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ASSOCIATE ELEMENTS IN COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity. For $a, b \in R$, define a and b to be *associates*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, so $a = rb$ and $b = sa$ for some $r, s \in R$. We are interested in the case where r and s can be taken or must be taken to be non zero-divisors or units. We study rings, R, called *strongly regular associate*, that have the property that, whenever $a \sim b$ for $a, b \in R$, then there exist non zero-divisors $r, s \in R$ with $a = rb$ and $b = sa$ and rings R, called *weakly présimplifiable*, that have the property that, for nonzero $a, b \in R$ with $a \sim b$, whenever $a = rb$ and $b = sa$, then r and s must be non zero-divisors.

Let R be a commutative ring with identity, and let $a, b \in R$. Then a and b are said to be *associates*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, or equivalently, if $Ra = Rb$. Thus, if $a \sim b$, there exist $r, s \in R$ with $ra = b$ and $sb = a$, and hence $a = sra$. So, if a is a regular element (i.e., non zero-divisor), $sr = 1$, and hence r and s are units. Hence, if a and b are regular elements of a commutative ring R with $a \sim b$, then $a = ub$ for some $u \in U(R)$, the group of units of R. For $a, b \in R$, let us write $a \approx b$ if $a = ub$ for some $u \in U(R)$. Of course, $a \approx b$ implies $a \sim b$ for elements a and b of any commutative ring R and for an integral domain the converse is true. In **[9]**, Kaplansky raised the question of when a commutative ring R satisfies the property that, for all $a, b \in R$, $a \sim b$ implies $a \approx b$. He remarked that Artinian rings, principal ideal rings and rings with $Z(R) \subseteq J(R)$ satisfy this property. (Here $Z(R)$ and $J(R)$ denote the set of zero-divisors and Jacobson radical of a ring R, respectively.) But he gave two examples of commutative rings that fail to satisfy this property. Let us recall these two examples and give a third example.

(1) Let $R = C([0, 3])$ be the ring of continuous functions on $[0, 3]$. Define $a(t), b(t) \in R$ by $a(t) = b(t) = 1 - t$ on $[0, 1]$,

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$a(t) = b(t) = 0$ on $[1, 2]$ and $a(t) = -b(t) = t - 2$ on $[2, 3]$. Then $a(t) \sim b(t)$ (for $c(t)a(t) = b(t)$ and $c(t)b(t) = a(t)$ where $c(t) = 1$ on $[0, 1]$, $c(t) = 3 - 2t$ on $[1, 2]$ and $c(t) = -1$ on $[2, 3]$), but $a(t) \not\approx b(t)$.

(2) Let $R = \{(n, f(X)) \in \mathbf{Z} \times GF(5)[X] | f(0) \equiv n \bmod 5\}$ be a subring of $\mathbb{Z} \times GF(5)[X]$. Then $(0, X) \sim (0, \overline{2}X)$, but $(0, X) \not\approx (0, \overline{2}X)$.

(3) (Fletcher [**7**]). Let K be a field and $R = K[X, Y, Z]/(X - XYZ)$. Then $\overline{X} \sim \overline{XY}$, but $\overline{X} \not\approx \overline{XY}$.

We define a commutative ring R with the property that, for all $a, b \in R$, $a \sim b$ implies $a \approx b$ to be *strongly associate.* These rings, called "associate rings," were introduced and studied by Spellman et al. [**10**] and later studied in [**1**]. The basis for the choice of the word "strongly associate" will become apparent from the next paragraph.

