ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 45, Number 6, 2015

A DYNAMICAL SYSTEM FOR A NONLOCAL PARABOLIC EQUATION WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. We consider a nonlocal parabolic and the corresponding elliptic equation with exponential nonlinearity. At first, we study the set of a stationary solution and compute its Morse index. Next, we obtain the time-global solution in the use of the Lyapunov function and define the dynamical system. Finally, we construct an exponential attractor by a squeezing property.

1. Introduction. We consider the parabolic equation

$$(1.1)\qquad \begin{cases} u_t = \Delta u + \lambda\left(\frac{e^u}{\int_\Omega e^u dx} - \frac{1}{|\Omega|}\right), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

and its stationary elliptic equation

$$(1.2)\qquad \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v dx} - \frac{1}{|\Omega|}\right) = 0, \quad x \in \Omega,$$

where $\Omega = \mathbb{R}^2/(a\mathbb{Z} \times b\mathbb{Z})$ is a flat 2-torus of sizes $a, b > 0$ and $\lambda > 0$. Here $|\Omega|$ denotes the measure of Ω in $\mathbb{R}^2$.

Equation (1.2) is known as a mean field equation because it is derived from the various Onsager's vortex theories (**[5, 6, 19]**). Meanwhile, (1.2) is concerned with the Chern-Simons gauge theory. As the Chern-Simons coupling constant tends to zero, the asymptotic behavior of the abelian Chern-Simons vortex condensates leads to (1.2). For more details, we refer to **[7, 22, 28, 31, 32, 34]** and the references therein.

2010 AMS *Mathematics subject classification.* Primary 35K55, Secondary 35J60, 37L25.

Keywords and phrases. Nonlocal, Morse index, Lyapunov function, exponential attractor, squeezing property.

Received by the editors on April 23, 2013, and in revised form on February 5, 2014.

DOI:10.1216/RMJ-2015-45-6-1897

If $a = b$ and $\lambda \in (0, 8\pi)$, we have a solution for (1.2) by the Morse-Trudinger inequality ([**27**]). If $\lambda > 8\pi$, (1.2) presents a lack of compactness and becomes delicate. If $\lambda \in (8\pi, 4\pi^2)$, we have a non-trivial solution ([**32**]) by a mountain pass structure around 0.

Another tool for deriving the existence of a non-trivial solution is the Leray-Schauder degree by [**20**]. By [**4, 21**], all solutions of (1.2) are bounded for all $\lambda \in K$, where K is a compact subset in

$$\mathbb{R}_+ \setminus \bigcup_{m \in \mathbb{N}} \{8m\pi\}$$

and

$$\mathbb{R}_+ = \{x \mid x > 0\}.$$

Moreover, in [**20**], it is proven that the Leray-Schauder degree is constant whenever $\lambda \in (8m\pi, 8(m+1)\pi)$ with $m \in \mathbb{N}$. Hence, it is important to study the behavior of the solution with $\lambda \to 8m\pi$.

In the case of $\lambda \to 8\pi$ or $\lambda \to 16\pi$, the symmetricity of solutions is considered in [**7**].

In this paper, we consider the corresponding one-dimensional problem to (1.1) and (1.2). After scaling, they are expressed as

$$\text{(1.3)} \quad \begin{cases} u_t = \Delta u + \lambda \left(\frac{e^u}{\int_{-1}^{1} e^u \, dx} - \frac{1}{2} \right), & x \in (-1,1), \ t > 0, \\ u(-1,t) = u(1,t), u_x(-1,t) = u_x(1,t), & t > 0, \\ u(x,0) = u_0(x), & x \in (-1,1) \end{cases}$$

and

$$\text{(1.4)} \quad \begin{cases} \Delta v + \lambda \left(\frac{e^v}{\int_{-1}^{1} e^v \, dx} - \frac{1}{2} \right) = 0, & x \in (-1,1), \\ v(-1) = v(1), v_x(-1) = v_x(1), \end{cases}$$

where $\lambda > 0$. It is clear that solutions of (1.3) and (1.4) can be extended to a smooth 2-periodic solution defined on $\mathbb{R}$. We shall not distinguish between a solution on $(-1,1)$ and its periodic extension. If $u(x,t)$ and $v(x)$ are solutions of (1.3) and (1.4), respectively, then their spatial shifts $u(x+\xi, t)$ and $v(x+\xi)$ are also solutions for all $\xi \in \mathbb{R}$ and

$$\frac{d}{dt} \int_{-1}^{1} u(x,t) \, dx = 0.$$

We normalize solutions by requiring

$$\int_{-1}^{1} u(x,t)\,dx = 0, \qquad \int_{-1}^{1} v(x)\,dx = 0.$$

Two solutions are said to be geometrically distinct if and only if they do not differ by a spatial shift. We define the solution set by

$$\mathcal{C} = \Big\{(\lambda, v) \in \mathbb{R}_+ \times \big(C^2(\Omega) \cap C(\overline{\Omega})\big) \mid v = v(x) \text{ solves (1.4) for } \lambda > 0\Big\},$$

where $\Omega = (-1,1)$. We set $\lambda_k = 2(k\pi)^2$ for $k \in \mathbb{N} \cup \{0\}$. Then we have the following multiple existence theorem in [**30**]:

Theorem 1.1. *Nontrivial solutions are contained in $\mathcal{C}$ if and only if $\lambda > \lambda_1$. Moreover, if $\lambda > \lambda_k$ holds for some $k \in \mathbb{N}$, at least k geometrically distinct solutions are contained in $\mathcal{C}$.*

We can prove this theorem by a bifurcation theory in [**8**]. By a simple rescaling argument, it follows that (1.2) has solutions of the form $u(x_1, x_2) = u(x_1)$ if and only if

$$\lambda > \frac{2\lambda_1 b}{a}.$$

Thus, a relevant consequence of Theorem 1.1 is that solutions obtained in [**32**] must be truly two-dimensional in the sense that they cannot be reduced to functions depending on only one variable.

