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CONTRACTIONS OF DEL PEZZO SURFACES TO $\mathbb{P}^2$ OR $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT. In this article, we consider $r-1$ disjoint lines given in a del Pezzo surface S_r and study how to determine if a contraction given by the lines produces a map to S_1 (one point blow up of $\mathbb{P}^2$) or $\mathbb{P}^1 \times \mathbb{P}^1$ by checking only the configuration of lines. Here, we show that we can determine if the disjoint lines produce a contraction to $\mathbb{P}^1 \times \mathbb{P}^1$ by combining a quartic rational divisor class to them. We also study the quartic rational divisor classes along the configuration of lines in del Pezzo surfaces.

1. Introduction. A *del Pezzo surface* is a smooth projective surface S_r whose anticanonical class $-K_{S_r}$ is ample. Each del Pezzo surface S_r can be constructed by blowing up $r \leq 8$-points from $\mathbb{P}^2$ unless it is $\mathbb{P}^1 \times \mathbb{P}^1$ **[2]**. Conversely, each r-disjoint line in S_r gives a contraction to $\mathbb{P}^2$. But, if we choose $r-1$ disjoint lines in S_r, the corresponding contraction produces a map to the blow-up of one point in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Therefrom, we consider the following main question:

Question 1.1. *When $r-1$ disjoint lines are given on a del Pezzo surface S_r, can we determine if a contraction given by the lines produces a map to S_1 (one point blow up of $\mathbb{P}^2$) or $\mathbb{P}^1 \times \mathbb{P}^1$ by checking the configuration of lines?*

Here, lines are rational curves with (-1)-self intersection which produce contractions of S_r. As a matter of fact, one can figure out the answer by performing contraction for the given lines indeed. Thus, the actual issue of the question is if we can determine the dichotomy

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in Question 1.1 by only checking configuration of the lines before we produce the related contraction.

For given lines l_i, $1 \leq i \leq r-1$, in a del Pezzo surface S_r, the dichotomy of two different types of contractions is related to the chance of finding another line l_r disjoint to each l_i so that the r lines produce a contraction to $\mathbb{P}^2$. Otherwise, the given $r-1$ lines must produce a contraction to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, the issue of the main Question 1.1 is equivalent to finding the characterization of the configuration of $r-1$ disjoint lines which can be a subset of r disjoint lines. We recall that, in [**5**], the divisor *classes of lines* (also called *lines*) corresponded to vertices of the Gosset polytope $(r-4)_{21}$ constructed in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ and a divisor class of $r-1$ disjoint lines; a skew $(r-1)$-line corresponds to an $(r-2)$-simplex in $(r-4)_{21}$. Therefore, the main Question 1.1 is equivalent to the following Question 1.2.

Question 1.2. *For a given $(r-2)$-simplex in $(r-4)_{21}$, can we determine whether the simplex is contained in an $(r-1)$-simplex in $(r-4)_{21}$ by checking the configuration of vertices of the $(r-2)$-simplex?*

In Section 2, we separate the $(r-2)$-simplexes into two types (A-type and B-type) of orbits of an $(r-2)$-simplex in $(r-4)_{21}$. Here, the orbit of A-type consists of an $(r-2)$-simplex in $(r-4)_{21}$ which is not contained in any $(r-1)$-simplex, and the one of B-type consists of $(r-2)$-simplexes where each $(r-2)$-simplex is in a $(r-1)$-simplex. To identify the type of each $(r-2)$-simplex by the configuration of vertices in the simplex, we consider a divisor class $q \in \operatorname{Pic} S_r$ satisfying $q^2 = 2$, $q \cdot K_{S_r} = -4$ which is called a *quartic rational divisor class.* The quartic rational divisor classes also consist of two types (I and II) of Weyl orbits. Here, the type I is the orbit containing $2h - e_1 - e_2$, and the type II is the Weyl orbit of $3h - \sum_{i=1}^{6} e_i + e_7$ which exist only for $r = 7, 8$ (see subsection 2.2). Then we combine A-type $(r-2)$-simplexes and type I quartic rational divisor classes to get the following theorem which gives an answer to Question 1.1 (equivalently Question 1.2).

