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GORENSTEIN CATEGORIES $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ AND DIMENSIONS

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ABSTRACT. Let $\mathscr{A}$ be an abelian category and $\mathscr{X},\mathscr{Y},\mathscr{Z}$ additive full subcategories of $\mathscr{A}$. We introduce and study the Gorenstein category $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ as a common generalization of some known modules such as Gorenstein projective (injective) modules [**5**], strongly Gorenstein flat modules [**3**] and Gorenstein FP-injective modules [**4**], and prove the stability of $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. We also establish Gorenstein homological dimensions in terms of the category $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

1. Introduction and preliminaries. Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ an additive full subcategory of $\mathscr{A}$. Sather-Wagstaff, Sharif and White [**10**] introduced the Gorenstein category $\mathcal{G}(\mathscr{C})$ which is defined as

(1.1)
$$\begin{aligned}\mathcal{G}(\mathscr{C}) = \{A \text{ is an object of } \mathscr{A} \mid & \text{ there is an exact sequence of objects} \\ & \text{in } \mathscr{C} \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \\ & \text{which is both } \mathrm{Hom}_{\mathscr{A}}(\mathscr{C},-)\text{-exact and } \mathrm{Hom}_{\mathscr{A}}(-,\mathscr{C})\text{-exact,} \\ & \text{such that } A \cong \mathrm{Im}\,(C_0 \to C^0)\}.\end{aligned}$$

This definition unifies the following notions: modules of Gorenstein dimension zero [**1**], Gorenstein projective (injective) modules [**5**], V-Gorenstein projective (injective) modules [**6**], and so on. It is well known that Gorenstein projective (injective) modules have nice properties when the ring in question is n-Gorenstein (a ring R is called *n-Gorenstein* if R is a left and right Noetherian ring with self-injective dimension at most n for an integer $n \geq 0$ on either side). Also there

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are many results of a homological nature which may be generalized from Noetherian to coherent rings. To this end, Ding, Li and Mao [**3**] introduced and studied the strongly Gorenstein flat R-modules as the modules of the form $\operatorname{Im}\delta_0$ for some exact sequence of projective R-modules

$$P:\cdots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} \cdots$$

such that the complex $\operatorname{Hom}_R(P,Q)$ is exact for each flat R-module Q. Bennis and Ouarghi [**2**] proved that some results in [**3**] remain true in the above definition whenever Q is considered to be in any class of modules containing all projective modules.

In this paper, we investigate the objects that arise from an iteration of this construction. Let $\mathscr{A}$ be an abelian category and $\mathscr{X},\mathscr{Y},\mathscr{Z}$ additive full subcategories of $\mathscr{A}$. We introduce the Gorenstein subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ of $\mathscr{A}$, which unifies the following notions: strongly Gorenstein flat modules [**3**], Gorenstein FP-injective modules [**4**], $\mathscr{X}$-Gorenstein projective modules [**2**], $\mathscr{Y}$-Gorenstein injective modules [**9**] and the Gorenstein category $\mathcal{G}(\mathscr{C})$ [**10**]. We give some general characterizations of the Gorenstein subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ and prove the stability of $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. We also establish Gorenstein homological dimensions in terms of the Gorenstein subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

Let $\mathscr{A}$ be an abelian category and $\mathscr{C}$ a full subcategory of $\mathscr{A}$. An exact sequence in $\mathscr{A}$ is called $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact if it remains still exact after applying the functor $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C},-)$. Let M be an object of $\mathscr{A}$. An exact sequence $\cdots\to C_1\to C_0\to M\to 0$ in $\mathscr{A}$ with all C_i in $\mathscr{C}$ is called a *proper $\mathscr{C}$-resolution* of M if it is $\operatorname{Hom}_{\mathscr{A}}(\mathscr{C},-)$-exact. Dually, the notions of a $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{C})$-exact sequence and a coproper $\mathscr{C}$-coresolution of M are defined.

For M, $\mathscr{X}$-pd(M) is defined as $\inf\{n\geq 0 \mid$ there is an exact sequence $0\to C_n\to\cdots\to C_0\to M\to 0$ in $\mathscr{A}$ with all C_i in $\mathscr{C}\}$, and set $\mathscr{X}$-pd(M) infinity if no such integer exists. We also define $\mathscr{Y}$-id(M) dually, and set

$$\begin{aligned}\operatorname{res}\widehat{\mathscr{X}} &= \text{ the subcategory of objects } M \text{ of } \mathscr{A} \text{ with } \mathscr{X}\text{-pd}(M)<\infty,\\ \operatorname{cores}\widehat{\mathscr{Y}} &= \text{ the subcategory of objects } N \text{ of } \mathscr{A} \text{ with } \mathscr{Y}\text{-id}(N)<\infty.\end{aligned}$$

Let $\mathscr{X},\mathscr{Y}$ be two additive full subcategories of $\mathscr{A}$. Write $\mathscr{X}\perp\mathscr{Y}$ if

$\mathrm{Ext}^{\geq 1}_{\mathscr{A}}(X,Y)=0$ for each object $X\in\mathscr{X}$ and each object $Y\in\mathscr{Y}$. We denote

$$\mathscr{X}^{\perp}=\{A\in\mathscr{A}\mid \mathrm{Ext}^1_{\mathscr{A}}(X,A)=0,\quad \text{for all } X\in\mathscr{X}\},$$
$$^{\perp}\mathscr{Y}=\{B\in\mathscr{A}\mid \mathrm{Ext}^1_{\mathscr{A}}(B,Y)=0,\quad \text{for all } Y\in\mathscr{Y}\}.$$

Given a class $\mathscr{F}$ of objects in $\mathscr{A}$ and an object $X\in\mathscr{A}$, a morphism $\varphi: F\to X$ with $F\in\mathscr{F}$ is called an *$\mathscr{F}$-precover* of X if $\mathrm{Hom}_{\mathscr{A}}(F',F)\to\mathrm{Hom}_{\mathscr{A}}(F',X)\to 0$ is exact for all $F'\in\mathscr{F}$. If, moreover, any f such that $\varphi f=\varphi$ is an automorphism of F, we say that $\varphi: F\to X$ is an $\mathscr{F}$-cover. $\mathscr{F}$-preenvelopes and $\mathscr{F}$-envelopes are defined dually.