According to [**5, 6, 19**], the analogous Dirichlet boundary problem for (1.2)

$$\begin{cases} \Delta v + \lambda \frac{e^v}{\int_\Omega e^v\,dx} = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega \end{cases} \tag{1.5}$$

is concerned with statistical mechanics of point vortices in the mean field limit, where $\Omega \subset \mathbb{R}^2$. If $0 < \lambda < 8\pi$ and Ω is simply connected, it is shown in [**33**] that (1.5) has a unique solution.

We can also establish the unique existence of solution of (1.2) for small $\lambda > 0$ ([**32**]). The difference between (1.2) and (1.5) is that (1.5) has no solution for $\lambda \geq \lambda_2$ with some $\lambda_2 > 0$ if Ω is a ball.

Equation (1.5) is studied for higher dimension in [**26**]. If Ω is a ball, there exist $0 < \lambda_1 < \lambda_2$ such that (1.5) has a unique solution for $0 < \lambda < \lambda_1$ and that no solution for $\lambda > \lambda_2$. For general non-local term

$$\frac{e^v}{\left(\int_\Omega e^v dx\right)^p} \quad \text{for } p > 0,$$

see [**24**].

Equation (1.4) can be reduced to the boundary value problem

$$\begin{cases} \Delta v + \frac{\lambda}{2}\left(\frac{e^v}{\int_I e^v dx} - 1\right) = 0, & x \in I \equiv (0,1), \\ v_x(0) = v_x(1) = 0, \\ \int_I v(x)\, dx = 0. \end{cases} \tag{1.6}$$

Clearly, if $v(x)$ is a solution of (1.6), then its even 2-periodic extension on $\mathbb{R}$ defines a solution of (1.4). According to the lemma in [**30**], the converse is also true. Hence, we consider equation (1.6) and the following parabolic problem instead of (1.4) and (1.3), respectively:

$$\begin{cases} u_t = \Delta u + \frac{\lambda}{2}\left(\frac{e^u}{\int_I e^u dx} - 1\right), & x \in I,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in I, \\ \int_I u(x,t)\, dx = 0, & t > 0. \end{cases} \tag{1.7}$$

To simplify the notation, we shall identify a solution of (1.6) with its even 2-periodic extension. We denote again by $\mathcal{C}$ the solution set for (1.6). Now we define the Morse index at $(\lambda, v) \in \mathcal{C}$, denoted by $i = i(\lambda, v)$, by the number of negative eigenvalues μ of

$$\begin{cases} \Delta\phi + \frac{\lambda}{2}\frac{e^v}{\int_I e^v dx}\phi - \frac{\lambda}{2}\frac{\int_I e^v \phi\, dx}{\left(\int_I e^v dx\right)^2} e^v = -\mu\phi, & x \in I, \\ \phi_x(0) = \phi_x(1) = 0, \\ \int_I \phi(x)\, dx = 0. \end{cases} \tag{1.8}$$

The first theorem in this paper is concerned with the bifurcation diagram of solution set $\mathcal{C}$ and its spectral property of the linearized operator. The advantage of the Neumann boundary condition in (1.6) is that the trivial solution $(\lambda, v) = (\lambda_n, 0)$ has a simple eigenvalue $\mu = 0$ in (1.8).

Theorem 1.2. *Equation* (1.6) *has a trivial solution* $(\lambda, v) = (\lambda, 0)$ *for any* $\lambda \in \mathbb{R}_+$. *From* $(\lambda, v) = (\lambda_n, 0)$ *with* $n \in \mathbb{N}$, *two continua* $\mathcal{S}_n^{\pm}$ *of a solution of* (1.6) *bifurcate. For* $(\lambda, v) = (\lambda, 0)$ *with* $\lambda_{n-1} < \lambda < \lambda_n$, *it holds that* $i(\lambda, 0) = n - 1$. *For* $(\lambda, v) \in \mathcal{S}_n^{\pm}$ *sufficiently close to* $(\lambda_n, 0)$, *it holds that* $i(\lambda, v) = n - 1$.

Now the Lyapunov function

$$J_\lambda(u) = \frac{1}{2} \int_I |\nabla u|^2 \, dx - \frac{\lambda}{2} \log \int_I e^u dx + \frac{\lambda}{2} \int_I u \, dx$$

plays an important role in obtaining a time-global solution. For the definitions of notion about a Lyapunov function and other dynamical properties, see **[9, 15, 16, 35]**. Let

$$X = \left\{ u \in H^1(I) \mid \int_I u \, dx = 0 \right\}$$

and

$$H_N^2(I) = \{u \in H^2(I) \mid u_x(0) = u_x(1) = 0\},$$

respectively.

The next two theorems are concerned with the global existence and regularity of the solution.

Theorem 1.3. *For* $u_0 \in X$, (1.7) *admits a unique global solution* $u = u(x, t)$ *such that*

$$u \in C\left([0, +\infty); X\right), \qquad u_t \in L^2\left((0, +\infty); L^2(I)\right).$$

For any $T > 0$, *we have*

$$u \in L^2\left((0, T); H_N^2(I)\right).$$

The associated nonlinear semigroup $T(t)$

$$T(t)u_0(\cdot) = u(\cdot, t)$$

defines a dynamical system on X.

Theorem 1.4. *For any* $\eta > 0$, *the orbit* $t \in [\eta, +\infty) \mapsto u(\cdot, t)$ *is compact in* X.

We obtain the dynamical properties of $T(t)$ by the existence of the Lyapunov function and regularity of the solution.

Theorem 1.5. *For $u_0 \in X$, $\omega(u_0)$ is nonempty, compact, invariant and connected in X. And it holds that $\omega(u_0) \subset \mathcal{C}$.*

For fixed $k > 0$, we set

$$X_k = \{u \in X \mid J_\lambda(u) \leq k\}.$$

Then, for $u_0 \in X_k$, we have $u(\cdot, t) \in X_k$ for all $t \geq 0$. Hence, a dynamical system $\{T(t) : X \to X\}$ is reduced to the subdynamical system $\{T(t) : X_k \to X_k\}$. By getting the estimates independent of initial value, we have the following:

Theorem 1.6. *The dynamical system $T(t)$ on X_k has a global attractor $\mathcal{A}$.*

The following theorem about an exponential attractor is the main in this paper:

Theorem 1.7. *There exists a compact absorbing and positively invariant set $\mathcal{X} \subset X_k$ such that a dynamical system $T(t)$ on $\mathcal{X}$ admits an exponential attractor $\mathcal{E}$ in $H^1(I)$.*

This paper is composed of six sections and one appendix. In Section 2, we show Theorem 1.2 by a bifurcation theory. The first statement of the theorem has been proved in **[30]**. However, to introduce the notation, we sketch the proof. Next we compute the Morse index.

