_2}\exp\left(-2\sqrt{3}p_1\operatorname{arctanh}\left(\frac{x}{\sqrt{3}y}\right)\right).$$

Note that an integer power of $H$ is a polynomial if and only if $p_1=0$ and $p_2=3(1-q)/(1+2q)$ with $q\in\mathbb{Q}^+$.

The second system in equation (1.5) has the first integral

$$H = x^{-3-2p_2}(3y^2+x^2)^{-3+p_2} \exp\left(-2\sqrt{3}p_1 \arctan\left(\frac{x}{\sqrt{3}y}\right)\right).$$

Note that an integer power of $H$ is a polynomial if and only if $p_1 = 0$ and $p_2 = 3(1-q)/(1+2q)$ with $q \in \mathbb{Q}^+$.

The third system in equation (1.5) has the first integral

$$H = x^{-2(3+p_1)} y^{-3+2p_1} \exp \frac{2p_2 y}{x}.$$

Note that an integer power of $H$ is a polynomial if and only if $p_2 = 0$ and $p_1 = 3(1-2q)/(2(1+q))$ with $q \in \mathbb{Q}^+$.

The fourth system in equation (1.5) has the first integral

$$H = x \exp\left(-y\frac{2p_1 x + p_2 y}{3x^2}\right).$$

Note that an integer power of $H$ is a polynomial if and only if $p_1 = p_2 = 0$.

The fifth system in equation (1.5) has the first integral $x/y$ which is never a polynomial. $\square$

**5. Weight exponent $\mathbf{s} = (1,3)$.** Performing a change of variables $(X,Y) = (x^3, y)$, the planar weight homogeneous systems of weight degree 4 and weight exponent $(1,3)$ become

$$\dot{X} = 3a_{40}X^2 + 3a_{11}XY, \qquad \dot{Y} = b_{60}X^2 + b_{31}XY + b_{02}Y^2. \tag{5.1}$$

Again, we shall use the inverse integrating factor $V = X\dot{Y} - Y\dot{X}$ for computing the first integrals of system (5.1).

It is clear that, if $a_{11} = a_{40} = 0$, then a polynomial first integral is $X$, and, if $b_{60} = b_{31} = b_{02} = 0$, then a polynomial first integral is $Y$.

Now, we consider the other cases.

*Case* (i). $3a_{11} - b_{02} \neq 0$ and $R = -(3a_{40} - b_{31})^2 + 4(-3a_{11} + b_{02})b_{60} \neq 0$. In this case, system (5.1) has the first integral

$$\frac{6}{\sqrt{R}}(a_{11}(3a_{40}+b_{31}) - 2a_{40}b_{02}) \arctan\left(\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{\sqrt{R}X}\right)$$

$$+2(3a_{11}-b_{02})\log X+3a_{11}\log\left(\frac{Y(3a_{40}X-b_{31}X+3a_{11}Y-b_{02}Y)}{X^2}-b_{60}\right).$$

Here, $\log A$ always means $\log|A|$ and, as usual, log is the logarithm in base $e$. Since this first integral must be a polynomial, we must have

$$a_{11}(3a_{40}+b_{31})-2a_{40}b_{02}=0. \tag{5.2}$$

Now, we consider different subcases.

If $3a_{11}-2b_{02}\neq 0$, then from equation (5.2), we obtain

$$a_{40}=-\frac{a_{11}b_{31}}{3a_{11}-2b_{02}}.$$

Therefore, using the exponential of the previous first integral, we obtain that the first integral is

$$H=X^{1-(3a_{11}/6a_{11}-2b_{02})}p(X,Y)^{3a_{11}/(6a_{11}-2b_{02})},$$

where

$$\begin{aligned}p(X,Y)&=(3a_{11}-2b_{02})b_{60}X^2+2(3a_{11}-b_{02})b_{31}XY\\&\quad-(9a_{11}^2-9a_{11}b_{02}+2b_{02}^2)Y^2.\end{aligned}$$

Note that, since $a_{11}-b_{02}\neq 0$ and $3a_{11}-2b_{02}\neq 0$, we have $9a_{11}^2-9a_{11}b_{02}+2b_{02}^2\neq 0$. Therefore, an integer power of $H$ is a polynomial first integral if and only if $3a_{11}/(6a_{11}-2b_{02})=m/n\in\mathbb{Q}^+$, and $m/n<1$. In this case, the first integral $H$ is

$$\begin{aligned}X^{n-m}((3a_{11}-2b_{02})b_{60}X^2&+2(3a_{11}-b_{02})b_{31}XY\\&-(9a_{11}^2-9a_{11}b_{02}+2b_{02}^2)Y^2)^m.\end{aligned}$$

If $3a_{11}-2b_{02}=0$, that is, $b_{02}=3a_{11}/2$. In this case, from equation (5.2), we obtain $a_{11}b_{31}=0$. Hence, either $a_{11}=0$ or $b_{31}=0$. However, if $a_{11}=0$, then $b_{02}=0$, which contradicts the fact that $3a_{11}-2b_{02}\neq 0$. Therefore, this case is not possible and we must have $b_{31}=0$. Then, the first integral is

$$H=\frac{-2b_{60}X^2+6a_{40}XY+3a_{11}Y^2}{X},$$

which is never a polynomial.