2. Gorenstein subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ of $\mathscr{A}$. Let $\mathscr{A}$ be an abelian category and $\mathscr{X},\mathscr{Y},\mathscr{Z}$ additive full subcategories of $\mathscr{A}$. In this section, we introduce and study the Gorenstein subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ of $\mathscr{A}$ and prove the stability of the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

Definition 2.1. The *Gorenstein subcategory* $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ of $\mathscr{A}$ is defined as

$$\begin{aligned}\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})=\{&A \text{ is an object of } \mathscr{A}\mid \text{ there is an exact}\\ &\text{sequence of objects in } \mathscr{X}\cdots\longrightarrow X_1\longrightarrow X_0\longrightarrow X^0\longrightarrow X^1\longrightarrow\cdots,\\ &\text{which is both } \mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)\text{-exact and}\\ &\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})\text{-exact, such that } A\cong \mathrm{Im}\,(X_0\to X^0)\}.\end{aligned}$$

Remark 2.2.

(1) It is clear that each object in $\mathscr{X}$ is in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. If

$$X:\cdots\longrightarrow X_1\longrightarrow X_0\longrightarrow X^0\longrightarrow X^1\longrightarrow\cdots$$

is a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$ and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$ exact sequence of objects in $\mathscr{X}$, then by symmetry, all the images, the kernels and the cokernels of X are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

(2) If $\mathscr{X}=\mathscr{Y}=\mathscr{Z}=\mathscr{C}$, then the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ is exactly the Gorenstein category $\mathcal{G}(\mathscr{C})$ in **[8, 10]**.

(3) If $\mathscr{A}$ is the category of R-modules and $\mathscr{X} = \mathscr{Y}$ is the class of projective R-modules, then the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ is exactly the class of $\mathscr{Z}$-Gorenstein projective modules in [**2**]. In particular, if $\mathscr{Z}$ is the class of flat R-modules, then the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ is exactly the class of strongly Gorenstein flat modules in [**3**].

(4) If $\mathscr{A}$ is the category of R-modules and $\mathscr{X} = \mathscr{Z}$ is the class of injective R-modules, then the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ is exactly the class of $\mathscr{Y}$-Gorenstein injective modules in [**9**]. In particular, if $\mathscr{Y}$ is the class of FP-injective R-modules, then the subcategory $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ is exactly the class of Gorenstein FP-injective modules in [**4**].

In what follows, we always assume that $\mathscr{X} \subseteq \mathscr{Y}$ and $\mathscr{X} \subseteq \mathscr{Z}$. The following result investigates the behavior of the object in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ in short exact sequences.

Theorem 2.3. *Given a both* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$*-exact and* $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$*-exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2.1}$$

in $\mathscr{A}$*. If any two of* A, B *and* C *are in* $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$*, then so is the third.*

Proof. Assume that A, C are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. Then there exist both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences of objects in $\mathscr{X}$:

$$X_A : \cdots \longrightarrow X_A^{-2} \longrightarrow X_A^{-1} \longrightarrow X_A^{0} \longrightarrow X_A^{1} \longrightarrow \cdots$$
$$\text{with } A \cong \mathrm{Im}\,(X_A^{-1} \longrightarrow X_A^{0}),$$
$$X_C : \cdots \longrightarrow X_C^{-2} \longrightarrow X_C^{-1} \longrightarrow X_C^{0} \longrightarrow X_C^{1} \longrightarrow \cdots$$
$$\text{with } C \cong \mathrm{Im}\text{—,}(X_C^{-1} \longrightarrow X_C^{0}).$$

Since $\mathscr{X} \subseteq \mathscr{Y}$ and $\mathscr{X} \subseteq \mathscr{Z}$, we have the sequence (2.1) is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact. Thus, there is an exact sequence of objects in $\mathscr{X}$:

$$X_B : \cdots \longrightarrow X_A^{-2} \oplus X_C^{-2} \longrightarrow X_A^{-1} \oplus X_C^{-1} \longrightarrow X_A^{0} \oplus X_C^{0} \longrightarrow X_A^{1} \oplus X_C^{1} \longrightarrow \cdots$$

such that $B \cong \operatorname{Im}(X_A^{-1} \oplus X_C^{-1} \to X_A^0 \oplus X_C^0)$. It follows by the fundamental lemma of homological algebra that the sequence X_B is both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact. This implies that B is in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

Assume that B, C are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. Then there exist both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences of objects in $\mathscr{X}$:

$$\begin{aligned} X_B &: \cdots \longrightarrow X_B^{-2} \longrightarrow X_B^{-1} \longrightarrow X_B^0 \longrightarrow X_B^1 \longrightarrow \cdots \\ &\quad \text{with } B \cong \operatorname{Im}(X_B^{-1} \longrightarrow X_B^0), \\ X_C &: \cdots \longrightarrow X_C^{-2} \longrightarrow X_C^{-1} \longrightarrow X_C^0 \longrightarrow X_C^1 \longrightarrow \cdots \\ &\quad \text{with } C \cong \operatorname{Im}(X_C^{-1} \longrightarrow X_C^0). \end{aligned}$$

By [**8,** Theorem 3.8], we have both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $0 \to A \to X_B^0 \to X_C^0 \oplus X_B^1 \to X_C^1 \oplus X_B^2 \to \cdots$. Consider the both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact short exact sequence $0 \to B^{-1} \to X_B^{-1} \to B \to 0$. We have the following pullback diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & B^{-1} & = & B^{-1} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & C' & \longrightarrow & X_B^{-1} & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

A simple diagram chasing argument shows that the second row and the first column in the above diagram are both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact. Consider both $\operatorname{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\operatorname{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact short exact sequence $0 \to C^{-1} \to X_C^{-1} \to C \to 0$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{-1} & \longrightarrow & X_C^{-1} & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C' & \longrightarrow & X_B^{-1} & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

Then we obtain both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $0 \to C^{-1} \to C' \oplus X_C^{-1} \to X_B^{-1} \to 0$, and so we get both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $\cdots \to X_C^{-4} \to X_C^{-3} \to X_C^{-2} \oplus X_B^{-1} \to C' \oplus X_C^{-1} \to 0$ by the preceding proof.