In Section 3, we obtain a time-local solution by a contraction mapping theorem, extend it globally and establish a time-global solution in Theorem 1.3. The boundedness of solution is given by the existence of the Lyapunov function.

In Section 4, using energy inequalities, we consider the regularity of solution. In Section 5, the statements of Theorems 1.4, 1.5 and 1.6 follow from Sections 3 and 4 through standard results in **[15, 16]**. In Section 6, we show that the dynamical system has a squeezing property (**[29, 36]**), which proves Theorem 1.6.

In the appendix, we discuss the difficulty of an equation with non-local term. We introduce the known results on the equation without a non-local term.

2. Stationary solution. At $(\lambda, v) = (\lambda_n, 0)$, (1.8) has a simple eigenvalue $\mu = 0$. We apply a bifurcation theory in [**8**]. And we obtain a curve of solution (λ, v) and parametrize $(\lambda, v) = (\lambda(s), v(\cdot, s)), \mu = \mu(s)$ and $\phi = \phi(\cdot, s)$. To compute the Morse index, we consider the signs of $(d/ds)\mu$ and $(d^2/ds^2)\mu$ at bifurcation points ([**24, 26**]).

Proof of Theorem 1.2. For $(\lambda, v) = (\lambda, 0)$ with $\lambda_{n-1} < \lambda < \lambda_n$, the kth eigenvalue and eigenfunction of (1.8) are given as

$$\mu^k = (k\pi)^2 - \frac{\lambda}{2} = \frac{1}{2}(\lambda_k - \lambda), \qquad \phi^k(x) = \cos k\pi x$$

for $k \in \mathbb{N}$. Note that a constant eigenfunction is not contained due to the integral condition. Thus, it holds that (1.8) has a simple eigenvalue $\mu = 0$ at $(\lambda, v) = (\lambda_n, 0)$ and that $i(\lambda, 0) = n-1$ for $\lambda_{n-1} < \lambda < \lambda_n$. We will show that the nontrivial solutions bifurcate from $(\lambda, v) = (\lambda_n, 0)$. We define

$$\mathcal{U} = \left\{ v \in C^2(\overline{I}) \mid v'(0) = v'(1) = 0 \quad \text{and} \quad \int_I v(x)\,dx = 0 \right\},$$
$$\mathcal{V} = \left\{ v \in C(\overline{I}) \mid \int_I v(x)\,dx = 0 \right\}$$

and a mapping $F : \mathbb{R} \times \mathcal{U} \to \mathcal{V}$ by

$$F(\lambda, v) = \Delta v + \frac{\lambda + \lambda_n}{2}\left(\frac{e^v}{\int_I e^v dx} - 1\right)$$

for $n \in \mathbb{N}$. Then $F(\lambda, 0) = 0$ and the Fréchet derivative is given as

$$F_v(\lambda, v)[w] = \Delta w + \frac{\lambda + \lambda_n}{2}\frac{e^v}{\int_I e^v dx} w - \frac{\lambda + \lambda_n}{2}\frac{\int_I e^v w dx}{\left(\int_I e^v dx\right)^2} e^v$$

for $w \in \mathcal{U}$. Since

$$F_v(0,0)[w] = \Delta w + (n\pi)^2 w,$$

the kernel of $F_v(0,0)$ is spanned only by $\cos n\pi x$. Hence, applying [**8,** Theorem 1.7] to this setting, we obtain two continua $\mathcal{S}_n^{\pm}$ of solutions

(λ, v) of (1.6) bifurcating from $(\lambda, v) = (\lambda_n, 0)$ satisfying

$$\mathcal{S}_n^+ = \left\{ (\lambda(s), v(\cdot, s)) \mid \lim_{s \to +0} (\lambda(s), v(\cdot, s)) = (\lambda_n, 0) \text{ and } s \in (0, \alpha) \right\}$$

and

$$\mathcal{S}_n^- = \left\{ (\lambda(s), v(\cdot, s)) \mid \lim_{s \to -0} (\lambda(s), v(\cdot, s)) = (\lambda_n, 0) \text{ and } s \in (-\alpha, 0) \right\}$$

in $\mathbb{R} \times \mathcal{U}$ with some $\alpha > 0$. Moreover, the mapping

$$s \in (-\alpha, \alpha) \longmapsto (\lambda(s), v(\cdot, s)) \in \mathbb{R}_+ \times \mathcal{U}$$

belongs to $C^2(-\alpha, \alpha)$ and $v(\cdot, s)$ is expressed as

$$v(\cdot, s) = s \cos n\pi x + s\psi(\cdot, s)$$

for a function $\psi(\cdot, s) : (-\alpha, \alpha) \to \mathcal{W}$ with C^2 dependence in s and $\psi(\cdot, 0) = 0$, where $\mathcal{W}$ is a complement of the kernel of $F_v(0, 0)$.