A general study of various associate relations was begun by Anderson and Valdes-Leon [**3**] in their study of factorization in commutative rings with zero-divisors. Let R be a commutative ring, and let $a, b \in R$. There a and b were defined to be *associates*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, *strong associates*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$, and *very strong associates*, denoted $a \cong b$, if $a \sim b$ and further when $a \neq 0$, $a = rb$ ($r \in R$) implies $r \in U(R)$. Clearly $a \cong b \Rightarrow a \approx b$ and $a \approx b \Rightarrow a \sim b$, but examples were given to show that neither of these implications could be reversed. Thus, it is of interest to study commutative rings R where for all $a, b \in R$ (i) $a \sim b \Rightarrow a \approx b$, (ii) $a \approx b \Rightarrow a \cong b$ or (iii) $a \sim b \Rightarrow a \cong b$. We have already defined a ring R satisfying (i) to be strongly associate. Following Bouvier [**6**], we define a commutative ring R to be *présimplifiable* if, for $x, y \in R$, $xy = x$ implies $x = 0$ or $y \in U(R)$. Commutative rings satisfying the equivalent condition (7) of Theorem 1 were studied by Fletcher [**8**] who called them "pseudo-domains." We first note that (ii) and (iii) are equivalent to R being présimplifiable. Note that, while $\sim$ and $\approx$ are both equivalence relations on R, the relation $\cong$ is an equivalence relation on R if and only if R is présimplifiable. The following theorem gives several conditions equivalent to a ring being présimplifiable. A proof may be found in [**1,** Theorem 1].

Theorem 1. *For a commutative ring R, the following conditions are equivalent.*

(1) *For all* $a, b \in R$, $a \sim b \Rightarrow a \cong b$.
(2) *For all* $a, b \in R$, $a \approx b \Rightarrow a \cong b$.
(3) *For all* $a \in R$, $a \cong a$.
(4) R *is présimplifiable.*
(5) $Z(R) \subseteq 1 - U(R) = \{1 - u | u \in U(R)\}$.
(6) $Z(R) \subseteq J(R)$.
(7) *For* $0 \neq r \in R$, $sRr = Rr \Rightarrow s \in U(R)$.

Our next theorem shows that, in one case when two elements are associate, we can say more. Recall that a nonunit a of a commutative ring R is *irreducible* or is an *atom* if, whenever $a = bc$, $b, c \in R$, either $a \sim b$ or $a \sim c$. This is equivalent to $(a) = (b)(c)$ implies $(a) = (b)$ or $(a) = (c)$.

Theorem 2. *Let* R *be a commutative ring and* $a \in R$ *an atom. Suppose that* $b \in R$ *with* $a \sim b$. *Then at least one of the following two conditions holds.*

(1) $a = rb$ *and* $b = sa$, $r, s \in R$, *imply that* r *and* s *are regular.*
(2) $a \approx b$.

Moreover, if (1) *does not hold, then* a *is prime and* $a = ue$ *where* u *is a unit and* e *is idempotent.*

Proof. Suppose that $a = rb$ where r is not regular. Now $(b) = (a) = (r)(b) \subseteq (r)$. If $(a) \subsetneq (r)$, then r is regular since a is an atom [**2,** Theorem 1], a contradiction. So $(a) = (r)$. Thus, $(a) = (r)(b) = (a)^2$. So $a = ta^2$ for some $t \in R$ and so $e = ta$ is idempotent with $(a) = (e)$. Write $R = R_1 \times R_2$ where $R_1 = Re$ and $R_2 = R(1-e)$ with $e = (1,0)$ and $a = (\alpha, \beta)$. Then $Ra = Re$ gives $\alpha \in U(R_1)$ and $\beta = 0$. Hence, $a = ue$ for some $u \in U(R)$. Also, a irreducible forces $\beta = 0$ to be irreducible in R_2; so R_2 is an integral domain, and hence a is prime. Likewise, $(b) = (a) = (e)$, so $b = ve$ where $v \in U(R)$. Thus $b = vu^{-1}a$ and hence $a \approx b$.

Now $a \sim b$ gives that b is an atom. So, likewise, if $b = sa$ where s is not regular, then $b = ue$ where $u \in U(R)$ and e is idempotent and $b \approx a$. Thus, (2) and the moreover statement hold. □

We next show that all possibilities in the previous theorem may occur.

Example 3.