Theorem. *For disjoint lines l_i, $1 \leq i \leq r-1$, on a del Pezzo surface S_r, they produce a contraction to $\mathbb{P}^1 \times \mathbb{P}^1$ if there is a quartic rational divisor class q on S_r satisfying*

$$2q + K_{S_r} \equiv l_1 + \cdots + l_{r-1}.$$

Moreover, the quartic rational divisor classes of type I *are bijectively related to A-type* $(r-2)$*-simplexes in* $(r-4)_{21}$.

Furthermore, we show that, for each type II quartic rational divisor class q in S_r, $r = 7, 8$, uniquely there exist a line l_q and a divisor class D_q such that $D_q \cdot l_q = 0$ and $q \equiv l_p + D_q$. Here, the divisor class D_q satisfies $D_q^2 = 3$ and $D_q \cdot K_{S_r} = -3$, which was studied as $A_2(1)$-divisor (1-degree 2-simplex divisor) in [**6**].

The lines in del Pezzo surfaces and their configurations have been studied from many different motivations [**3, 7, 8, 9**]. This article gives an application of previous study [**5, 6**] where the configurations of lines are described via subpolytopes in Gosset polytopes. The configuration of r or $r-1$ lines in S_r and related contractions to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ in Question 1.1 is one of the typical questions to appear in the study of the Cox rings of del Pezzo surfaces [**1**]. Furthermore, the quartic rational divisor classes in the article are also considered in the Mysterious Duality related to twice-wrapped M5-brane [**4**].

2. Contractions of del Pezzo surfaces. In this section, we give an answer to Questions 1.1 and 1.2 by considering rational quartic divisors and also describe the configuration of the rational quartic divisor classes.

In the following subsection, we review basic facts about the comparison between subpolytopes in Gosset polytopes and special divisors in del Pezzo surfaces from [**5**], and provide an answer for Question 1.2.

2.1. Gosset polytopes and del Pezzo surfaces. In this subsection, we review general facts on del Pezzo surfaces S_r and Gosset polytopes $(r-4)_{21}$ in the Picard groups $\operatorname{Pic} S_r$ given by Weyl actions [**2, 5**]. In addition, we introduce two Weyl orbits in the Picard group related to $(r-2)$-simplexes in $(r-4)_{21}$.

2.1.1. Gosset polytopes in Picard groups of del Pezzo surfaces. We consider a del Pezzo surface S_r given by blowing up $r \leq 8$-points from $\mathbb{P}^2$ and the corresponding blow up by $\pi_r : S_r \to \mathbb{P}^2$. In addition, $K_{S_r}^2 = 9 - r$ is called the degree of the del Pezzo surface. Each exceptional curve and the corresponding class given by blowing up are denoted by e_i, and both the class of $\pi_r^*(h)$ in S_r and the class of a line h in $\mathbb{P}^2$ are referred to as h. Then, we have

$$h^2 = 1, \qquad h \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij} \quad \text{for } 1 \leq i, j \leq r,$$

and the Picard group of S_r is $\operatorname{Pic} S_r \simeq \mathbb{Z}h \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ with the signature $(1, -r)$. In addition,

$$K_{S_r} \equiv -3h + \sum_{i=1}^{r} e_i.$$

The inner product given by the intersection on $\operatorname{Pic} S_r$ induces a negative definite symmetric bilinear form on $(\mathbb{Z}K_{S_r})^\perp$ in $\operatorname{Pic} S_r$ where we can also define natural reflections. To define reflections on $(\mathbb{Z}K_{S_r})^\perp$ in $\operatorname{Pic} S_r$, we consider a root system

$$R_r := \{d \in \text{Pic } S_r \mid d^2 = -2,\ d \cdot K_{S_r} = 0\},$$

with simple roots $d_0 = h - e_1 - e_2 - e_3$, $d_i = e_i - e_{i+1}$, $1 \leq i \leq r-1$. Each element d in R_r defines a reflection on $(\mathbb{Z}K_{S_r})^\perp$ in $\operatorname{Pic} S_r$