We set $\mathcal{C}_n = \mathcal{S}_n^- \cup \{(\lambda_n, 0)\} \cup \mathcal{S}_n^+$. We denote the kth eigenvalue $\mu^k(\lambda, v(\cdot))$ and corresponding eigenfunction $\phi^k(\cdot, \lambda, v(\cdot))$ of (1.8) at $(\lambda(s), v(\cdot, s)) \in \mathcal{C}_n$ by

$$\mu_n^k (\lambda(s), v(\cdot, s))$$

and

$$\phi_n^k (\cdot, \lambda(s), v(\cdot, s)),$$

respectively. It follows from a perturbation theory in **[18]** that

$$\mu_n^k (\lambda(s), v(\cdot, s)) = \mu_n^k (s)$$

and

$$\phi_n^k (\cdot, \lambda(s), v(\cdot, s)) = \phi_n^k (\cdot, s)$$

are with C^2 dependence in s. For the simplicity, we write

$$\lambda(s) = \lambda, \quad v(\cdot, s) = v, \quad \mu_n^n(s) = \mu, \quad \mu_n^k(s) = \mu^k,$$
$$\phi_n^n (\cdot, s) = \phi, \qquad \phi_n^k (\cdot, s) = \phi^k,$$
$$\lambda(0) = 2 (n\pi)^2, \qquad v(\cdot, 0) = 0,$$
$$\mu_n^n(0) = \mu_0 = 0, \quad \mu_n^k(0) = \mu_0^k = \left(k^2 - n^2\right) \pi^2$$

and

$$\phi_n^n(\cdot,0) = \phi(0) = \phi_0 = \cos n\pi x,$$
$$\phi_n^k(\cdot,0) = \phi^k(0) = \phi_0^k = \cos k\pi x$$

for $(\lambda(s), v(\cdot,s)) \in \mathcal{C}_n$. Under this notation, we have

$$\dot{v}(\cdot,0) = \phi(0) = \cos n\pi x,$$

where $\dot{v}$ stands for $(d/ds)v(\cdot,s)$ and $\dot{v}(\cdot,0) = (d/ds)v(\cdot,s)|_{s=0}$.

Lemma 2.1. *We have the following*:

$$\int_I \left(\phi_0^k\right)^2 dx = \frac{1}{2}, \qquad \int_I \left(\phi_0^k\right)^3 dx = 0$$

and

$$\int_I \left(\phi_0^k\right)^4 dx = \frac{3}{8}$$

for $k \in \mathbb{N}$.

Proof of Lemma 2.1. Since $\phi_0^k = \cos k\pi x$, it is obvious. □

Differentiating (1.6) with respect to s, we have

$$(2.1)\qquad \begin{cases} \Delta\dot{v} + \frac{\dot{\lambda}}{2}\left(\frac{e^v}{\int_I e^v\,dx} - 1\right) + \frac{\lambda}{2}\frac{e^v}{\int_I e^v\,dx}\dot{v} & \\ \qquad -\frac{\lambda}{2}\frac{\int_I e^v\dot{v}\,dx}{\left(\int_I e^v\,dx\right)^2}e^v = 0, & x \in I, \\ \dot{v}_x(0) = \dot{v}_x(1) = 0, & \\ \int_I \dot{v}(x)dx = 0. & \end{cases}$$

Differentiating (2.1) with respect to s, putting $s = 0$, multiplying by ϕ_0 and integrating it over I, we have

$$\dot{\lambda}(0)\int_I (\phi_0)^2\,dx + \frac{\lambda(0)}{2}\int_I (\phi_0)^3\,dx = 0,$$

and hence

$$\dot{\lambda}(0) = 0$$

by Lemma 2.1.

Next, differentiating (1.8) with respect to s, putting $s = 0$ and $k = n$, multiplying by ϕ_0 and integrating it over I, we have

$$\frac{\dot{\lambda}(0)}{2}\int_I (\phi_0)^2\,dx + \frac{\lambda(0)}{2}\int_I (\phi_0)^3\,dx = -\dot{\mu}(0)\int_I (\phi_0)^2\,dx,$$

and hence

$$\dot{\mu}(0) = 0.$$

Lemma 2.2. *We have the following*:

$$\int_I \dot{\phi}(0)\,(\phi_0)^2\,dx = \frac{1}{24}.$$

Proof of Lemma 2.2. Noting that

$$(\phi_0)^2 = \cos^2 n\pi x = \frac{1}{2}(\cos 2n\pi x + 1) = \frac{1}{2}\left(\phi_0^{2n} + 1\right), \tag{2.2}$$

we have

$$\int_I \dot{\phi}(0)\,(\phi_0)^2\,dx = \frac{1}{2}\int_I \left(\phi_0^{2n} + 1\right)\dot{\phi}(0)\,dx = \frac{1}{2}\int_I \phi_0^{2n}\dot{\phi}(0)\,dx.$$

Differentiating (1.8) with respect to s, putting $s = 0$ and $k = n$, multiplying by ϕ_0^{2n} and integrating it over I, we have

$$\int_I \dot{\phi}(0)\left(\Delta\phi_0^{2n} + \frac{\lambda(0)}{2}\phi_0^{2n}\right)dx + \frac{\lambda(0)}{2}\int_I \phi_0^{2n}\,(\phi_0)^2\,dx = 0$$

and, moreover by (2.2) and $\Delta\phi_0^{2n} = -2\lambda(0)\cos 2n\pi x = -2\lambda(0)\phi_0^{2n}$,

$$-\frac{3}{2}\lambda(0)\int_I \phi_0^{2n}\dot{\phi}(0)dx + \frac{1}{4}\lambda(0)\int_I \left(\phi_0^{2n}\right)^2 dx = 0.$$

Hence, it holds that

$$\int_I \dot{\phi}(0)\,(\phi_0)^2\,dx = \frac{1}{12}\int_I \left(\phi_0^{2n}\right)^2 dx = \frac{1}{24}. \qquad \square$$

Differentiating (1.8) and (2.1) twice with respect to s, putting $s = 0$ and $k = n$, multiplying by ϕ_0, integrating it over I and eliminating $\int_I (\phi_0)^2 \ddot{v}dx$, we have

$$\int_I (\phi_0)^4\,dx + 3\int_I (\phi_0)^2\,\dot{\phi}(0)\,dx - 3\left(\int_I (\phi_0)^2\,dx\right)^2 = \frac{-3\ddot{\mu}(0)}{\lambda(0)}\int_I (\phi_0)^2\,dx,$$

and hence,

$$\ddot{\mu}(0) = \frac{(n\pi)^2}{3} > 0.$$

It holds that

$$\mu(0) = \dot{\mu}(0) = 0 \quad \text{and} \quad \ddot{\mu}(0) > 0,$$

which implies that $\mu(s)$ is nonnegative sufficiently close to $s = 0$. $\square$

3. Global solution. First we establish a time-local solution of (1.7) in the same way as **[25]** by a contraction mapping theorem. Hence we omit it. We prove the uniform boundedness of local solutions by the Lyapunov function.