(1) Let R be an integral domain. If $0 \neq a \in R$ is an atom and $b \in R$ with $a \sim b$, then both (1) and (2) of Theorem 2 hold. For example, take $a = 2$, $b = -2$ in $\mathbb{Z}$.
(2) Let F be a field and $R = F[X, Y, Z]/(X - XYZ) = F[x, y, z]$. Then $x \in R$ is an atom and $x = xyz$ gives $x \sim xy$. But $x \not\approx xy$ [**3,** Example 2.3]. So (2) of Theorem 2 fails and hence (1) holds.
(3) Let F be a field, and take $a = (1, 0) \in R = F \times F$. So a is an atom, even prime. Take $b = a$, so certainly (2) of Theorem 2 holds, but $a = aa$ where a is not regular, so (1) fails.

Theorem 2 motivates the following definitions.

Definition 4. Let R be a commutative ring and $a, b \in R$. We say that a and b are *strongly regular associates*, denoted $a \approx_r b$, if there exist regular elements $r, s \in R$ with $a = rb$ and $b = sa$ and a and b are *very strongly regular associates*, denoted $a \cong_r b$, if $a \sim b$ and either (1) $a = b = 0$ or (2) $a = rb$ implies r is regular. A ring R is said to be *strongly regular associate* if whenever $a \sim b$ for $a, b \in R$, $a \approx_r b$.

It is easily seen that $\approx_r$ is an equivalence relation on R, even a congruence. It is also easily seen that $\cong_r$ is transitive and in fact $\cong_r$ is symmetric. For, suppose $a \cong_r b$, where we can assume $a \neq 0$. Let $b = sa$, so we need s regular. Now $a \sim b$, so $a = tb$. Thus, $a = tb = t(sa) = (ts)a = (ts)tb = (tst)b$. Since $a \cong_r b$, tst is regular, and hence so is s. However, $\cong_r$ need not be reflexive. For if $e \in R$ is an idempotent with $e \neq 0, 1$, then $e = e^2$ shows that $e \not\cong_r e$. Note that $\cong_r$ is reflexive if and only if, for $x, y \in R$, $x = xy$ implies $x = 0$ or y is regular. With this in mind we make the following definition.

Definition 5. Let R be a commutative ring. Then R is *weakly présimplifiable* if, for $x, y \in R$, $x = xy$ implies $x = 0$ or y is regular.

We next give a weakly présimplifiable analog of Theorem 1. For a commutative ring R, $\operatorname{reg}(R)$ is the set of regular elements (i.e., non zero-divisors) of R.

Theorem 6. *For a commutative ring R the following conditions are equivalent.*

(1) *For all $a, b \in R$, $a \sim b$ implies $a \cong_r b$.*
(2) *For all $a, b \in R$, $a \approx_r b$ implies $a \cong_r b$.*
(3) *For all $a, b \in R$, $a \approx b$ implies $a \cong_r b$.*
(4) *For all $a \in R$, $a \cong_r a$.*
(5) *R is weakly présimplifiable.*
(6) *$Z(R) \subseteq 1 - \operatorname{reg}(R)$ $(= 1 + \operatorname{reg}(R))$.*
(7) *For (prime) ideals $P, Q \subseteq Z(R)$, $P + Q \neq R$.*
(8) *For $a, b \in Z(R)$, $(a, b) \neq R$.*
(9) *For $a \in R$, either a or $a - 1$ is regular.*
(10) *For $0 \neq r \in R$, $sRr = Rr$ implies s is regular.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Clear.

(4) $\Rightarrow$ (1). Suppose that $a \sim b$. We need to show that $a \cong_r a$ implies that $a \cong_r b$. As the case $a = 0$ is trivial, we assume that $a \neq 0$. Suppose that $a = rb$. Now $a \sim b$ gives $b = sa$; so $a = rsa$. Hence rs and thus r itself is regular.

(4) $\Leftrightarrow$ (5). This has already been noted.

(5) $\Rightarrow$ (6). Let $y \in Z(R)$, so there exists $0 \neq x \in R$ with $xy = 0$. Then $x = x(1 - y)$, so $1 - y \in \operatorname{reg}(R)$, and hence $y \in 1 - \operatorname{reg}(R)$.