Consider the exact sequence $0 \to X_C^{-1} \to C' \oplus X_C^{-1} \to C' \to 0$. Then [**8,** Theorem 3.6] yields both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $\cdots \to X_C^{-4} \to X_C^{-3} \oplus X_C^{-1} \to X_C^{-2} \oplus X_B^{-1} \to C' \to 0$. Applying [**8,** Theorem 3.6] again for the first column in the first diagram, we get both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $\cdots \to X_C^{-4} \oplus X_B^{-3} \to X_C^{-3} \oplus X_C^{-1} \oplus X_B^{-2} \to X_C^{-2} \oplus X_B^{-1} \to A \to 0$. Therefore, the following exact sequence of objects in $\mathscr{X}$

$$\begin{aligned} X_A : \cdots \longrightarrow X_C^{-4} \oplus X_B^{-3} \longrightarrow X_C^{-3} \oplus X_C^{-1} \\ \oplus X_B^{-2} \longrightarrow X_C^{-2} \oplus X_B^{-1} \longrightarrow X_B^0 \longrightarrow X_C^0 \\ \oplus X_B^1 \longrightarrow \cdots \end{aligned}$$

is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact, such that $A \cong \mathrm{Im}(X_C^{-2} \oplus X_B^{-1} \to X_B^0)$. It follows that A is in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

Assume that A, B are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. Then there exist both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences of objects in $\mathscr{X}$:

$$\begin{aligned} &X_A : \cdots \longrightarrow X_A^{-2} \longrightarrow X_A^{-1} \longrightarrow X_A^0 \longrightarrow X_A^1 \longrightarrow \cdots \\ &\qquad \text{with } A \cong \mathrm{Im}\,(X_A^{-1} \longrightarrow X_A^0), \\ &X_B : \cdots \longrightarrow X_B^{-2} \longrightarrow X_B^{-1} \longrightarrow X_B^0 \longrightarrow X_B^1 \longrightarrow \cdots \\ &\qquad \text{with } B \cong \mathrm{Im}\,(X_B^{-1} \longrightarrow X_B^0). \end{aligned}$$

By [**8,** Theorem 3.6], we have both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $\cdots \to X_B^{-3} \oplus X_A^{-2} \to X_B^{-2} \oplus X_A^{-1} \to X_B^{-1} \to C \to 0$. Consider both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact short exact sequence $0 \to B \to X_B^0 \to B^1 \to 0$. We have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & X_B^0 & \longrightarrow & A' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B^1 & = & B^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

A simple diagram chasing argument shows that the second row and the third column in the above diagram are both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact. Consider both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact short exact sequence $0 \to A \to X_A^0 \to A^1 \to 0$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & X_B^0 & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & X_A^0 & \longrightarrow & A^1 & \longrightarrow & 0.
\end{array}$$

Then we obtain both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact sequence $0 \to X_B^0 \to A' \oplus X_A^0 \to A^1 \to 0$, and so we get both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact sequence $0 \to A' \oplus X_A^0 \to X_A^1 \oplus X_B^0 \to X_A^2 \to X_A^3 \to \cdots$ by the preceding proof.

Consider the exact sequence $0 \to A' \to A' \oplus X_A^0 \to X_A^0 \to 0$. Then [**8,** Theorem 3.8] yields both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact sequence $0 \to A' \to X_A^1 \oplus X_B^0 \to X_A^0 \oplus X_A^2 \to X_A^3 \to \cdots$. Applying [**8,** Theorem 3.8] again for the third column in the first diagram, we get both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact sequence $0 \to C \to X_A^1 \oplus X_B^0 \to X_A^0 \oplus X_A^2 \oplus X_B^1 \to X_A^3 \oplus X_B^2 \to \cdots$.

Therefore, the following exact sequence of objects in $\mathscr{X}$

$$X_C : \cdots \longrightarrow X_B^{-2} \oplus X_A^{-1} \longrightarrow X_B^{-1} \longrightarrow X_A^1 \oplus X_B^0 \longrightarrow X_A^0 \oplus X_A^2 \oplus X_B^1 \longrightarrow X_A^3 \oplus X_B^2 \longrightarrow \cdots$$

is both $\Longrightarrow$ $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\Longrightarrow$ $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact, such that $C \cong\Longrightarrow \mathrm{Im}\,(X_B^{-1} \to X_A^1 \oplus X_B^0)$. It follows that C is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$. □

Lemma 2.4. *Assume that A is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$. If $\mathscr{X} \perp \mathscr{Y}$, then $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(Y, A) = 0$ for any $Y \in \mathrm{cores}\,\widehat{\mathscr{Y}}$. Also if $\mathscr{X} \perp \mathscr{Z}$, then $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(A, Z) = 0$ for any $Z \in \mathrm{res}\,\widehat{\mathscr{Z}}$.*

Proof. It is easy. □

Corollary 2.5. *Given a short exact sequence of objects in $\mathscr{A}$*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0. \tag{2.2}$$

(i) *Assume that $\mathscr{X} \perp \mathscr{Y}, \mathscr{X} \perp \mathscr{Z}$. If A, C are in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$, then B is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$;*
(ii) *Assume that $\mathscr{X} \perp \mathscr{Z}$. If C is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ and the sequence (2.2) is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact, then A is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ if and only if B is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$;*
(iii) *Assume that $\mathscr{X} \perp \mathscr{Y}$. If A is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ and the sequence (2.2) is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact, then C is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ if and only if B is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$.*

Proof.

(i) Since A, C are in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ and $\mathscr{X} \perp \mathscr{Y}, \mathscr{X} \perp \mathscr{Z}$, it follows that $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(Y, A) = 0 = \mathrm{Ext}_{\mathscr{A}}^{\geq 1}(C, Z)$ for any $Y \in \mathscr{Y}$ and $Z \in \mathscr{Z}$. So the sequence (2.2) is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact. Thus, Theorem 2.3 implies that B is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$.
(ii) Since C is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ and $\mathscr{X} \perp \mathscr{Z}$, we get $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(C, Z) = 0$ for any $Z \in \mathscr{Z}$, and so the sequence (2.2) is $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{Z})$-exact. Hence, Theorem 2.3 shows our desired result.
(iii) Since A is in $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ and $\mathscr{X} \perp \mathscr{Y}$, we get $\mathrm{Ext}_{\mathscr{A}}^{\geq 1}(Y, A) = 0$ for any $Y \in \mathscr{Y}$, and so the sequence (2.2) is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y}, -)$-exact. Hence, Theorem 2.3 shows our desired result. □

Corollary 2.6. ([**10**, Theorem4.12]). *Assume that* $\mathscr{X}\perp\mathscr{X}$.