Proof of Theorem 1.3. We have the Poincaré-Wirtinger inequality (**[3]**)

$$\|w - \overline{w}\|_2 < C_1 \|\nabla w\|_2 \tag{3.1}$$

for $w \in H^1(I)$, where $\|\cdot\|_2$ denotes the standard L^2 norm,

$$\overline{w} = \int_I w\, dx$$

and C_1 is a positive constant depending only on I.

Now $w \in X$ implies that $\overline{w} = 0$. Thus, we define the norm in X as $\|w\|_X = \|\nabla w\|_2$ for $w \in X$. We can prove the local existence of a solution $u \in C([0, \delta]; X)$ of (1.7) for sufficiently small $\delta > 0$.

Next we show that we can extend it globally in time. We have the Lyapunov function

$$J_\lambda(u) = \frac{1}{2}\int_I |\nabla u|^2\, dx - \frac{\lambda}{2}\log \int_I e^u dx + \frac{\lambda}{2}\int_I u\, dx.$$

In fact, we have

$$J_\lambda(u(t_2)) - J_\lambda(u(t_1)) = \int_{t_1}^{t_2} \frac{d}{dt} J_\lambda(u)\, ds = -\int_{t_1}^{t_2} \|u_t\|_2^2\, ds \le 0$$

for $0 \le t_1 < t_2$. In particular, it holds that

$$J_\lambda(u(t)) \le J_\lambda(u_0)$$

for all $t > 0$. Since $H^1(I) \subset C(\overline{I})$ and (3.1),

$$\|w\|_\infty \leq C_2 \|w\|_{H^1} \leq C_3 \|w\|_X$$

for $w \in X$, where $C_2 > 0$ and $C_3 = C_2\sqrt{1 + C_1^2}$ depend only on an interval I and $\|w\|_{H^1} = \sqrt{\|w\|_2^2 + \|\nabla w\|_2^2}$ for $w \in H^1(I)$. Eventually, we obtain

$$\begin{aligned}
\|u\|_X^2 &= 2J_\lambda(u) + \lambda \log \int_I e^u dx \\
&\leq 2J_\lambda(u_0) + \lambda \|u\|_\infty \\
&\leq \|u_0\|_X^2 + \lambda \|u_0\|_\infty + \lambda \|u\|_\infty \\
&\leq \|u_0\|_X^2 + C_3\lambda \|u_0\|_X + C_3\lambda \|u\|_X ,
\end{aligned}$$

and hence

$$\|u\|_X \leq C_3\lambda + \|u_0\|_X \equiv C_4 < +\infty,$$

where $C_4 > 0$ depends only on λ, C_3 and $\|u_0\|_X$. Moreover, we have

$$\begin{aligned}
\int_0^t \|u_t\|_2^2 \, ds &= -J_\lambda(u) + J_\lambda(u_0) \\
&\leq \frac{1}{2} \|u_0\|_X^2 + \frac{\lambda C_3}{2} \left(\|u_0\|_X + \|u\|_X\right) \leq C_5
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \|\Delta u\|_2^2 \, ds &= \int_0^t \left\| u_t - \frac{\lambda}{2}\left(\frac{e^u}{\int_I e^u dx} - 1\right)\right\|_2^2 ds \\
&\leq 2\int_0^t \|u_t\|_2^2 \, ds + \frac{\lambda^2}{2}\left\|\frac{e^u}{\int_I e^u dx} - 1\right\|_\infty^2 t \\
&\leq 2C_5 + \frac{\lambda^2}{2}\left(e^{2C_3\|u\|_X} + 1\right)^2 t,
\end{aligned}$$

where $C_5 > 0$ depends only on λ, C_3 and $\|u_0\|_X$. If a local solution exists, the unique solution $u = u(x,t)$ exists for all time $t > 0$ satisfying $u \in C([0,\infty); X)$, $u_t \in L^2((0,\infty); L^2(I))$ and $u \in L^2((0,T); H_N^2(I))$ for any $T > 0$.

Then we denote by $T(t)$ the mapping which gives a solution $u = u(\cdot,t) \in X$ for given $u_0 \in X$. To prove that $T(t)$ defines a dynamical system on X, we have only to show that it depends continuously on an initial function u_0. In fact, due to boundedness of the solution of (1.7), we can utilize estimates similar to those used in establishing a local solution (for details, see the proof of Theorem 3.4.1 in **[16]**). Hence, Theorem 1.3 follows. □

4. Regularity. From the Jensen inequality (**[36]**), we note that

$$\int_I e^u dx \geq e^{\int_I u\,dx} = 1$$

for all $u \in X$.

Proposition 4.1. *For any $\eta_2 > 0$ and $u_0 \in X$,*

$$u(\cdot, t) \in H_N^2(I)$$

for $t \geq \eta_2$.

Proof of Proposition 4.1. It holds that

$$\begin{aligned}
&\eta_2 \|u_t(\cdot,\eta_2)\|_2^2 \\
&= \int_0^{\eta_2} s \frac{d}{dt} \|u_t\|_2^2 \, ds + \int_0^{\eta_2} \|u_t\|_2^2 \, ds \\
&\leq 2 \int_0^{\eta_2} s \int_I u_t \left\{ \Delta u + \frac{\lambda}{2} \left(\frac{e^u}{\int_I e^u dx} - 1 \right) \right\}_t dx\, ds + C_5 \\
&= 2 \int_0^{\eta_2} s \int_I \left(-|\nabla u_t|^2 + \frac{\lambda}{2} \frac{e^u u_t^2}{\int_I e^u dx} - \frac{\lambda}{2} \frac{\int_I e^u u_t dx}{\left(\int_I e^u dx\right)^2} e^u u_t \right) dx\, ds + C_5 \\
&\leq \lambda \eta_2 \sup_{0 \leq t \leq \eta_2} e^{\|u(\cdot,t)\|_\infty} \int_0^{\eta_2} \|u_t\|_2^2 \, ds + C_5 \\
&\leq \lambda \eta_2 e^{C_3 C_4} C_5 + C_5,
\end{aligned}$$

which implies $u_t(\cdot, \eta_2) \in L^2(I)$. For $t \geq \eta_2$, we have similarly

$$\|u_t\|_2^2 = \int_{\eta_2}^t \frac{d}{dt} \|u_t\|_2^2 \, ds + \|u_t(\cdot,\eta_2)\|_2^2 \leq \lambda e^{C_3 C_4} C_5 + \|u_t(\cdot,\eta_2)\|_2^2,$$

which implies $u_t(\cdot, t) \in L^2(I)$ and $u(\cdot, t) \in H_N^2(I)$ for $t \geq \eta_2$. □

