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LOWER BOUND FOR THE HIGHER MOMENT OF SYMMETRIC SQUARE L-FUNCTIONS

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ABSTRACT. Let $\mathcal{S}_k(N)$ be the space of holomorphic cusp forms of weight k, level N and let $\mathcal{B}_k(N)$ be an orthogonal basis of $\mathcal{S}_k(N)$ consisting of newforms. Let $L(s, \mathrm{sym}^2 f)$ be the symmetric square L-function of $f \in \mathcal{B}_k(N)$. In this paper, the lower bound of the higher moment of $L(1/2, \mathrm{sym}^2 f)$ is established, i.e., for any even positive number r,

$$\sum_{f\in\mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)^r \gg (\log N)^{r(r+1)/2}$$

holds for $N \to \infty$.

1. Introduction and statement of results. An important problem in number theory is to determine the asymptotic formula of the moments of central values of L-functions varying in a family. This problem has been intensively studied in recent years.

Katz and Sarnak [**7**] have introduced the idea of a family of L-functions with an associated symmetry type, and they gave strong evidence that the symmetry group governs many properties of the distribution of zeros of the L-functions. Later, Conrey and Farmer [**3**] proved that the symmetry group also governs the behavior of the mean values of the L-functions.

The general conjecture for the moment of L-functions can be obtained by random matrix theory. For a family $\mathcal{F}$ of L-functions with functional equations $s \mapsto 1 - s$ and appropriate gamma factors,

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$L(1/2,f)$ is the central value. The conjecture is that, as $D \to \infty$,

$$\frac{1}{D^*} \sum_{\substack{f \in \mathcal{F} \\ c(f) \leq D}} L\left(\frac{1}{2}, f\right)^r \sim \frac{a_r g_r}{\Gamma(1+B(r))} \left(\log D^A\right)^{B(r)},$$

where the family $\mathcal{F}$ is partially ordered by the conductor $c(f)$, and D^* the number of elements with $c(f) \leq D$, for details, see **[3]**. Let $L(s, \mathrm{sym}^2 f)$ be the symmetric square L-function of $f \in \mathcal{B}_k(N)$. In the context of the moment of $L(1/2, \mathrm{sym}^2 f)$, random matrix theory says that one should have, as $N \to \infty$,

$$\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)^r \sim C(k,r)(\log N)^{r(r+1)/2}.$$

The above asymptotic formulas for $r = 1, 2$ are known, see **[2]**.

Following the idea of Rudnick and Soundararajan **[11]**, we shall consider the lower bounds for the moments of symmetric square L-functions in this paper. Our main theorem is as follows.

Theorem 1.1. *Let k be a fixed even integer and N a squarefree number. Let $L(s, \mathrm{sym}^2 f)$ be the symmetric square L-function of $f \in \mathcal{B}_k(N)$. For any even positive number r, we have*

$$\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)^r \gg (\log N)^{r(r+1)/2}$$

holds for $N \to \infty$.

The research on lower bounds for the moments of the central values of L-functions began with Ramachandra **[9]** and Heath-Brown **[4]**, and has been studied by many authors, see **[1, 6, 8, 10, 11, 12]**. In particular, Rudnick and Soundararajan **[11]** and Tang **[12]** obtained the lower bound of the moments of $L(s,f)$ and $L(s, \mathrm{sym}^2 f)$ in the weight k aspect for the full modular group $SL(2, \mathbb{Z})$, respectively. And in the paper, **[8]**, Radziwill and Soundararajan extended the idea of Rudnick and Soundararajan to obtain lower bounds of the Riemann zeta function for all real numbers $r \geq 1$. Following the method of Rudnick and Soundararajan, we can obtain lower bounds for these moments in Theorem 1.1 for all rational numbers $r \geq 1$.