$$\sigma_d(D) := D + (D \cdot d)d \quad \text{for } D \in (\mathbb{Z}K_{S_r})^\perp$$

and the corresponding Weyl group $W(S_r)$ is E_r where $3 \leq r \leq 8$. Here, the extended list of E_r includes $E_3 = A_1 \times A_2$, $E_4 = A_4$ and $E_5 = D_5$. This reflection on $K_{S_r}^\perp$ can be extended to $\operatorname{Pic} S_r$. The divisor classes D satisfying $D \cdot K_{S_r} = \alpha$, $D^2 = \beta$ where α and β are integers are preserved by the extended action of $W(S_r)$. Thus, the subsets of special divisors below can be naturally understood according to the representation of $W(S_r)$.

Now, we want to construct Gosset polytopes $(r-4)_{21}$ in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ as the convex hull of the set of special classes in $\operatorname{Pic} S_r$, which is known as lines. A *line* in $\operatorname{Pic} S_r$ is equivalently a divisor class l with $l^2 = -1$ and $K_{S_r} \cdot l = -1$, and the set of lines is given as

$$L_r := \{l \in \text{Pic } S_r \mid l^2 = -1,\ K_{S_r} \cdot l = -1\}.$$

As the Weyl group $W(S_r)$ acts as an affine reflection group on the affine hyperplane given by $D \cdot K_{S_r} = -1$, $W(S_r)$ acts on the set of lines in $\operatorname{Pic} S_r$. Therefrom, we construct a semiregular polytope in $\operatorname{Pic} S_r \otimes \mathbb{Q}$ whose vertices are exactly the lines in $\operatorname{Pic} S_r$. Since the symmetry group of the polytope is $W(S_r)$, the polytope is actually a

Gosset polytope $(r-4)_{21}$ which is an r-dimensional convex semiregular uniform polytope given by the symmetry group of E_r-type.

For a Gosset polytope $(r-4)_{21}$, faces are regular simplexes except the facets which consist of $(r-1)$-simplexes and $(r-1)$-crosspolytopes. Since the faces in $(r-4)_{21}$ are basically configurations of vertices, we obtain a natural characterization of faces in $(r-4)_{21}$ as divisor classes in $\operatorname{Pic} S_r$. Here, to identify each face in $(r-4)_{21}$, we want to use the barycenter of the face. Since each vertex of the polytope $(r-4)_{21}$ represents a line in S_r, and the honest centers of simplexes (respectively, crosspolytopes) are written as $(l_1+\cdots+l_k)/k$ (respectively, $(l'_1+l'_2)/2$) which may not be elements in $\operatorname{Pic} S_r$. Therefore, alternatively, we choose $(l_1+\cdots+l_k)$ as the center of a face so that $(l_1+\cdots+l_k)$ is in $\operatorname{Pic} S_r$.

We use the algebraic geometry of del Pezzo surfaces to identify the divisor classes corresponding to the faces in $(r-4)_{21}$. For this purpose, we consider the following set of divisor classes which are called *skew a-lines, exceptional systems* and *rulings* in $\operatorname{Pic} S_r$.

$$L_r^a := \{D \in \operatorname{Pic} S_r \mid D = l_1+\cdots+l_a,\ l_i \text{ disjoint lines in } S_r\}$$
$$\mathcal{E}_r := \{e \in \operatorname{Pic} S_r \mid e^2 = 1,\ K_{S_r}\cdot e = -3\}$$
$$F_r := \{f \in \operatorname{Pic}\ S_r \mid f^2 = 0,\ K_{S_r}\cdot f = -2\}.$$

In particular, a *skew a-line* in L_r^a is an extension of the definition of lines in S_r. Each skew a-line represents an $(a-1)$-simplex in an $(r-4)_{21}$ polytope. Furthermore, for each skew a-line, there is only one set of disjoint lines in L_r^a to present it. The skew a-lines also have $D^2=-a$ and $D\cdot K_{S_r}=-a$, and the divisor classes with these conditions are equivalently skew a-lines for $a\le 3$, see [**5**] for details.