Proposition 4.2. *For any $\eta_3 > \eta_2$ and $u_0 \in X$,*

$$u(\cdot, t) \in H^3(I)$$

for $t \geq \eta_3$.

Proof of Proposition 4.2. It holds that

$$\begin{aligned}
\int_{\eta_2}^{\eta_3} \|\nabla u_t\|_2^2 \, ds &= -\int_{\eta_2}^{\eta_3} \int_I u_t \Delta u_t dx \, ds \\
&= -\int_{\eta_2}^{\eta_3} \int_I u_t \left\{ u_t - \frac{\lambda}{2} \left(\frac{e^u}{\int_I e^u dx} - 1 \right) \right\}_t dx \, ds \\
&= \int_{\eta_2}^{\eta_3} \int_I \left(-u_t u_{tt} + \frac{\lambda}{2} \frac{e^u u_t^2}{\int_I e^u dx} \right. \\
&\qquad \left. - \frac{\lambda}{2} \frac{\int_I e^u u_t dx}{\left(\int_I e^u dx\right)^2} e^u u_t \right) dx \, ds \\
&\leq \frac{1}{2} \|u_t(\cdot, \eta_2)\|_2^2 + \frac{\lambda}{2} e^{C_3 C_4} C_5 \leq C_6,
\end{aligned}$$

where $C_6 > 0$ depends only on λ, η_2, C_3 and $\|u_0\|_X$ by Proposition 4.1, and that

$$\begin{aligned}
&(\eta_3 - \eta_2) \|\nabla u_t(\cdot, \eta_3)\|_2^2 \\
&= \int_{\eta_2}^{\eta_3} (s - \eta_2) \frac{d}{dt} \|\nabla u_t\|_2^2 \, ds + \int_{\eta_2}^{\eta_3} \|\nabla u_t\|_2^2 \, ds \\
&\leq 2 \int_{\eta_2}^{\eta_3} (s - \eta_2) \int_I \nabla u_t \\
&\quad \cdot \nabla \left\{ \Delta u + \frac{\lambda}{2} \left(\frac{e^u}{\int_I e^u dx} - 1 \right) \right\}_t dx \, ds + C_6 \\
&\leq \lambda \int_{\eta_2}^{\eta_3} (s - \eta_2) \int_I \frac{e^u}{\int_I e^u dx} (u_t \nabla u + \nabla u_t) \cdot \nabla u_t \, dx \, ds \\
&\quad - \lambda \int_{\eta_2}^{\eta_3} (s - \eta_2) \int_I \frac{\int_I e^u u_t dx}{\left(\int_I e^u dx\right)^2} e^u \nabla u \cdot \nabla u_t \, dx \, ds + C_6,
\end{aligned}$$

which implies $\nabla u_t(\cdot, \eta_3) \in L^2(I)$ due to

$$\sup_{\eta_2 \leq t \leq \eta_3} \|\nabla u(\cdot, t)\|_\infty \leq C_2 \sup_{\eta_2 \leq t \leq \eta_3} \|u(\cdot, t)\|_{H^2} < +\infty.$$

For $t \geq \eta_3$, we have

$$\|\nabla u_t\|_2^2 = \int_{\eta_3}^{t} \frac{d}{dt} \|\nabla u_t\|_2^2 \, ds + \|\nabla u_t(\cdot, \eta_3)\|_2^2,$$

which implies $\nabla u_t(\cdot,t) \in L^2(I)$ and $u(\cdot,t) \in H^3(I)$ for $t \geq \eta_3$. $\square$

5. Dynamical properties. All theorems concerned with dynamical properties follow from the results in Sections 3 and 4 along with standard theorems in [**16, 35**]. The existence of the Lyapunov function is crucial for the proof.

Proof of Theorem 1.4. The inclusion $H^2(I) \subset H^1(I)$ is compact by [**16,** Theorem 1.4.8]. According to Proposition 4.1, $\cup_{t\geq\eta} T(t)u_0$ is bounded in $H^2(I)$ for any $\eta > 0$. $\square$

Proof of Theorem 1.5. The first statement of theorem follows from [**16,** Theorem 4.3.3]. Next, the existence of the Lyapunov function implies that $\omega(u_0) \subset \mathcal{C}$ by [**16,** Theorem 4.3.4]. $\square$

Proof of Theorem 1.6. As we derived in the proof of Theorem 1.3, we have

$$\|u\|_X^2 \leq 2J_\lambda(u_0) + \lambda \|u\|_\infty \leq 2k + \lambda C_3 \|u\|_X\,,$$

and eventually

$$\|u\|_X \leq \frac{C_3\lambda + \sqrt{(C_3\lambda)^2 + 8k}}{2},$$

which is independent of the initial value. Hence, we have an absorbing set in X_k. We apply [**35,** Theorem 1.1, Chapter I] to guarantee the existence of global attractor $\mathcal{A} \subset X_k$. $\square$