(6) $\Rightarrow$ (5). Suppose that $x = xy$ with $x \neq 0$. Then $x(1 - y) = 0$ so $1 - y \in Z(R) \subseteq 1 - \operatorname{reg}(R)$, and hence $y \in \operatorname{reg}(R)$.

(6) $\Rightarrow$ (7). Suppose $P + Q = R$ so there exist $p \in P$ and $q \in Q$ with $p + q = 1$. Now $q = 1 - r$ where $r \in \operatorname{reg}(R)$. Hence, $1 - p = q = 1 - r$ gives that $p = r$ is regular, a contradiction.

(7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (6). Clear.

(5) $\Rightarrow$ (10). $sRr = Rr$ implies $r = str$ for some $t \in R$. Then st, and hence s is regular.

(10) $\Rightarrow$ (5). Suppose $r = sr$ where $r \neq 0$. Then $sRr = Rr$; so s is regular. □

Corollary 7. *A weakly présimplifiable ring R is strongly regular associate.*

Definition 8. A commutative ring R is called a *bounded factorization ring* (BFR) if, for each nonzero nonunit $a \in R$, there exists a natural number $N(a)$ so that, for any factorization $a = a_1 \cdots a_n$ of a where each a_i is a nonunit, we have $n \leq N(a)$. A commutative ring R is called a z-BFR if, for each nonzero zero-divisor $a \in R$, there exists a natural number $N_Z(a)$ so that for any factorization $a = b_1 \cdots b_n$ of a where each b_j is a zero-divisor, we have $n \leq N_Z(a)$.

Certainly, R a BFR implies R is a z-BFR. Also, a z-BFR R is weakly présimplifiable. For suppose that, in R, $0 \neq x = xy$ with $x, y \in Z(R)$. Then $x = xy = xy^2 = \cdots$, so x has arbitrarily long factorizations involving zero-divisors, a contradiction.

Theorem 9. *For a Noetherian ring R, the following conditions are equivalent.*

(1) *R is a BFR (z-BFR).*
(2) *R is (weakly) présimplifiable.*
(3) $\cap_{n=1}^{\infty}(y^n) = 0$ *for each nonunit $y \in R$ ($y \in Z(R)$).*
(4) $\cap_{n=1}^{\infty} I^n = 0$ *for each proper ideal I (contained in $Z(R)$).*

Proof. The BFR case is given in [**3,** Theorem 3.9]. We do the z-BRR case, which is similar. We have already observed that (1) $\Rightarrow$ (2).

Certainly (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

By the Krull intersection theorem, $\cap_{n=1}^{\infty} I^n = 0_{1-I} = \{x \in R \mid xi = x$ for some $i \in I\}$, so (2) $\Rightarrow$ (4).

We show that (4) $\Rightarrow$ (1). Let $0 \neq x \in R$ be a zero-divisor, and let $Z(R) = P_1 \cup \cdots \cup P_n$, a finite union of prime ideals. Suppose that x has arbitrarily long factorizations involving zero-divisors. If $x = a_1 \cdots a_m$ where $m \geq kn$ and each a_i is a zero-divisor, then each a_i is in some P_j and hence $x \in P_i^k$ for some $i \leq i \leq n$. So, for each k, there exists a $1 \leq i(k) \leq n$ with $x \in P_{i(k)}^k$. Thus, for some $1 \leq l \leq n$, there are infinitely many k with $i(k) = l$. Then $x \in \cap_{m=1}^{\infty} P_l^m = 0$, a contradiction. $\square$

Theorem 10. *Let R be a commutative ring with the property that, for each ideal I $(\subseteq Z(R))$, $\cap_{n=1}^{\infty} I^n = \{x \in R \mid x = xi$ for some $i \in I\}$. Then the following statements are equivalent.*

(1) $\cap_{n=1}^{\infty} I^n = 0$ *for each proper ideal* I *(contained in* $Z(R)$*).*
(2) $\cap_{n=1}^{\infty} (y^n) = 0$ *for each nonunit* $y \in R$ $(y \in Z(R))$*.*
(3) R *is (weakly) présimplifiable.*

Proof. The présimplifiable case is [**3,** Theorem 3.10]. We do the weakly présimplifiable case.