2. Preliminaries.

2.1. Holomorphic cusp forms. For a fixed even positive integer k and a squarefree number N, the space $\mathcal{S}_k(N)$ of the holomorphic cusp forms of the weight k for the Hecke congruence subgroup $\Gamma_0(N)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^{k-2}\mathrm{d}x\,\mathrm{d}y, \quad f, g \in \mathcal{S}_k(N).$$

Let $\mathcal{B}_k(N)$ be an orthogonal basis of $\mathcal{S}_k(N)$ consisting of newforms, that is, normalized eigenforms for all Hecke operators. We write the Fourier expansion of a newform $f \in \mathcal{B}_k(N)$

$$f(z) = \sum_{n\geq 1} \lambda_f(n)n^{(k-1)/2}e(nz).$$

Then the Ramanujan conjecture, now a theorem due to Deligen, says

$$|\lambda_f(n)| \leq \tau(n),$$

where $\tau(n)$ is the number of divisors of n. The Hecke eigenvalues $\lambda_f(n)$ satisfy the relation

$$\lambda_f(1) = 1, \qquad \lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right). \tag{2.1}$$

2.2. Peterssson trace formula. Write the weight function

$$\omega_f = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}\langle f, f\rangle = \frac{(k-1)N}{2\pi^2}L(1, \mathrm{sym}^2 f),$$

where $\langle f, f\rangle$ is the Petersson inner product and $L(s, \mathrm{sym}^2 f)$ is the symmetric square L-function of $f(z)$. The following bound is well known

$$\frac{1}{N\log^3 N} \ll \omega_f^{-1} \ll \frac{\log N}{N}.$$

The Petersson trace formula states that

$$\sum_{f\in\mathcal{B}_k(N)} \omega_f^{-1}\lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^{-k}\sum_{N|c}\frac{S(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $\delta(m,n)$ is the diagonal symbol, $J_{k-1}(x)$ is the standard J-Bessel function and $S(m,n;c)$ is the classical Kloosterman sum defined by

$$S(m,n;c)=\sum_{d\bar{d}\equiv 1 (\operatorname{mod} c)} e\left(\frac{md+n\bar{d}}{c}\right).$$

Using the estimate $J_{k-1}(x) \ll \min(1,x/k)$ and the Weil bound for Kloosterman sums, we have the following result, see [**5,** Corollary 14.24].

Lemma 2.1. *With notation as above, we have for any $m,n\geq 1$,*

$$\sum_{f\in\mathcal{B}_k(N)} \omega_f^{-1}\lambda_f(m)\lambda_f(n)$$
$$=\delta(m,n)+O\left(\tau_3((m,n))(m,n,N)^{1/2}(mn)^{1/4}\frac{\tau(N)}{N\sqrt{k}}\log\left(1+\frac{(mn)^{1/4}}{\sqrt{Nk}}\right)\right),$$

where the implied constant is absolute.

2.3. Symmetric square L-functions. For a newform $f\in\mathcal{B}_k(N)$, the symmetric square L-function is given by

$$L(s,\operatorname{sym}^2 f)=\zeta^{(N)}(2s)\sum_{n\geq 1}\frac{\lambda_f(n^2)}{n^s},$$

where $\zeta^{(N)}(s)$ is the Riemann zeta function with the local factor at the prime N removed. The completed L-function $\Lambda(s,\operatorname{sym}^2 f) = N^s L_\infty(s,\operatorname{sym}^2 f)L(s,\operatorname{sym}^2 f)$ is entire and it satisfies the functional equation

$$\Lambda(s,\operatorname{sym}^2 f)=\Lambda(1-s,\operatorname{sym}^2 f),$$

where

$$L_\infty(s,\operatorname{sym}^2 f)=\pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k+1}{2}\right)\Gamma\left(\frac{s+k}{2}\right),$$

is the local factor at the infinity place. The first moment of central values of the symmetric square L-functions is established, see [**2**].