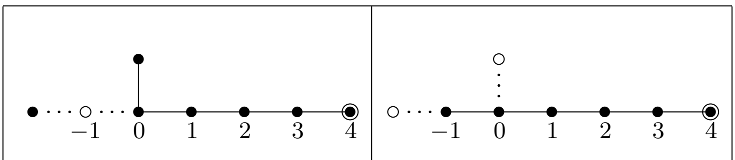
After proper comparison between divisor classes obtained from the geometry of the polytope $(r-4)_{21}$ and those given by the geometry of a del Pezzo surface, we come to the correspondences in Table 1.

Remark 2.1. In particular, in this article, it is a useful fact that the set of skew a-lines in $\operatorname{Pic} S_r$, $1\le a\le r$, is bijective to the set of $(a-1)$-simplexes in the Gosset polytope $(r-4)_{21}$.

Table 1. Correspondences between special divisors and subpolytopes.

del Pezzo surface S_r	Gosset polytopes $(r-4)_{21}$
lines	vertices
skew a-lines $1 \leq a \leq r$	$(a-1)$-simplexes $1 \leq a \leq r$
exceptional systems	$(r-1)$-simplexes $(r < 8)$
rulings	$(r-1)$-crosspolytopes

2.1.2. Two orbits of $(r-2)$-simplexes in a Gosset $(r-4)_{21}$. To make sense of the above Question 1.2, there should be an $(r-2)$-simplex in $(r-4)_{21}$ which is not in an $(r-1)$-simplex. Indeed, there are two types of $(r-2)$-simplexes in Gosset $(r-4)_{21}$ given as two orbits of Weyl action $W(S_r)$ on $\operatorname{Pic} S_r$ where one of the orbits consists of such $(r-2)$-simplexes in $(r-4)_{21}$. For example, for 4_{21}, there are two types of 6-simplexes in it, and the total number of them, $N_6^{4_{21}}$, can be calculated as:



A-type B-type

$$
\begin{aligned}
N_6^{4_{21}} &= [E_8 : A_6 \times A_1] + [E_8 : A_6] \\
&= \frac{2^{14}3^5 5^2 7}{7! \times 2!} + \frac{2^{14}3^5 5^2 7}{7!} \\
&= 69120 + 138240 = 207360.
\end{aligned}
$$

Here, we observe that A-type 6-simplexes cannot be extended to 7-simplexes by an argument using the Coxeter-Dynkin diagram. Moreover, we call an $(r-2)$-simplex in $(r-4)_{21}$ B-*type* (respectively, A-*type*) if it is contained in an $(r-1)$-simplex in $(r-4)_{21}$ (respectively, if there is no $(r-1)$-simplex in $(r-4)_{21}$ containing the $(r-2)$-simplex).

By performing the calculation of 6-simplexes in 4_{21} to the other Gosset polytopes $(r-4)_{21}$, we get Table 2, which shows $(r-2)$-simplexes in $(r-4)_{21}$ according to two types of orbits.

TABLE 2. Total number of $(r-2)$-simplexes in $(r-4)_{21}$.

$(r-4)_{21}$	-1_{21}	0_{21}	1_{21}	2_{21}	3_{21}	4_{21}
total #	9	30	120	648	6048	207360
A, B	$3, 6$	$10, 20$	$40, 80$	$216, 432$	$2016, 4032$	$69120, 138240$

TABLE 3. Total number of quartic rational divisor classes.

r	3	4	5	6	7	8
total #	3	10	40	216	2072	82560
I, II	3, 0	10, 0	40, 0	216, 0	2016, 56	69120, 13440

According to the correspondence between $(r-2)$-simplexes in $(r-4)_{21}$ and skew $(r-1)$-lines in $\operatorname{Pic} S_r$ (Table 1), we conclude a skew $(r-1)$-line corresponding to an A-type $(r-2)$-simplex produces a contraction to $\mathbb{P}^1 \times \mathbb{P}^1$ and the other case (i.e., B-type) produces a contraction to S_1. Note that, although we have identified two orbits of $(r-2)$-simplexes in $(r-4)_{21}$ via the Weyl action, it is a very complicated question of group action to identify the related orbit to a given $(r-2)$-simplex by checking the configuration of vertices in it. Thus, we transfer back the configuration of vertices to the configuration of lines via the correspondences in Table 1 so that we can use the study in [**5, 6**].