(1) $\Rightarrow$ (2). This is always true.

(2) $\Rightarrow$ (1). Let $z \in \cap_{n=1}^{\infty} I^n$. Then $z = zi$ for some $i \in I$, so $z \in \cap_{n=1}^{\infty}(i^n) = 0$.

(2) $\Rightarrow$ (3). Suppose that $xy = x$ and $y \in Z(R)$. Then $x \in \cap_{n=1}^{\infty}(y^n) = 0$.

(3) $\Rightarrow$ (2). Let $y \in Z(R)$ and $x \in \cap_{n=1}^{\infty}(y^n)$. Then $x = x(ry)$ for some $r \in R$. Then $ry \in Z(R)$ forces $x = 0$ and hence $\cap_{n=1}^{\infty}(y^n) = 0$. $\square$

Let's revisit the three examples of rings mentioned in the first paragraph.

Example 11.

(1) The ring $R = C([0,3])$ is not weakly présimplifiable. For define $f(t) \in R$ by $f(t) = 1$ on $[0,1]$, $f(t) = 2-t$ on $[1,2]$ and $f(t) = 0$ on $[2,3]$. Then $f(t)$ and $f(t) - 1$ are both zero-divisors. Note that the function $c(t)$ in the example is regular, so $a(t) \approx_r b(t)$. Our next theorem will show that $C([a,b])$ is strongly regular associate (but not strongly associate).
(2) Let $R = \{(n, f(X)) \in \mathbf{Z} \times GF(5)[X] | f(0) \equiv n \bmod 5\}$. Now $Z(R) = 5\mathbf{Z} \times \{0\} \cup \{0\} \times X\,GF(5)[X]$. So, for $a \in R$, a or $a-1$ is regular. So R is weakly présimplifiable and hence strongly regular associate, but not strongly associate and hence not présimplifiable.
(3) Let $R = K[X,Y,Z]/(X - XYZ) = K[x,y,z]$, K a field. Here $Z(R) = (x) \cup (1 - yz)$. Since $(x) + (1 - yz) \neq R$, R is weakly présimplifiable and hence strongly regular associate, but not strongly associate and hence not présimplifiable.

Theorem 12. *The ring $C([a,b])$ is strongly regular associate.*

Proof. Let $R = C([a,b])$, $a < b$. First, observe $f \in Z(R)$ if and only if there exist α, β, $a \leq \alpha < \beta \leq b$ with $f(t) = 0$ on $[\alpha, \beta]$. Also, note that, if $f(t) = 0$ on $[\alpha, \beta]$, then there is a maximal closed interval $[\alpha', \beta']$, $[\alpha, \beta] \subseteq [\alpha', \beta'] \subseteq [a, b]$ with $f(t) = 0$ on $[\alpha', \beta']$. Suppose that $a(t), b(t) \in R$ with $a(t) \sim b(t)$. Choose $c(t) \in R$ with $a(t)c(t) = b(t)$. Note that $c^{-1}(0) \subseteq a^{-1}(0) = b^{-1}(0)$. Suppose that $c(t)$ is not regular. Let $[\alpha, \beta]$ be a maximal closed subinterval on which $c(t) = 0$. Modify $c(t)$ on $[\alpha, \beta]$ to $t - \alpha$ on $[\alpha, (\alpha+\beta)/2]$ and $-t + \beta$ on $[(\alpha+\beta)/2, \beta]$. Make this modification on each such maximal subinterval to obtain a new $c_1(t) \in R$ which is regular. Then $c_1(t)a(t) = b(t)$. Similarly, there is a regular element $c_2(t) \in R$ with $c_2(t)b(t) = a(t)$. □