*Case* (ii). $b_{02} = 3a_{11}$ and $b_{31} - 3a_{40} \neq 0$. In this case, system (5.1) has the first integral

$$(b_{31} - 3a_{40})^2 \log X + \frac{3a_{11}(3a_{40} - b_{31})Y}{X} \\ + 3(-3a_{40}^2 + b_{31}a_{40} - a_{11}b_{60}) \log\left(-\frac{b_{60}X - 3a_{40}Y + b_{31}Y}{X}\right).$$

In order for the first integral to be a polynomial, we must have $a_{11}(3a_{40} - b_{31}) = 0$, that is, $a_{11} = 0$ (and hence, $b_{02} = 0$). Then, using the exponential of the previous first integral, we obtain the following first integral

$$X^{1/3}\left(\frac{b_{60}X - 3a_{40}Y + b_{31}Y}{X}\right)^{a_{40}/(3a_{40}-b_{31})}.$$

Then, we must have $b_{31}/a_{40} = -m/n$ with $m/n \in \mathbb{Q}^+$. In this case, the previous first integral becomes $H = X^m(b_{60}X - 3a_{40}Y + b_{31}Y)^{3n}$.

*Case* (iii). $b_{02} = 3a_{11}$ and $b_{31} = 3a_{40}$. System (5.1) has the first integral

$$\frac{-2b_{60}X^2 \log X + 6a_{40}YX + 3a_{11}Y^2}{6X^2}.$$

Since the case $a_{40} = a_{11} = 0$ has been studied, we have that in this case the first integral is never a polynomial.

*Case* (iv). $3a_{11} - b_{02} \neq 0$ and $R = 0$. Then,

$$b_{06} = \frac{-(3a_{40} - b_{31})^2}{4(3a_{11} - b_{02})},$$

and system (5.1) has the first integral

$$\frac{3(-3a_{11}a_{40} + 2b_{02}a_{40} - a_{11}b_{31})X}{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y} + (3a_{11} - b_{02}) \log(36a_{11}X - 12b_{02}X) \\ + 3a_{11} \log\left(-\frac{3a_{40}X - b_{31}X + 6a_{11}Y - 2b_{02}Y}{X}\right).$$

In order to show that it is a polynomial, we must have

$$2a_{40}b_{02} - a_{11}(3a_{40} + b_{31}) = 0. \tag{5.3}$$

We will now consider two different subcases.

If $3a_{11} \neq 2b_{02}$, then condition (5.3) becomes

$$a_{40} = -\frac{a_{11}b_{31}}{3a_{11} - 2b_{02}}.$$

Using the exponential of the previous first integral, we obtain the following first integral:

$$X^{1/(36a_{11}-12b_{02})}\left(\frac{b_{31}X - 3a_{11}Y + 2b_{02}Y}{X}\right)^{a_{11}/4(b_{02}-3a_{11})^2}.$$

From this first integral, we obtain the first integral:

$$X(b_{31}X - 3a_{11}Y + 2b_{02}Y)^{-3a_{11}/b_{02}}.$$

So, if $b_{02} \neq 0$, then $3a_{11}/b_{02} = -m/n$ with $m/n \in \mathbb{Q}^+$, and the polynomial first integral is

$$X^n(b_{31}X - 3a_{11}Y + 2b_{02}Y)^m.$$

If $b_{02} = 0$, then $H = b_{31}X - 3a_{11}Y$ is a polynomial first integral. This concludes the proof of Theorem 1.4. $\square$

**6. Weight exponent $\mathbf{s} = (1, 4)$.** We introduce the change $(X, Y) = (x^4, y)$ in the planar weight homogeneous polynomial differential systems (1.1) of weight degree 4 with weight exponent $(1, 4)$. In these new variables $(X, Y)$, the systems with weight exponent $(1, 4)$ become, after introducing the new independent variable $d\tau = x^3\, dt$, as follows:

$$X' = 4(a_{40}X + a_{01}Y), \qquad Y' = b_{70}X + b_{31}Y, \tag{6.1}$$

where the prime denotes the derivative with respect to $\tau$. If $a_{40} = a_{01} = 0$, then a polynomial first integral is $X$, and, if $b_{70} = b_{31} = 0$, then a polynomial first integral is $Y$.