(i) *If* $\mathscr{X}$ *is closed under taking kernels of epimorphisms, then so is* $\mathcal{G}(\mathscr{X})$.
(ii) *If* $\mathscr{X}$ *is closed under taking cokernels of monomorphisms, then so is* $\mathcal{G}(\mathscr{X})$.

Let $\mathscr{C}$ be a class of objects in $\mathscr{A}$. Assume that $\mathscr{A}$ has enough projective objects and injective objects. We call $\mathscr{C}$ *projectively resolving* [**7**] if (1) it contains all projective objects; (2) for every short exact sequence $0\to C'\to C\to C''\to 0$ with $C''\in\mathscr{C}$, the conditions $C'\in\mathscr{C}$ and $C\in\mathscr{C}$ are equivalent. An injectively resolving class is defined dually.

Corollary 2.7. *If* $\mathscr{Z}$ *is a class of R-modules that contains all projective R-modules, then the class of* $\mathscr{Z}$*-Gorenstein projective R-modules is projectively resolving. In particular, the class of strongly Gorenstein flat R-modules and the class of Gorenstein projective R-modules are projectively resolving.*

Corollary 2.8. *If* $\mathscr{Y}$ *is a class of R-modules that contains all injective R-modules, then the class of* $\mathscr{Y}$*-Gorenstein injective R-modules is injectively resolving. In particular, the class of Gorenstein FP-injective R-modules and the class of Gorenstein injective R-modules are injectively resolving.*

Theorem 2.9. *The subcategory* $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ *is closed under direct summands.*

Proof. Let $A_1\oplus A_2=A$ be in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. Then there exist both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence of objects in $\mathscr{X}$:

$$X_A:\cdots\longrightarrow X_A^{-2}\longrightarrow X_A^{-1}\longrightarrow X_A^{0}\longrightarrow X_A^{1}\longrightarrow\cdots$$
$$\text{with } A\cong \mathrm{Im}\,(X_A^{-1}\longrightarrow X_A^{0}).$$

Consider the both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact short exact sequence $0\to A^{-1}\to X_A^{-1}\to A\to 0$. We have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^{-1} & = & A^{-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & D & \longrightarrow & X_A^{-1} & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A_2 & \longrightarrow & A & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

A simple diagram chasing argument shows that the middle row is both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact. Similarly, we have both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $0 \to D' \to X_A^{-1} \to A_2 \to 0$. Consider the exact sequence $0 \to A_i \to A \to A_j \to 0$ for $i,j = 1,2$. Then [**8,** Theorem 3.6] yields both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences $X_A^{-1} \oplus X_A^{-2} \to X_A^{-1} \to A_1 \to 0$ and $X_A^{-1} \oplus X_A^{-2} \to X_A^{-1} \to A_2 \to 0$. Again, [**8,** Theorem 3.6] provides both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences $X_A^{-1} \oplus X_A^{-2} \oplus X_A^{-3} \to X_A^{-1} \oplus X_A^{-2} \to X_A^{-1} \to A_1 \to 0$ and $X_A^{-1} \oplus X_A^{-2} \oplus X_A^{-3} \to X_A^{-1} \oplus X_A^{-2} \to X_A^{-1} \to A_2 \to 0$. Continuing this process, we get both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences

$$\cdots \longrightarrow X_A^{-1} \oplus X_A^{-2} \oplus X_A^{-3} \longrightarrow X_A^{-1} \oplus X_A^{-2} \longrightarrow X_A^{-1} \longrightarrow A_1 \longrightarrow 0,$$

$$\cdots \longrightarrow X_A^{-1} \oplus X_A^{-2} \oplus X_A^{-3} \longrightarrow X_A^{-1} \oplus X_A^{-2} \longrightarrow X_A^{-1} \to A_2 \longrightarrow 0.$$

Dually, repeated applications of [**8,** Theorem 3.8] yields both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences

$$0 \longrightarrow A_1 \longrightarrow X_A^0 \longrightarrow X_A^0 \oplus X_A^1 \longrightarrow X_A^0 \oplus X_A^1 \oplus X_A^2 \longrightarrow \cdots,$$

$$0 \longrightarrow A_2 \longrightarrow X_A^0 \longrightarrow X_A^0 \oplus X_A^1 \longrightarrow X_A^0 \oplus X_A^1 \oplus X_A^2 \longrightarrow \cdots.$$

Consequently, A_1 and A_2 are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$. $\square$

Set $\mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})=\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$, and inductively set $\mathcal{G}^{n+1}(\mathscr{X},\mathscr{Y},\mathscr{Z})=\mathcal{G}(\mathcal{G}^n(\mathscr{X},\mathscr{Y},\mathscr{Z}),\mathcal{G}^n(\mathscr{Y}),\mathcal{G}^n(\mathscr{Z}))$ for any $n\geq 1$, where $\mathcal{G}^n(\mathscr{Y})=\mathcal{G}(\mathcal{G}^{n-1}(\mathscr{Y}))$ and $\mathcal{G}^0(\mathscr{Y})=\mathscr{Y}$. Let $\mathscr{C}$ be an additive full subcategory of $\mathscr{A}$. Huang [**8**] provided a method to construct a proper $\mathscr{C}$-resolution (respectively, coproper $\mathscr{C}$-coresolution) of one term in a short exact sequence in $\mathscr{A}$ from those of the other two terms. By using these constructions, he answered affirmatively an open question on the stability of the Gorenstein category $\mathcal{G}(\mathscr{C})$ posed by Sather-Wagstaff, Sharif and White [**10**]. Now we get the following result.

Theorem 2.10. *$\mathcal{G}^n(\mathscr{X},\mathscr{Y},\mathscr{Z})=\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$ for any $n\geq 1$.*

Proof. It is easy to see that $\mathscr{X}\subseteq\mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})\subseteq\mathcal{G}^2(\mathscr{X},\mathscr{Y},\mathscr{Z})\subseteq\cdots$ is an ascending chain of additive subcategories of $\mathscr{A}$.