Example 13. Let $R = K[X_1, \ldots, X_n]/(f_1^{s_1} \cdots f_n^{s_n})$, K a field, where $f_i \in K[X_i]$ is irreducible and $s_i \geq 0$ with at least one $s_i \geq 1$. Then R is weakly présimplifiable but is présimplifiable if and only if exactly one $s_i > 0$. Note that $Z(R) = \cup\{(\overline{f_i}) \mid s_i \geq 1\}$ and $J(R) = \operatorname{nil}(R) = \cap\{(\overline{f_i}) | s_i \geq 1\}$ since R is a Hilbert ring. Now $\sum\{(\overline{f_i}) | s_i \geq 1\} \neq R$; so R is weakly présimplifiable by Theorem 6. But R is présimplifiable if and only if $Z(R) \subseteq J(R)$, which occurs when exactly one $s_i \geq 1$.

We have yet to give an example of a ring that is not strongly regular associate. We do so using the method of idealization. Let R be a commutative ring and M an R-module. The idealization or trivial extension $R(+)M$ of R and M is the ring $R \bigoplus M$ with addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. For a good introduction to idealization, see **[5]**. We recall the following:

(1) every prime (maximal) ideal of $R(+)M$ has the form $P \bigoplus M$ where P is a prime (maximal) ideal of R,
(2) $J(R(+)M) = J(R) \bigoplus M$,
(3) $\operatorname{nil}(R(+)M) = \operatorname{nil}(R) \bigoplus M$,
(4) $Z(R(+)M) = \{Z(R) \cup Z(M)\} \bigoplus M$,
(5) $\operatorname{reg}(R(+)M) = \{\operatorname{reg}(R) \cap (R - Z(M))\} \bigoplus M$, and
(6) $U(R(+)M) = U(R) \bigoplus M$. Before studying $R(+)M$, we give the promised example.

Example 14. Let R be a commutative ring that is not strongly associate, e.g., one of the three examples in the first paragraph. Let $M = \bigoplus_{\mathcal{M}\in \max(R)} R/\mathcal{M}$. Then $R(+)M$ is not strongly regular associate. Note that $Z(R(+)M) = (R - U(R))(+)M = R(+)M - U(R(+)M)$, so $R(+)M$ is a total quotient ring. Since R is not strongly associate, there exist $a, b \in R$ with $a \sim b$, but $a \not\approx b$. Then $(a, 0) \sim (b, 0)$, but $(a, 0) \not\approx_r (b, 0)$ as elements of $R(+)M$. For, if $(b, 0) = (r, m)(a, 0)$ for some regular $(r, m) \in R(+)M$, then $r \in U(R)$ so $b = ra$ and hence $a \approx b$, a contradiction.

A (weakly) présimplifiable ring R must be indecomposable as $e \not\cong_r e$ for an idempotent $e \in R$ with $e \neq 0, 1$. Example 14 can be used to construct indecomposable rings that are not strongly regular associate.

In [**4**] the associate relations defined on commutative rings were extended to modules as follows. Let M be an R-module. For $m, n \in M$, define $m \sim n$ if $Rm = Rn$, $m \approx n$ if $m = un$ for some $u \in U(R)$, and $m \cong n$ if $m \sim n$ and either $m = n = 0$ or $m = rn$ implies $r \in U(R)$. Then M is *strongly associate* (*présimplifiable*) if $m \sim n$ implies $m \approx n$ ($m \cong n$). Theorem 1 may be appropriately extended to modules. We note that the following are equivalent for an R-module M:

(1) for $m, n \in M$, $m \sim n$ implies $m \cong n$,
(2) $m = rm \neq 0$ implies $r \in U(R)$, and
(3) $Z(M) \subseteq J(R)$.

For $m, n \in M$, we further define $m \approx_r n$ if $m = rn$ and $sm = n$ for some $r, s \in R - \{Z(M) \cup Z(R)\}$ and $m \cong_r n$ if $m \sim n$ and either $m = n = 0$ or $m = rn$ implies $r \in R - \{Z(M) \cup Z(R)\}$. Then M is *strongly regular associate* (*weakly présimplifiable*) if $m \sim n$ implies $m \approx_r n$ ($m \cong_r n$). So M is weakly présimplifiable if and only if $m = rm \neq 0$ implies $r \notin Z(M) \cup Z(R)$. Finally, we say that R is ***M**-strongly regular associate* (***M**-weakly présimplifiable*) if, for $a, b \in R$, $a \sim b$ implies $ra = b$ and $sb = a$ for some $r, s \in R - \{Z(M) \cup Z(R)\}$ ($a = b = 0$ or $a = rb$ implies $r \notin Z(M) \cup Z(R)$, or equivalently, $a = ra \neq 0$ implies $r \notin Z(M) \cup Z(R)$.)