In fact, skew $(r-1)$-lines in $\operatorname{Pic} S_r$ produce one of the Weyl orbits satisfying divisor equations $D \cdot K_{S_r} = -(r-1)$, $D^2 = -(r-1)$. Recall that, when $k = 1, 2, 3$, the divisors with the equations $D \cdot K_{S_r} = -k$, $D^2 = -k$, consist of one orbit which is given by skew k-lines, see [**5**]. But, if k is bigger, there are more orbits which are not well known. In the following, we introduce a special divisor which happens to be in one Weyl orbit corresponding to an A-type $(r-2)$-simplex and determine the dichotomy of Question 1.1.

2.2. Contractions and quartic rational divisor classes. We consider a divisor class $q \in \operatorname{Pic}(S_r)$ satisfying $q^2 = 2$, $q \cdot K_{S_r} = -4$. We call the divisor class q quartic rational divisor class. The total number of such divisor classes in S_r is finite and given as in Table 3.

By applying the representation of the Weyl action $W(S_r)$ on $\operatorname{Pic} S_r$, we deduce that the set of quartic rational divisor classes in S_r is one Weyl orbit containing $2h - e_1 - e_2$, except $r = 7, 8$, which have one more Weyl orbit. We call the orbit of $2h-e_1-e_2$ type I and the other one of $3h-\sum_{i=1}^{6} e_i+e_7$ in $r=7,8$ type II. The sizes of orbits are listed in Table 3. Here, we observe the list of type I matched with the list of A-type of $(r-2)$-simplexes in $(r-4)_{21}$, and we get the following theorem which gives an answer to Question 1.1.

Theorem 2.2. *For disjoint lines l_i, $1\le i\le r-1$, on del Pezzo surface S_r, they produce a contraction to $\mathbb{P}^1\times\mathbb{P}^1$ if there is a quartic rational divisor class q on S_r satisfying*

$$2q+K_{S_r}\equiv l_1+\cdots+l_{r-1}.$$

Moreover, the quartic rational divisor classes of type I *corresponding to A-type $(r-2)$-simplexes in $(r-4)_{21}$.*

Proof. We consider a skew $(r-1)$-line $D:=l_1+\cdots+l_{r-1}$ for the given disjoint lines l_i, $1\le i\le r-1$. According to the relationship in Table 1, the skew $(r-1)$-line D is bijectively related to an $(r-2)$-simplex in $(r-4)_{21}$ which is contained in one of two types in Table 2.

Here we want to show the $(r-2)$-simplex related to a quartic rational divisor class is indeed an A-type so that the corresponding contraction produces a map to $\mathbb{P}^1\times\mathbb{P}^1$.

Suppose D corresponds to a B-type $(r-2)$-simplex in $(r-4)_{21}$. Then there is another line l_r disjoint to each l_i, $1\le i\le r-1$, so that l_i, $1\le i\le r$, gives an $(r-1)$-simplex in $(r-4)_{21}$. Thus, we have

$$l_r\cdot(2q)=l_r\cdot(l_1+\cdots+\ l_{r-1}-K_{S_r})=1$$

and $l_r\cdot q=1/2$, which gives the contradiction.

Thus, we conclude each skew $(r-2)$-line D, which is injectively related to a quartic rational divisor class, corresponds to an A-type $(r-2)$-simplex in $(r-4)_{21}$.