Let M be an object in $\mathcal{G}^2(\mathscr{X},\mathscr{Y},\mathscr{Z})$, and

$$\cdots\longrightarrow G_1\longrightarrow G_0\longrightarrow G^0\longrightarrow G^1\longrightarrow\cdots$$

both $\mathrm{Hom}_{\mathscr{A}}(\mathcal{G}(\mathscr{Y}),-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathcal{G}(\mathscr{Z}))$-exact sequences in $\mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})$ with $M\cong\mathrm{Im}\,(G_0\to G^0)$. Then for any $j\geq 0$, there exist both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequences:

$$\cdots\longrightarrow X_j^i\longrightarrow\cdots\longrightarrow X_j^1\longrightarrow X_j^0\longrightarrow G_j\longrightarrow 0,$$
$$0\longrightarrow G^j\longrightarrow Y_0^j\longrightarrow Y_1^j\longrightarrow\cdots\longrightarrow Y_i^j\longrightarrow\cdots$$

with all X_j^i and Y_i^j in $\mathscr{X}$. By [**8,** Corollary 3.7 and 3.9], we get exact sequences:

$$\cdots\longrightarrow\bigoplus_{j=0}^{n}X_j^{n-j}\longrightarrow\cdots\longrightarrow X_0^1\oplus X_1^0\longrightarrow X_0^0\longrightarrow M\longrightarrow 0,$$

$$0\longrightarrow M\longrightarrow Y_0^0\longrightarrow Y_1^0\oplus Y_0^1\longrightarrow\cdots\longrightarrow\bigoplus_{j=0}^{n}Y_{n-j}^j\longrightarrow\cdots,$$

which are both $\mathrm{Hom}_{\mathscr{A}}(\mathscr{Y},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact. So

$$\cdots \longrightarrow \bigoplus_{j=0}^{n} X_j^{n-j} \longrightarrow \cdots \longrightarrow X_0^1 \oplus X_1^0 \longrightarrow X_0^0$$

$$\longrightarrow Y_0^0 \longrightarrow Y_1^0 \oplus Y_0^1 \longrightarrow \cdots \longrightarrow \bigoplus_{j=0}^{n} Y_{n-j}^{j} \longrightarrow \cdots$$

is an exact sequence in $\mathscr{X}$ with $M \cong \mathrm{Im}(X_0^0 \to Y_0^0)$, and hence M is in $\mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})$ and $\mathcal{G}^2(\mathscr{X},\mathscr{Y},\mathscr{Z}) \subseteq \mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})$. This implies that $\mathcal{G}^2(\mathscr{X},\mathscr{Y},\mathscr{Z}) = \mathcal{G}^1(\mathscr{X},\mathscr{Y},\mathscr{Z})$. By using induction on n we easily get the assertion. □

Corollary 2.11. ([**8,** Theorem 4.1]). *$\mathcal{G}^n(\mathscr{X}) = \mathcal{G}(\mathscr{X})$ for any $n \geq 1$.*

Corollary 2.12. *If $\mathscr{Z}$ is a class of R-modules that contains all projective R-modules, then the class of $\mathscr{Z}$-Gorenstein projective R-modules is stable. Dually, if $\mathscr{Y}$ is a class of R-modules that contains all injective R-modules, then the class of $\mathscr{Y}$-Gorenstein injective R-modules is stable.*

3. Gorenstein homological dimensions. In this section, we establish Gorenstein homological dimensions in terms of the Gorenstein category $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z})$.

Proposition 3.1. *Assume that $\mathscr{X} = \mathscr{Y}$, $\mathscr{X} \perp \mathscr{Z}$ and every object in $\mathscr{A}$ has an epic $\mathscr{X}$-precover. Consider the following $Hom_{\mathscr{A}}(\mathscr{X},-)$-exact sequences*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{K}_n \longrightarrow \widetilde{G}_{n-1} \longrightarrow \cdots \longrightarrow \widetilde{G}_0 \longrightarrow M \longrightarrow 0$$

in $\mathscr{A}$, where each G_i and $\widetilde{G}_i$ are in $\mathcal{G}(\mathscr{X},\mathscr{Y},\mathscr{Z}) = \mathcal{G}(\mathscr{X},\mathscr{Z})$. Then K_n is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ if and only if $\widetilde{K}_n$ is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Proof. In view of our assumption, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence:

$$0 \longrightarrow L_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0,$$

where each X_i is in $\mathscr{X}$. Then we get the following commutative diagrams:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & L_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & K_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & L_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & \widetilde{K}_n & \longrightarrow & \widetilde{G}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \widetilde{G}_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

From these two diagrams, we have the following $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequences:

$$\begin{aligned}
0 \longrightarrow L_n \longrightarrow K_n \oplus X_{n-1} \longrightarrow \cdots \\
\longrightarrow G_1 \oplus X_0 \longrightarrow G_0 \longrightarrow 0, \\
0 \longrightarrow L_n \longrightarrow \widetilde{K}_n \oplus X_{n-1} \longrightarrow \cdots \\
\longrightarrow \widetilde{G}_1 \oplus X_0 \longrightarrow \widetilde{G}_0 \longrightarrow 0.
\end{aligned}$$

If K_n is in $G(\mathscr{X},\mathscr{Z})$, then Corollary 2.5 implies that L_n is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, and so $\widetilde{K}_n$ is also in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Similarly, if $\widetilde{K}_n$ is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, then K_n is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. $\square$

Definition 3.2. Assume $\mathscr{X} = \mathscr{Y}$, $\mathscr{X} \perp \mathscr{Z}$ and every object in $\mathscr{A}$ has an epic $\mathscr{X}$-precover. We say that an object M of $\mathscr{A}$ has $\mathcal{G}(\mathscr{X},\mathscr{Z})$-projective dimension less than or equal to n, denoted by $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(M) \leq n$, if there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

in $\mathscr{A}$ with each G_i in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. If no such finite sequence exists, define $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(M) = \infty$; otherwise, if n is the least such integer, define $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(M) = n$.