Theorem 15. *Let R be a commutative ring and M an R-module.*

(1) *$R(+)M$ is présimplifiable if and only if R is présimplifiable and $Z(M) \subseteq J(R)$ (i.e., M is présimplifiable), or equivalently,*

$Z(M) \cup Z(R) \subseteq J(R)$.

(2) *The following are equivalent.*
   (a) *$R(+)M$ is weakly présimplifiable.*
   (b) *$Z(M) \cup Z(R) \subseteq 1 - reg\,(R) \cap (R - Z(M))$.*
   (c) *For (prime) ideals $P, Q \subseteq Z(M) \cup Z(R)$, $P + Q \neq R$.*
   (d) *For $a \in R$, a or $a - 1 \notin Z(M) \cup Z(R)$.*
   (e) *R is M-weakly présimplifiable and M is weakly présimplifiable.*

(3) *If $R(+)M$ is strongly (regular) associate, then R is strongly associate (M-strongly regular associate) and M is strongly (regular) associate.*

(4) *Suppose that R is présimplifiable (M-weakly présimplifiable). Then $R(+)M$ is strongly (regular) associate if and only if M is strongly (regular) associate.*

Proof. (1) This is given in [**4**]. It follows from Theorem 1 since $Z(R(+)M) = \{Z(M) \cup Z(R)\} \bigoplus M$ and $J(R(+)M) = J(R) \bigoplus M$.

(2) This easily follows from Theorem 6 since $Z(R(+)M) = \{Z(M) \cup Z(R)\} \bigoplus M$ and $\operatorname{reg}(R(+)M) = \{\operatorname{reg}(R) \cap (R - Z(M))\} \bigoplus M$.

(3) The strongly associate case is given in [**1,** Theorem 14]. The proof of the strongly regular associate case is similar.

(4) The case where R is présimplifiable is given in [**1,** Theorem 14]. The proof of the case where R is M-weakly présimplifiable is similar. □

Corollary 16. *Let G be an abelian group with torsion subgroup G_t and let $R = \mathbf{Z}(+)G$.*

(1) *R is présimplifiable $\Leftrightarrow$ G is présimplifiable $\Leftrightarrow$ G is torsion-free.*

(2) *R is strongly associate $\Leftrightarrow$ G is strongly associate $\Leftrightarrow$ $G = F \bigoplus G_t$ where F is torsion-free and $4G_t = 0$ or $6G_t = 0$.*

(3) *R is weakly présimplifiable $\Leftrightarrow$ G is weakly présimplifiable $\Leftrightarrow$ $G_t = 0$ or G_t is p-primary (i.e., $Z(G) = (p)$) for some prime p.*

(4) *R is strongly regular associate $\Leftrightarrow$ G is strongly regular associate. The group G is strongly regular associate if $Z(G)$ is a finite union of prime ideals.*

Proof. (1) and (2) [**1,** Theorem 15 and Corollary 16].

(3) This follows from the equivalence of $(a), (c)$ and (e) of Theorem 15(2).