Lemma 2.2. *For $(m, N) = 1$ and any $\epsilon > 0$, one has*

$$\begin{aligned}\sum_{f \in \mathcal{B}_k(N)} & \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right) \lambda_f(m^2) \\ &= \frac{1}{\sqrt{m}} \left(2\gamma + \frac{L'_\infty}{L_\infty}\left(\frac{1}{2}, \mathrm{sym}^2 f\right) + \log \frac{\sqrt{N}}{m} + \sum_{p \mid N} \frac{2 \log p}{p-1} \right) \\ &\qquad + O_{k,\epsilon}\left(\left(\frac{m}{N^{3/4}} + \frac{1}{N^{1/4}} \right) (mN)^\epsilon \right).\end{aligned}$$

3. Proof of the theorem.

Proof. Let $x = N^{1/2r}$ and consider

$$A(f) = A(f, x) = \sum_{n \le x} \frac{\lambda_f(n^2)}{\sqrt{n}}.$$

Define

$$S_1 = \sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right) A(f)^{r-1},$$

and

$$S_2 = \sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} A(f)^r,$$

where r is an even positive integer. By Hölder's inequality, we have

$$\begin{aligned}&\left(\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right) A(f)^{r-1} \right)^r \\ &\qquad \le \left(\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)^r \right) \left(\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} A(f)^r \right)^{r-1},\end{aligned}$$

which gives

$$\sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)^r \ge \frac{S_1^r}{S_2^{r-1}}.$$

We will prove the main theorem by finding the asymptotic orders of the magnitude of S_1 and S_2.

In order to get the asymptotic formula of S_2, we need to study the combinatorics of the Fourier coefficient $\lambda_f(n^2)$. Following from the Hecke multiplicity (2.1), we may write

$$\lambda_f(n_1^2)\lambda_f(n_2^2)\cdots\lambda_f(n_r^2) = \sum_{t|n_1n_2\cdots n_r} b_t(n_1,n_2,\ldots,n_r)\lambda_f(t^2),$$

where $b_t(n_1,n_2,\ldots,n_r)$ are certain non-negative integers which satisfy

$$b_t(n_1,n_2,\ldots,n_r) \ll (n_1n_2\cdots n_r)^{\epsilon}.$$

It is easy to see that b_1 satisfies a multiplicative property. If

$$\left(\prod_{i=1}^{r} m_i, \prod_{i=1}^{r} n_i\right) = 1,$$

then

$$b_1(m_1n_1, m_2n_2,\ldots,m_rn_r) = b_1(m_1,m_2,\ldots,m_r)\, b_1(n_1,n_2,\ldots,n_r).$$

Denote

$$B_r(n) = \sum_{\substack{n_1,n_2,\ldots,n_r\\ n_1n_2\cdots n_r=n}} b_1(n_1,n_2,\ldots,n_r),$$

then we have that $B_r(n)$ is a multiplicative function. Note that $B_r(p^a)$ is independent of p and grows at most polynomially in a and

$$B_r(p)=0,\ B_r(p^2)=\frac{r(r-1)}{2},$$

which follows from

$$b_1(p,1,\ldots,1) = b_1(p^2,1,\ldots,1) = 0$$

and

$$b_1(p,p,\ldots,1) = 1.$$

Consequently, we can estimate S_2 as follows:

$$S_2 = \sum_{n_1,n_2,\ldots,n_r\le x} \frac{1}{\sqrt{n_1n_2\cdots n_r}}$$

$$\begin{aligned}
&\sum_{t|n_1n_2\cdots n_r} b_t(n_1,n_2,\ldots,n_r) \sum_{f\in\mathcal{B}_k(N)} \omega_f^{-1}\lambda_f(t^2) \\
&= \sum_{n_1,n_2,\ldots,n_r\le x} \frac{b_1(n_1,n_2,\ldots,n_r)}{\sqrt{n_1n_2\cdots n_r}} \\
&\quad + O\left(N^{-3/2+\epsilon}x^r \sum_{n_1,n_2,\ldots,n_r\le x} \frac{1}{(n_1n_2\cdots n_r)^{1/2-\epsilon}}\right) \\
&= \sum_{n_1,n_2,\ldots,n_r\le x} \frac{b_1(n_1,n_2,\ldots,n_r)}{\sqrt{n_1n_2\cdots n_r}} \\
&\quad + O\left(N^{-3/4+\epsilon}\right),
\end{aligned}$$