Proposition 3.3. *Assume that $\mathscr{X} = \mathscr{Y}$, $\mathscr{X} \perp \mathscr{Z}$ and every object in $\mathscr{A}$ has an epic $\mathscr{X}$-precover. Let M be an object in $\mathscr{A}$ with*

$\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(M)=n$. Then there exist $Hom_{\mathscr{A}}(\mathscr{X},-)$-exact sequences

$$0\longrightarrow H\longrightarrow G\longrightarrow M\longrightarrow 0,$$
$$0\longrightarrow M\longrightarrow H'\longrightarrow G'\longrightarrow 0,$$

with G in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, $\mathscr{X}$-pd$(H)\leq n-1$ and G' in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, $\mathscr{X}$-pd$(H')\leq n$.

Proof. We will prove the desired result by induction on n. If $n=0$, then M is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Thus, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence

$$0\longrightarrow 0\longrightarrow M\longrightarrow M\longrightarrow 0.$$

We also have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence

$$0\longrightarrow M\longrightarrow X\longrightarrow G'\longrightarrow 0,$$

with X in $\mathscr{X}$ and G' in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Now, let $n=1$, and let $0\to L_1\to L_0\to M\to 0$ be a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence with each L_i in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. By the case $n=0$, we know that there is a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0\to L_1\to X_0\to H_0\to 0$ with X_0 in $\mathscr{X}$ and H_0 in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L_1 & \longrightarrow & L_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X_0 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_0 & = & H_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

A simple diagram chasing argument shows that the second row and the second column in the above diagram are $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact. But L_0, H_0 are in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, so Corollary 2.5 implies that G_0 is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Thus, we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to X_0 \to G_0 \to M \to 0$ with X_0 in $\mathscr{X}$ and G_0 in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Also, there is a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to G_0 \to X_1 \to G_1 \to 0$ with X_1 in $\mathscr{X}$ and G_1 in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X_0 & = & X_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G_0 & \to & X_1 & \to & G_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & H_1 & \to & G_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

From the middle column, we know that $\mathscr{X}\text{-pd}(H_1) \leq 1$. Thus, we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to M \to H_1 \to G_1 \to 0$ with $\mathscr{X}\text{-pd}(H_1) \leq 1$ and G_1 in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Suppose $n > 1$. Then we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ with each G_i in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Let $K_1 = \mathrm{Im}\,(G_1 \to G_0)$. Then we have $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequences $0 \to K_1 \to G_0 \to M \to 0$ and $0 \to G_n \to \cdots \to G_1 \to K_1 \to 0$, i.e., $\mathcal{G}(\mathscr{X},\mathscr{Z})\text{-pd}(K_1) = n-1$.

By the induction hypothesis, there is a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to K_1 \to H \to \overline{G} \to 0$ with $\mathscr{X}\text{-pd}(H) \leq n-1$ and $\overline{G}$ in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bar{G} & = & \bar{G} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Note that the middle column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact. Then Corollary 2.5 implies that G is in $\mathcal{G}(\mathscr{X}, \mathscr{Z})$. Thus, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact sequence $0 \to G \to X \to G' \to 0$ with X in $\mathscr{X}$ and G' in $\mathcal{G}(\mathscr{X}, \mathscr{Z})$. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & X & \longrightarrow & H' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Since $X \in \mathscr{X}$ and $\mathscr{X}$-pd$(H) \leq n-1$, we get $\mathscr{X}$-pd$(H') \leq n$ by the middle row. A simple diagram chasing argument shows that the first row and the third column in the above diagram are $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X}, -)$-exact. Therefore, the first row and the third column are the desired exact sequences. $\square$

Theorem 3.4. *Assume that $\mathscr{X} = \mathscr{Y}$, $\mathscr{X} \perp \mathscr{Z}$ and every object in $\mathscr{A}$ has an epic $\mathscr{X}$-precover. Let M be an object in $\mathscr{A}$ with finite*

$\mathcal{G}(\mathscr{X},\mathscr{Z})$-projective dimension. Then the following are equivalent for a nonnegative integer n:

(i) $\mathcal{G}(\mathscr{X},\mathscr{Z})\text{-pd}(M)\leq n$;
(ii) *There is a* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$*-exact sequence* $0\to G\to X_{n-1}\to\cdots\to X_0\to M\to 0$ *with each* X_i *in* $\mathscr{X}$ *and* G *in* $\mathcal{G}(\mathscr{X},\mathscr{Z})$;
(iii) M *has a proper* $\mathcal{G}(\mathscr{X},\mathscr{Z})$*-resolution of length* n;
(iv) *There is a* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$*-exact sequence* $0\to X_n\to\cdots\to X_1\to G\to M\to 0$ *with each* X_i *in* $\mathscr{X}$ *and* G *in* $\mathcal{G}(\mathscr{X},\mathscr{Z})$;
(v) *There is a* $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$*-exact sequence* $0\to X_n\to\cdots\to X_{i+1}\to G\to X_{i-1}\to\cdots\to X_0\to M\to 0$ *with each* X_i *in* $\mathscr{X}$ *and* G *in* $\mathcal{G}(\mathscr{X},\mathscr{Z})$;
(vi) $\mathrm{Ext}^i_{\mathscr{A}}(M,Z)=0$ *for all* $i>n$ *and all* $Z\in\mathscr{Z}$;
(vii) $\mathrm{Ext}^i_{\mathscr{A}}(M,L)=0$ *for all* $i>n$ *and all* $L\in\mathrm{res}\,\widehat{\mathscr{Z}}$;
(viii) $\mathrm{Ext}^{n+1}_{\mathscr{A}}(M,L)=0$ *for all* $L\in\mathrm{res}\,\widehat{\mathscr{Z}}$.

Furthermore, we have that

$$\begin{aligned}\mathcal{G}(\mathscr{X},\mathscr{Z})\text{-pd}(M)&=\sup\{i\in\mathbb{N}\mid \mathrm{Ext}^i_{\mathscr{A}}(M,L)\neq 0 \text{ for some } L\in\mathrm{res}\,\widehat{\mathscr{Z}}\}\\&=\sup\{i\in\mathbb{N}\mid \mathrm{Ext}^i_{\mathscr{A}}(M,Z)\neq 0 \text{ for some } Z\in\mathscr{Z}\}.\end{aligned}$$

Proof. The case $n=0$ is trivial. We may assume $n\geq 1$.