Now, we consider the other cases.

*Case* (i). $R = -(4a_{40} - b_{31})^2 - 16a_{01}b_{70} \neq 0$. Now, a first integral of system (6.1) is:

$$\frac{2(4a_{40} + b_{31})}{\sqrt{R}} \arctan\left(\frac{(4a_{40} - b_{31})X + 8a_{01}Y}{\sqrt{R}X}\right) + \log(-b_{70}X^2 + (4a_{40} - b_{31})XY + 4a_{01}Y^2).$$

Since it must be a polynomial, we must have $b_{31} = -4a_{40}$ such that the polynomial first integral is $H = -b_{70}X^2 + 8a_{40}XY + 4a_{01}Y^2$.

*Case* (ii). $R = 0$. We consider different subcases.

First, we study when $b_{70} \neq 0$. Then, from $R = 0$, we obtain

$$a_{01} = -\frac{(4a_{40} - b_{31})^2}{16b_{70}}. \tag{6.2}$$

If $b_{31} - 4a_{40} \neq 0$, then the first integral is:

$$\frac{2(4a_{40} + b_{31})b_{70}X}{-2b_{70}X + (4a_{40} - b_{31})Y} + (4a_{40} - b_{31}) \log(2b_{70}X + (-4a_{40} + b_{31})Y),$$

which is a polynomial if and only if $b_{31} = -4a_{40}$. The polynomial first integral is $b_{70}X - 4a_{40}Y$.

If $b_{31} = 4a_{40}$, then $a_{40} \neq 0$ (otherwise, $b_{31} = 0$ and, from equation (6.2), we also have $a_{01} = 0$, which has already been considered), and the first integral of equation (6.1) is

$$H = \frac{Y}{X} - \frac{b_{70} \log X}{4a_{40}},$$

which is never a polynomial.

If $b_{70} = 0$, then from $R = 0$, we obtain $b_{31} = 4a_{40}$. We only consider the case $a_{40} \neq 0$, where the first integral of equation (6.1) is:

$$-\frac{a_{40}X}{Y} + a_{01} \log Y,$$

which is never a polynomial. This completes the proof of Theorem 1.5. $\square$

**7. Weight exponent $s = (2,3)$.** We prove Theorem 1.6. The planar weight homogeneous polynomial differential systems (1.1) with weight degree 4 and weight-exponent $(2,3)$ are:

$$\dot{x} = a_{11}xy, \qquad \dot{y} = b_{30}x^3 + b_{02}y^2. \tag{7.1}$$

If $a_{11}(3a_{11} - 2b_{02}) \neq 0$, then the first integral of system (7.1) is:

$$H = x^{-2b_{02}/a_{11}} \Big(2b_{30}x^3 - 3a_{11}y^2 + 2b_{02}y^2\Big).$$

Then, an integer power of $H$ is a polynomial first integral if and only if $-2b_{02}/a_{11} \in \mathbb{Q}^+$.

If $a_{11} = 0$, then $H = x$ is a polynomial first integral of system (7.1). If $b_{30} = b_{02} = 0$, then $H = y$ is a polynomial first integral of system (7.1). If $a_{11} \neq 0$ and $3a_{11} = 2b_{02}$, then the first integral of system (7.1) is:

$$H = \frac{y^4}{x^3} - \frac{2b_{30}}{a_{11}} \log x,$$

which is never a polynomial. This completes the proof of Theorem 1.6. □

**8. Weight exponent s** $= (2,5)$**.** Here, we prove Theorem 1.7. The planar weight homogeneous polynomial differential systems (1.1) of weight degree 4 with weight exponent $(2,5)$ are:

$$\dot{x} = a_{01} y, \qquad \dot{y} = b_{40} x^4.$$

It is straightforward to prove that $H = 2b_{40}x^5 - 5a_{01}y^2$ is a polynomial first integral. □

**9. Weight exponent s** $= (6,9)$**.** Now, we prove Theorem 1.8. The planar weight homogeneous polynomial differential systems (1.1) of weight degree 4 with weight exponent $(6,9)$ are:

$$\dot{x} = a_{01} y, \qquad \dot{y} = b_{20} x^2.$$

It is straightforward to prove that $H = 2b_{20}x^3 - 3a_{01}y^2$ is a polynomial first integral. This completes the proof of Theorem 1.8. □

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