6. Exponential attractor. We prepare an orthogonal projection and related notions for the squeezing property. Since $-\Delta + 1$ on $H_N^2(I) \subset L^2(I)$ is a positive definite operator, we can write $-\Delta + 1$ as A^2, where A is a self-adjoint positive operator. It can be given explicitly by

$$Au = \sum_{n=1}^{\infty} \eta_n^{1/2}\,(u, e_n)\, e_n$$

for $u \in \mathcal{D}(A) = H^1(I)$ with $e_1 = 1, e_n = \sqrt{2}\cos(n-1)\pi x$ for $n \geq 2$ being orthonormal eigenfunctions of $-\Delta+1$ associated with eigenvalues $\eta_n = (n-1)^2\pi^2 + 1$ and (u, e_n) being the inner product in $L^2(I)$. We can take

$$\|u\|_{H^1}^2 = \|Au\|_2^2 = \sum_{n=1}^{\infty} \left| \eta_n^{1/2} (u, e_n) \right|^2.$$

We denote by H^N the vector space spanned by $e_1, e_2, \ldots, e_N$. Let

$$P_N : L^2(I) \longrightarrow H^N$$

be the orthogonal projection onto H^N and

$$Q_N = I_d - P_N,$$

where I_d is an identity mapping in $L^2(I)$. We note that

$$\begin{aligned} \|u\|_2^2 &= \sum_{n=p+1}^{\infty} |(u, e_n)|^2 \leq \frac{1}{\eta_{p+1}} \sum_{n=p+1}^{\infty} \left| \eta_n^{1/2} (u, e_n) \right|^2 \\ &= \frac{1}{\eta_{p+1}} \|u\|_{H^1}^2 \end{aligned}$$

for $u \in Q_p(L^2(I))$ with $p \in \mathbb{N}$. We have an absorbing set in X_k for $T(t)$ denoted by $\mathcal{B}$. We take $t_{\mathcal{B}}$ which satisfies $T(t)\mathcal{B} \subset \mathcal{B}$ for all $t \geq t_{\mathcal{B}}$. Let

$$\mathcal{X} = \overline{\cup_{t \geq t_{\mathcal{B}}} T(t)\mathcal{B}}.$$

Then, since $\mathcal{X}$ is a compact, invariant and absorbing set in X_k, we can consider the subdynamical system $T(t) : \mathcal{X} \to \mathcal{X}$. To construct an exponential attractor, we apply [**9,** Theorem 3.1].

Proof of Theorem 1.7. Let u_1 and u_2 be two solutions of (1.7) with initial values $u_{1,0}(x)$ and $u_{2,0}(x)$, respectively. Setting $w = u_1 - u_2$, we have

$$\begin{cases} w_t + A^2 w = w + \frac{\lambda}{2} \left(\frac{e^{u_1}}{\int_I e^{u_1}\, dx} - \frac{e^{u_2}}{\int_I e^{u_2}\, dx} \right), & x \in I,\ t > 0, \\ w_x(0,t) = w_x(1,t) = 0, & t > 0, \\ w(x,0) = u_{1,0}(x) - u_{2,0}(x), & x \in I, \\ \int_I w(x,t)\, dx = 0, & t > 0. \end{cases}$$

Let $z = Q_N w$. Then z satisfies

$$\begin{cases} z_t + A^2 z = z + \frac{\lambda}{2} Q_N \left(\frac{e^{u_1}}{\int_I e^{u_1}\, dx} - \frac{e^{u_2}}{\int_I e^{u_2}\, dx} \right), & x \in I,\ t > 0, \\ z_x(0,t) = z_x(1,t) = 0, & t > 0, \\ z(x,0) = Q_N \left(u_{1,0}(x) - u_{2,0}(x) \right), & x \in I. \end{cases}$$

Then we have

$$\begin{aligned}
&\frac{d}{dt}\|z\|_{H^1}^2 + 2\left\|A^2 z\right\|_2^2 \\
&= 2\left(A^2 z, z + \frac{\lambda}{2} Q_N\left(\frac{e^{u_1}}{\int_I e^{u_1}\,dx} - \frac{e^{u_2}}{\int_I e^{u_2}\,dx}\right)\right) \\
&\leq \left\|A^2 z\right\|_2^2 + 2\|z\|_2^2 + \frac{\lambda^2}{2}\left\|Q_N\left(\frac{e^{u_1}}{\int_I e^{u_1}\,dx} - \frac{e^{u_2}}{\int_I e^{u_2}\,dx}\right)\right\|_2^2 \\
&\leq \left\|A^2 z\right\|_2^2 + \frac{2}{\eta_{N+1}}\|z\|_{H^1}^2 + \frac{\lambda^2}{2\eta_{N+1}}\left\|\frac{e^{u_1}}{\int_I e^{u_1}\,dx} - \frac{e^{u_2}}{\int_I e^{u_2}\,dx}\right\|_{H^1}^2.
\end{aligned}$$

By the expression of the nonlinear term and $u_1, u_2 \in \mathcal{X} \subset \mathcal{B} \subset X$, we have

$$\left\|\frac{e^{u_1}}{\int_I e^{u_1}\,dx} - \frac{e^{u_2}}{\int_I e^{u_2}\,dx}\right\|_{H^1}^2 \leq C_7\|w\|_{H^1}^2,$$

where $C_7 > 0$ is the constant depending only on λ, I, k and $\mathcal{B}$. Hence, we have

$$\frac{d}{dt}\|z\|_{H^1}^2 + \eta_{N+1}\|z\|_{H^1}^2 \leq \frac{d}{dt}\|z\|_{H^1}^2 + \left\|A^2 z\right\|_2^2 \leq \frac{1}{\eta_{N+1}}\left(2 + \frac{C_7\lambda^2}{2}\right)\|w\|_{H^1}^2$$

and solve this differential inequality to obtain

$$\begin{aligned}
\|z\|_{H^1}^2 &\leq e^{-\eta_{N+1}t}\|z_0\|_{H^1}^2 + \frac{1}{\eta_{N+1}}\left(2 + \frac{C_7\lambda^2}{2}\right)\int_0^t \|w(s)\|_{H^1}^2\,ds \\
&\leq e^{-\eta_1 t}\|w_0\|_{H^1}^2 + \frac{1}{\eta_{N+1}}\left(2 + \frac{C_7\lambda^2}{2}\right)\int_0^t \|w(s)\|_{H^1}^2\,ds. \qquad (6.1)
\end{aligned}$$