(i) $\Rightarrow$ (ii). By (i), there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0\to N\to G_0\to M\to 0$ with G in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ and $\mathcal{G}(\mathscr{X},\mathscr{Z})\text{-pd}(N)\leq n-1$. For G, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0\to G_0'\to X_0\to G_0\to 0$ with G_0' in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ and X_0 in $\mathscr{X}$. Then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_0' & = & G_0' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

A simple diagram chasing argument shows that the first column is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact. For N, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to K \to G_1 \to N \to 0$ with G_1 in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ and $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(K) \leq n-2$. Then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_0' & \longrightarrow & G & \longrightarrow & G_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_0' & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

A simple diagram chasing argument shows that the second row and the second column are $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact; thus, G is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ by Corollary 2.5 and $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(H) \leq n-1$. It follows that we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to H \to X_0 \to M \to 0$ with X_0 in $\mathscr{X}$ and $\mathcal{G}(\mathscr{X},\mathscr{Z})$-pd$(H) \leq n-1$. By repeating this process, we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to G \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ with each X_i in $\mathscr{X}$ and G in $\mathcal{G}(\mathscr{X},\mathscr{Z})$.

(ii) $\Rightarrow$ (iii). Suppose M satisfies (ii). For G in (ii), there is both a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{Z})$-exact sequence $0 \to G \to X^0 \to \cdots \to X^{n-1} \to G' \to 0$ with each X^i in $\mathscr{X}$ and G' in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. Since $\mathscr{X} \subseteq \mathscr{Z}$, we have the following commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & G & \longrightarrow & X^0 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & G' & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Then we have a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence

$$X : 0 \longrightarrow X^0 \longrightarrow X_{n-1} \oplus X^1 \longrightarrow \cdots \longrightarrow X_1 \oplus X^{n-1} \longrightarrow X_0 \oplus G' \longrightarrow M \longrightarrow .$$

But each cokernel of X except M has a finite $\mathscr{X}$-resolution, so X is $\mathrm{Hom}_{\mathscr{A}}(\mathcal{G}(\mathscr{X},\mathscr{Z}),-)$-exact by Lemma 2.4. Therefore, X is a proper $\mathcal{G}(\mathscr{X},\mathscr{Z})$-resolution of M of length n.

(ii) $\Rightarrow$ (iv). Note that X in the proof of (ii) $\Rightarrow$ (iii) is just the desired exact sequence.

(iii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) are obvious.

(i) $\Rightarrow$ (v) is immediate by the equivalence of (i) and (iv).

(i) $\Rightarrow$ (vi). By assumption, there exists a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ with each G_i in $\mathcal{G}(\mathscr{X},\mathscr{Z})$. So $\mathrm{Ext}^{n+j}_{\mathscr{A}}(M,Z) \cong \mathrm{Ext}^{j}_{\mathscr{A}}(G_n,Z) = 0$ for all $j \geq 1$ and all $Z \in \mathscr{Z}$ by Lemma 2.4.

(vi) $\Rightarrow$ (vii) follows from the usual dimension shifting argument.

(vii) $\Rightarrow$ (viii) is clear.

(viii) $\Rightarrow$ (i). By hypothesis, let $\mathcal{G}(\mathscr{X},\mathscr{Z})\text{-pd}(M) = m < \infty$. If $m \leq n$, there is nothing to prove. So we assume $m > n$. Then there is a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $0 \to X_m \to \cdots \to X_1 \to G \to M \to 0$ with each X_i in $\mathscr{X}$ and G in $\mathcal{G}(\mathscr{X},\mathscr{Z})$ by the equivalence of (i) and (iv). Let $K_i = \mathrm{coker}\,(X_{i+1} \to X_i)$ for $1 \leq i \leq m-1$. If $n = 0$, then $\mathrm{Ext}^{n+j}_{\mathscr{A}}(M,K_1) = 0$ by (viii) since $\mathscr{X} \subseteq \mathscr{Z}$. Thus, the exact sequence $0 \to K_1 \to G \to M \to 0$ is split, and M is in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, as desired.

Let $n \geq 1$. Since $\mathscr{X}\text{-pd}(K_{n+1}) < \infty$, we have that $\mathrm{Ext}^1_{\mathscr{A}}(K_n,K_{n+1}) \cong \mathrm{Ext}^{n+1}_{\mathscr{A}}(M,K_{n+1}) = 0$ by Lemma 2.4 and (viii). So the exact sequence

$0 \to K_{n+1} \to X_n \to K_n \to 0$ splits. Thus, K_n is in $\mathcal{G}(\mathscr{X}, \mathscr{Z})$ and (i) follows.

The last claim is an immediate consequence of the equivalences of (i), (vi) and (vii). $\square$

Proposition 3.5. *Assume that* $\mathscr{X} = \mathscr{Y}$, $\mathscr{X} \perp \mathscr{Z}$ *and every object in* $\mathscr{A}$ *has an epic* $\mathscr{X}$*-precover. Then every object in* $\mathscr{A}$ *with finite* $\mathcal{G}(\mathscr{X}, \mathscr{Z})$*-projective dimension has a special* $\mathcal{G}(\mathscr{X}, \mathscr{Z})$*-precover.*

Proof. Let M be an object in $\mathscr{A}$ with finite $\mathcal{G}(\mathscr{X}, \mathscr{Z})$-projective dimension. Then Proposition 3.3 yields an exact sequence $0 \to H \to G \to M \to 0$ with G in $\mathcal{G}(\mathscr{X}, \mathscr{Z})$ and $\mathscr{X}\text{-pd}(H) \leq \mathcal{G}(\mathscr{X}, \mathscr{Z})\text{-pd}(M) - 1$. Now, if G' is in $\mathcal{G}(\mathscr{X}, \mathscr{Z})$, then $\mathrm{Ext}^1_{\mathscr{A}}(G', H) = 0$ which shows that $G \to M$ is a special $\mathcal{G}(\mathscr{X}, \mathscr{Z})$-precover of M. $\square$

Corollary 3.6. ([**9,** Proposition 3.16]). *Assume that* $\mathscr{Z}$ *is a class of* R*-modules that contains all projective* R*-modules. Then every* R*-module with finite* $\mathscr{Z}$*-Gorenstein projective dimension has a special* $\mathscr{Z}$*-Gorenstein projective precover.*

The dual results are given by the next results.