where we have used Lemma 2.1 in the second step. As for the main term, we have

$$\sum_{n\le x}\frac{B_r(n)}{\sqrt{n}} \le \sum_{n_1,n_2,\ldots,n_r\le x} \frac{b_1(n_1,n_2,\ldots,n_r)}{\sqrt{n_1n_2\cdots n_r}} \le \sum_{n\le x^r}\frac{B_r(n)}{\sqrt{n}}.$$

By the properties of $B_r(n)$, we have that the generating function satisfies

$$\sum_{n=1}^{\infty}\frac{B_r(n)}{n^s} = \zeta(2s)^{r(r-1)/2}D(s),$$

where $D(s)$ is a Dirichlet series which converges absolutely in $\Re s > 1/3+\epsilon$. A standard argument gives that, for a positive constant C_r,

$$\sum_{n\le z}\frac{B_r(n)}{\sqrt{n}} \sim C_r(\log z)^{r(r-1)/2}.$$

Finally, we get

$$S_2 \asymp (\log x)^{r(r-1)/2} \asymp (\log N)^{r(r-1)/2}. \tag{3.1}$$

Returning to S_1, note that

$$A(f)^{r-1} = \sum_{n_1,\ldots,n_{r-1}\le x} \frac{1}{\sqrt{n_1\cdots n_{r-1}}} \sum_{t|n_1\cdots n_{r-1}} b_t(n_1,\ldots,n_{r-1})\lambda_f(t^2).$$

By bound (2.2), we obtain

$$
\begin{aligned}
S_1 = & \sum_{n_1,n_2,\dots,n_{r-1}\le x} \frac{\sum_{t|n_1n_2\cdots n_{r-1}} b_t(n_1,n_2,\dots,n_{r-1})}{\sqrt{n_1n_2\cdots n_{r-1}}} \\
& \sum_{f\in\mathcal{B}_k(N)} \omega_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right)\lambda_f(t^2) \\
= & \frac{1}{2}\log N \sum_{n_1,n_2,\dots,n_{r-1}\le x} \frac{1}{\sqrt{n_1n_2\cdots n_{r-1}}} \\
& \sum_{t|n_1n_2\cdots n_{r-1}} \frac{b_t(n_1,n_2,\dots,n_{r-1})}{\sqrt{t}} + O(N^{-1/4r+\epsilon}).
\end{aligned}
$$

Note that

$$
b_1(n_1,n_2,\dots,n_{r-1},t) = \begin{cases} b_t(n_1,n_2,\dots,n_{r-1}) & \text{if } t \mid n_1n_2\cdots n_{r-1}, \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore,

$$
S_1 = \frac{1}{2}\log N \sum_{\substack{n_1,n_2,\dots,n_{r-1}\le x \\ n_r\le x^{r-1}}} \frac{b_1(n_1,n_2,\dots,n_{r-1},n_r)}{\sqrt{n_1n_2\cdots n_{r-1}n_r}} + O(N^{-1/4r+\epsilon}).
$$

Using $b_1(n_1,n_2,\dots,n_{r-1},n_r)\ge 0$, we have that $S_1 \gg (\log N)S_2$. Arguing, as in the case of S_2, we finally get that

$$
S_1 \gg (\log N)^{1+r(r-1)/2}. \tag{3.2}
$$

From (3.1) and (3.2), we have

$$
\frac{S_1^r}{S_2^{r-1}} \gg (\log N)^{r(r+1)/2},
$$

where the implied constant depends on r. This completes the proof. □

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