Now, since the set of skew $(r-1)$-lines in $\operatorname{Pic} S_r$ is bijective to the set of $(r-2)$-simplexes in the Gosset polytope $(r-4)_{21}$, by comparing Tables 2 and 3 via Weyl action we conclude the corresponding quartic rational divisor class must be type I and obtain the bijective relationship between the set of the quartic rational divisor classes of type I and the set of A-type $(r-2)$-simplexes in $(r-4)_{21}$. □

Before we state the converse of the theorem for quartic rational divisor classes, we show the following theorem about type II quartic rational divisor classes.

Theorem 2.3. *Each pair of disjoint lines l_a and l_b on S_8 (respectively, line l on S_7) gives two type* II *quartic rational divisor classes $2l_a + l_b - K_{S_8}$ and $l_a + 2l_b - K_{S_8}$ (respectively, $2l - K_{S_7}$). Conversely, for each type* II *quartic rational divisor class q on S_8 (respectively, S_7), there is a unique pair of disjoint lines l_q and l'_q (respectively, unique line l_q) such that $q \equiv 2l_q + l'_q - K_{S_8}$ (respectively, $q \equiv 2l_q - K_{S_7}$).*

Proof. For the case of S_8, we consider a set of ordered pairs of disjoint lines l_a and l_b on S_8:

$$\widetilde{L}_8^2 := \{(l_a, l_b) \mid l_a \text{ and } l_b \text{ are disjoint lines in } S_8\},$$

where $|\widetilde{L}_8^2| = (\#\text{ of 2-skew lines}) \times 2 = 13440$, and we consider a map $\phi : \widetilde{L}_8^2 \to \operatorname{Pic} S_8$, defined by $\phi((l_a, l_b)) := 2l_a + l_b - K_{S_8}$.

Since $(2l_a + l_b - K_{S_8})^2 = 2$ and $(2l_a + l_b - K_{S_8}) \cdot K_{S_8} = -4$, the range of ϕ in $\operatorname{Pic} S_8$ consists of quartic rational divisor classes. Moreover, ϕ is one-to-one because of the following reason.

Suppose two pairs of disjoint lines produce the same quartic rational divisor class, such as

$$2l_a + l_b - K_{S_8} \equiv 2l_c + l_d - K_{S_8}.$$

Then we consider

$$l_c \cdot (2l_a + l_b) = l_c \cdot (2l_c + l_d) = -2,$$

and we conclude $l_c \cdot l_a = -1$ and $l_c \cdot l_b = 0$ since two lines in S_8 may have intersection $-1, 0, 1, 2, 3$ (see **[5, 6]**). Thus, we get $l_c = l_a$ and then $l_b = l_d$.

Recall that the Weyl group $W(S_8)$ transitively acts on lines and 2-skew lines in S_8 as well as preserve K_{S_8}, see **[5]**. Thus, the quartic rational divisor classes mapped by ϕ form a single orbit of $W(S_8)$ action, i.e., the image of $\phi \in \operatorname{Pic} S_8$ is a single orbit of $W(S_8)$. Since

$$3h - \sum_{i=1}^{6} e_i + e_7 = 2e_7 + e_8 - K_{S_8}$$

is a typical element in the orbit, we conclude the map ϕ is a bijection between the Weyl orbit of type II quartic rational divisor classes and $\widetilde{L}_8^2$.

Similarly, one can show the case of S_7. □

Remark 2.4. A quartic divisor $l_a + 2l_b - K_{S_8}$ in S_8 is mapped to a quartic divisor $2l_b - K_{S_7}$ in S_7 via $\pi^8_{l_a} : S_8 \to S_7$, which is a blow down map given by an exceptional curve in l_a.