Proposition 3.7. *Assume that* $\mathscr{X} = \mathscr{Z}$, $\mathscr{X} \perp \mathscr{Y}$ *and every object in* $\mathscr{A}$ *has a monic* $\mathscr{X}$*-preenvelope. Consider the following* $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$*-exact sequences*:

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow H_n \longrightarrow 0,$$
$$0 \longrightarrow M \longrightarrow \widetilde{G}^0 \longrightarrow \cdots \longrightarrow \widetilde{G}^{n-1} \longrightarrow \widetilde{H}_n \longrightarrow 0$$

in $\mathscr{A}$, *where each* G^i *and* $\widetilde{G}^i$ *are in* $\mathcal{G}(\mathscr{X}, \mathscr{Y}, \mathscr{Z}) = \mathcal{G}(\mathscr{X}, \mathscr{Y})$. *Then* H_n *is in* $\mathcal{G}(\mathscr{X}, \mathscr{Y})$ *if and only if* $\widetilde{H}_n$ *is in* $\mathcal{G}(\mathscr{X}, \mathscr{Y})$.

Definition 3.8. Assume $\mathscr{X} = \mathscr{Z}$, $\mathscr{X} \perp \mathscr{Y}$ and every object in $\mathscr{A}$ has a monic $\mathscr{X}$-preenvelope. We say that an object N of $\mathscr{A}$ has a $\mathcal{G}(\mathscr{X}, \mathscr{Y})$-injective dimension less than or equal to n, denoted by $\mathcal{G}(\mathscr{X}, \mathscr{Y})\text{-id}(N) \leq n$, if there exists a $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{X})$-exact sequence

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0$$

in $\mathscr{A}$ with each G^i in $\mathcal{G}(\mathscr{X},\mathscr{Y})$. If no such finite sequence exists, define $\mathcal{G}(\mathscr{X},\mathscr{Y})\text{-id}(N)=\infty$; otherwise, if n is the least such integer, define $\mathcal{G}(\mathscr{X},\mathscr{Y})\text{-id}(N)=n$.

Proposition 3.9. *Assume that $\mathscr{X}=\mathscr{Z}$, $\mathscr{X}\perp\mathscr{Y}$ and every object in $\mathscr{A}$ has a monic $\mathscr{X}$-preenvelope. Let N be an object in $\mathscr{A}$ with $\mathcal{G}(\mathscr{X},\mathscr{Y})\text{-id}(N)=n$. Then there exist $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact sequences*

$$0\longrightarrow N\longrightarrow G\longrightarrow H\longrightarrow 0,$$
$$0\longrightarrow G'\longrightarrow H'\longrightarrow N\longrightarrow 0$$

with G in $\mathcal{G}(\mathscr{X},\mathscr{Y})$, $\mathscr{X}\text{-id}(H)\leq n-1$ and G' in $\mathcal{G}(\mathscr{X},\mathscr{Z})$, $\mathscr{X}\text{-id}(H')\leq n$.

Proposition 3.10. *Assume that $\mathscr{X}=\mathscr{Z}$, $\mathscr{X}\perp\mathscr{Y}$ and every object in $\mathscr{A}$ has a monic $\mathscr{X}$-preenvelope. Let N be an object in $\mathscr{A}$ with finite $\mathcal{G}(\mathscr{X},\mathscr{Y})$-injective dimension. Then the following are equivalent for a nonnegative integer n:*

(i) $\mathcal{G}(\mathscr{X},\mathscr{Y})\text{-id}(N)\leq n$;
(ii) *There is a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact sequence $0\to N\to X^0\to\cdots\to X^{n-1}\to G\to 0$ with each X^i in $\mathscr{X}$ and G in $\mathcal{G}(\mathscr{X},\mathscr{Y})$;*
(iii) *N has a coproper $\mathcal{G}(\mathscr{X},\mathscr{Y})$-coresolution of length n;*
(iv) *There is a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact sequence $0\to N\to G\to X^1\to\cdots\to X^n\to 0$ with each X^i in $\mathscr{X}$ and G in $\mathcal{G}(\mathscr{X},\mathscr{Y})$;*
(v) *There is a $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact sequence $0\to N\to X^0\to\cdots\to X^{i-1}\to G\to X^{i+1}\to\cdots\to X^n\to 0$ with each X^i in $\mathscr{X}$ and G in $\mathcal{G}(\mathscr{X},\mathscr{Z})$;*
(vi) *$\mathrm{Ext}^i_{\mathscr{A}}(Y,N)=0$ for all $i>n$ and all $Y\in\mathscr{Y}$;*
(vii) *$\mathrm{Ext}^i_{\mathscr{A}}(L,N)=0$ for all $i>n$ and all $L\in\mathrm{cores}\,\widehat{\mathscr{Y}}$;*
(viii) *$\mathrm{Ext}^{n+1}_{\mathscr{A}}(L,N)=0$ for all $L\in\mathrm{cores}\,\widehat{\mathscr{Y}}$;*

Furthermore, we have that

$$\begin{aligned}\mathcal{G}(\mathscr{X},\mathscr{Y})\text{-id}(N)&=\sup\{i\in\mathbb{N}\mid \mathrm{Ext}^i_{\mathscr{A}}(L,N)\neq 0 \text{ for some } L\in\mathrm{cores}\,\widehat{\mathscr{Y}}\}\\&=\sup\{i\in\mathbb{N}\mid \mathrm{Ext}^i_{\mathscr{A}}(Y,N)\neq 0 \text{ for some } Y\in\mathscr{Y}\}.\end{aligned}$$

Proposition 3.11. *Assume that $\mathscr{X}=\mathscr{Z}$, $\mathscr{X}\perp\mathscr{Y}$, and every object in $\mathscr{A}$ has a monic $\mathscr{X}$-preenvelope. Then every object in $\mathscr{A}$ with finite $\mathcal{G}(\mathscr{X},\mathscr{Y})$-injective dimension has a special $\mathcal{G}(\mathscr{X},\mathscr{Y})$-preenvelope.*

Corollary 3.12. ([**9,** Proposition 2.17]). *Assume that $\mathscr{Y}$ is a class of R-modules that contains all injective R-modules. Then every R-module with finite $\mathscr{Y}$-Gorenstein injective dimension has a special $\mathscr{Y}$-Gorenstein injective preenvelope.*

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