In the same manner, we deduce

$$\frac{d}{dt}\|w\|_{H^1}^2 + 2\left\|A^2 w\right\|_2^2 \leq \left\|A^2 w\right\|_2^2 + \frac{1}{\eta_1}\left(2 + \frac{C_7\lambda^2}{2}\right)\|w\|_{H^1}^2,$$

and hence,

$$\|w\|_{H^1}^2 \leq e^{1/\eta_1(2+(C_7\lambda^2/2))t}\|w_0\|_{H^1}^2. \qquad (6.2)$$

Substituting (6.2) into (6.1), we have

$$\|z\|_{H^1}^2 \leq \left(e^{-\eta_1 t} + \frac{\eta_1}{\eta_{N+1}} e^{1/\eta_1(2+(C_7\lambda^2/2))t}\right)\|w_0\|_{H^1}^2.$$

Now we choose $t_* > 0$ and $N_0 \in \mathbb{N}$ such that

$$t_* = \frac{9 \log 2}{\eta_1} \quad \text{and} \quad \eta_{N+1} \geq 2^9 \eta_1 e^{1/\eta_1 (2+(C_7\lambda^2/2))t*},$$

if $N \geq N_0$. To show the squeezing property, we only have to prove that

$$\|w\|_{H^1} \leq \frac{1}{8} \|w_0\|_{H^1} \quad \text{if } \|P_N w\|_{H^1} \leq \|Q_N w\|_{H^1} = \|z\|_{H^1}$$

for $t = t_*$. Indeed, for $t = t_*$,

$$\|w\|_{H^1} \leq 2 \|z\|_{H^1} \leq 2\sqrt{\frac{1}{2^9} + \frac{1}{2^9}} \|w_0\|_{H^1} \leq \frac{1}{8} \|w_0\|_{H^1} ,$$

which means that the dynamical system $T(t)$ has the squeezing property.

Next we prove the Lipschitz continuity.

$$\begin{aligned} \|u_1(s) - u_2(t)\|_{H^1} &\leq \|w(s)\|_{H^1} + \|u_2(s) - u_2(t)\|_{H^1} \\ &\leq e^{1/(2\eta_1)(2+(C_7\lambda^2/2))t} \|w_0\|_{H^1} + \left\| \int_t^s \frac{du_2}{dt}(p)\, dp \right\|_{H^1} \\ &\leq e^{1/(2\eta_1)(2+(C_7\lambda^2/2))t} \|w_0\|_{H^1} + \int_t^s \left\| \frac{du_2}{dt}(p) \right\|_{H^1} dp \end{aligned}$$

since $u_2(0) \in \mathcal{X}$, du_2/dt belongs to $H^1(I)$ by Proposition 4.2. We have

$$\begin{aligned} \|u_1(s) - u_2(t)\|_{H^1} \leq e^{1/(2\eta_1)(2+(C_7\lambda^2/2))t} \|w_0\|_{H^1} \\ + C_8 |s-t| , \end{aligned}$$

where C_8 is the constant depending only on λ, I, k and $\mathcal{B}$. Thus, we appeal to **[9,** Theorem 3.1] to construct an exponential attractor. $\square$

APPENDIX

A. Connecting orbit. We consider the parabolic equation

$$\text{(6.3)} \qquad \begin{cases} u_t = \Delta u + f(x,u), & x \in I,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \in H_0^1(I), & x \in I. \end{cases}$$

Here we assume that $f = f(x,u)$ is sufficiently smooth so that (6.3) has a unique classical solution $u \in C([0,+\infty); H_0^1(I))$. Moreover, assume that the corresponding stationary problem,

$$\begin{cases} \Delta v + f(x,v) = 0, & x \in I, \\ v(0) = v(1) = 0, \end{cases}$$

has at least two hyperbolic solutions $v_1, v_2, \ldots, v_k, \ldots$. Here a stationary solution v is said to be hyperbolic if the linearized problem,

$$\begin{cases} \Delta \phi + f_u(x,v)\phi = -\mu\phi, & x \in I, \\ \phi(0) = \phi(1) = 0, \end{cases}$$

does not have $\mu = 0$ as an eigenvalue. As shown in **[1]**, if the connecting orbit from hyperbolic stationary solution v_i to v_j exists, then the Morse index of v_i is greater than that of v_j. The proof is based on the fact that the zero in the x direction of the solution of the linearized equation for (6.3) does not increase in number with time, which is called *Matano's principle* (**[2, 23]**). However, by the lack of comparison principle, we cannot apply Matano's principle to (1.7). Thus, it is not obvious to determine which stationary solutions are connected by a heteroclinic orbit.

In **[10, 11, 12]**, the Dirichlet boundary problem without non-local term is considered:

$$\begin{cases} u_t = \Delta u + \lambda e^u, & x \in B,\ t > 0, \\ u(x,t) = 0, & x \in \partial B,\ t > 0, \\ u(x,0) = u_0(x) \in X, & x \in B, \end{cases} \tag{6.4}$$

and

$$\begin{cases} \Delta v + \lambda e^v = 0, & x \in B, \\ v = 0, & x \in \partial B, \end{cases} \tag{6.5}$$

where $\lambda > 0$, $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$ with $n \in [3, 9]$ and X is imbedded in $C^1(\overline{B})$. Since all solutions of (6.5) are radially symmetric by **[14]**, (6.5) is converted to the one-dimensional case. Hence, we can apply Matano's principle.

As shown in **[13, 17]**, there is some interval $L \subset \mathbb{R}_+$ such that (6.5) has at least two stationary solutions for all $\lambda \in L$, and we can compute their Morse indices. Then, in **[10, 11, 12]**, they prove that a classical connection from v_i to v_j exists if and only if $i(\lambda, v_i) > i(\lambda, v_j)$. Moreover, they define the notion of an L^1-solution of (6.4) and construct it in **[12]**. They prove that the L^1-connection from v_i to v_j exists if and only if $i(\lambda, v_i) > i(\lambda, v_j) + 2$.

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