For S_7 (respectively, S_8), we observe that $D = l_b - K_{S_7}$ (respectively, $l_a + l_b - K_{S_8}$) is a divisor class satisfying $D^2 = 3$ and $D \cdot K_{S_7} = -3$. In **[6]**, it is shown that such a divisor class can be written as $D \equiv l_1 + l_2 + l_3$ where l_i, $i = 1, 2, 3$, are lines with intersection 1. As lines in $\operatorname{Pic} S_r$ present vertices in Gosset polytope $(r-4)_{21} \in \operatorname{Pic} S_r$, the divisor class $D \equiv l_1 + l_2 + l_3$ is the center of corresponding 2-simplex, and we call such a divisor class an $A_2(1)$-*divisor* (1-degree 2-simplex divisor). Note such a divisor class exists when $r = 6, 7, 8$ (see **[6]** for details).

By considering $A_2(1)$-*divisor*, we obtain the following corollary.

Corollary 2.5. *For each type* II *quartic rational divisor class q on S_7 and S_8, uniquely there exist a line l_q and an $A_2(1)$-divisor D_q such that $D_q \cdot l_q = 0$ and $q = l_q + D_q$. In particular, for S_7, l_q and D_q, determine each other via $l_q = D_q + K_{S_7}$.*

Proof. For a type II quartic rational divisor class q on S_7, we consider $D_q := l_q - K_{S_7}$ as in the previous theorem. Since $D_q^2 = 3$ and $D_q \cdot K_{S_7} = -3$, the divisor class D_q is an $A_2(1)$-*divisor* and it satisfies $D_q \cdot l_q = (l_q - K_{S_7}) \cdot l_q = 0$. Similarly, for S_8, we consider $D_q :\equiv l_q + l'_q - K_{S_8}$, and this divisor is an $A_2(1)$-*divisor* satisfying

$$D_q \cdot l_q = \left(l_q + l'_q - K_{S_8}\right) \cdot l_q = 0.$$ □

In summary, we have the following theorem.

Theorem 2.6. *For each quartic rational divisor class q in S_r, $3 \leq r \leq 6$, there are $r-1$ disjoint lines l_i, $1 \leq i \leq r-1$ satisfying $2q + K_{S_r} = l_1 + \cdots + l_{r-1}$. For the quartic rational divisor class q in S_r, $r = 7, 8$, either there are $r-1$ disjoint lines l_i, $1 \leq i \leq r-1$,* *satisfying*

$$2q + K_{S_r} \equiv l_1 + \cdots + l_{r-1},$$

or uniquely there exist a line l_q *and an* $A_2(1)$*-divisor* D_q *such that* $D_q \cdot l_q = 0$ *and* $q = l_q + D_q$.

REFERENCES

1. V.V. Batyrev and O.N. Popov, *The Cox ring of a del Pezzo surface*, in *Arithmetic of higher-dimensional algebraic varieties*, B. Poonen and Y. Tschinkel, eds., Progr. Math. **226**, Birkhäuser, Boston, 2004.

2. M. Demazure, *Surfaces de Del Pezzo* I, II, III, IV, V, in *Séminaire sur les singularités des surfaces*, M. Demazure, H. Pinkham and B. Teissier, eds., Lect. Notes Math. **777**, Springer-Verlag, New York, 1980.

3. P. Du Val, *On the directrices of a set of points in a plane*, Proc. Lond. Math. Soc. **35** (1931), 23–74.

4. A. Iqbal, A. Neitzke and C. Vafa, *A mysterious duality*, Adv. Theor. Math. Phys. **5** (2001), 769–807.

5. J.H. Lee, *Gosset polytopes in Picard groups of del Pezzo surfaces*, Canad. J. Math. **64** (2012), 123–150.

6. ______, *Configurations of lines in del Pezzo surfaces with Gosset polytopes*, Trans. Amer. Math. Soc. **366** (2014), 4939–4967.

7. N.C. Leung and J.J. Zhang, *Moduli of bundles over rational surfaces and elliptic curves* I: *Simply laced cases*, J. Lond. Math. Soc. **80** (2009), 750–770.

8. Y. Manin, *Cubic forms*: *Algebra, geometry, arithmetic*, Nauka, Moscow, 1972; North-Holland, Amsterdam, 1974, second edition, 1986 (in English).

9. L. Manivel, *Configurations of lines and models of Lie algebras*, J. Algebra **304** (2006), 457–486.

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