(4) The first statement follows from Theorem 15 (4). Suppose that $Z(G) = Z(G_t) = (p_1) \cup \cdots \cup (p_s)$, where $p_1, \ldots, p_s$ are distinct primes. We show that G is strongly regular associate. Suppose that $0 \neq a \sim b$ in G_t. So $\langle a \rangle, = \langle b \rangle \approx \mathbf{Z}_n$ where the primes dividing n are a subset of $\{p_1, \ldots, p_s\}$. With a change of notation, for $l, m \in \mathbf{Z}$ with $[l, n] = [m, n] = 1$, we need a $k \in \mathbf{Z}$ with $kl \equiv m \bmod n$ and $[k, p_i] = 1$ for any p_i that doesn't divide n. Now $\bar{l}$ and $\overline{m}$ are units in $\mathbf{Z}_n$, so there is a $k_0 \in \mathbf{Z}$ with $\overline{k}_0 = (\bar{l})^{-1}\overline{m}$. Now by the Chinese remainder theorem, the system $x \equiv k_0 \bmod n$, $x \equiv 1 \bmod p_i$ for $p_i \in \{p_1, \ldots, p_s\}$, $p_i \nmid n$, has a solution k. Then $ka = b$ and $k \notin Z(G)$. So G is strongly regular associate. □

We next investigate the stability of the four properties présimplifiable, weakly présimplifiable, strongly associate and strongly regular associate under various standard ring constructions.

Theorem 17.

(1) *Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a nonempty family of commutative rings. Then $R = \prod_{\alpha \in \Lambda} R_\alpha$ is strongly (regular) associate if and only if each R_α is strongly (regular) associate. However, R is not (weakly) présimplifiable whenever $|\Lambda| > 1$.*

(2) *Let $(\Lambda, \leq)$ be a directed quasi-ordered set, and let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a direct system of rings. If each R_α is strongly associate (respectively, présimplifiable, weakly présimplifiable), then the direct limit $\varinjlim R_\alpha$ is strongly associate (respectively, présimplifiable, weakly présimplifiable). Further, suppose that for $\alpha < \beta$, the map $\lambda_\beta^\alpha : R_\alpha \to R_\beta$ preserves regular elements, then if each R_α is strongly regular associate, then $\varinjlim R_\alpha$ is strongly regular associate.*

(3) *Let $(\Lambda, \leq)$ be a directed quasi-ordered set, and let $\{R_\alpha\}_{\alpha \in \Lambda}$ be an inverse system of rings. If each R_α is (weakly) présimplifiable, then the inverse limit $R = \varprojlim R_\alpha$ is (weakly) présimplifiable.*

(4) *Let $\mathfrak{T}$ be an ultrafilter on Λ where $\{R_\alpha\}_{\alpha \in \Lambda}$ is a nonempty family of commutative rings. Then the ultraproduct $\prod R_\alpha / \mathfrak{T}$ is présimplifiable (respectively, strongly associate, weakly présimplifiable, strongly regular associate) $\Leftrightarrow$ $\{\alpha \in \Lambda \mid R_\alpha$ is*

présimplifiable (respectively, strongly associate, weakly présimplifiable, strongly regular associate)$\}\in\mathfrak{T}$. Hence, an ultraproduct of présimplifiable (respectively, strongly associate, weakly présimplifiable, strongly regular associate) rings is again présimplifiable (respectively, strongly associate, weakly présimplifiable, strongly regular associate).

Proof. (1) The strongly associate case is given in [**1,** Theorem 3 (1)]. The strongly regular associate case is similar. The "however" statement follows since a (weakly) présimplifiable ring is indecomposable.

(2) The strongly associate and présimplifiable cases are given in [**1,** Theorem 3 (2)]. The weakly présimplifiable case is similar. We do the strongly regular associate case. Let $x, y \in R$ with $x \sim y$. Let $x = ay$ and $y = bx$. For $\alpha \in \Lambda$, let $\lambda_\alpha : R_\alpha \to R$ be the natural map. Now there exists $\alpha_0 \in \Lambda$ and $x_{\alpha_0}, y_{\alpha_0}, a_{\alpha_0}, b_{\alpha_0}$ with $\lambda_{\alpha_0}(x_{\alpha_0}) = x$, $\lambda_{\alpha_0}(y_{\alpha_0}) = y$, $\lambda_{\alpha_0}(a_{\alpha_0}) = a$, $\lambda_{\alpha_0}(b_\alpha) = b_{\alpha_0}$, $x_{\alpha_0} = a_{\alpha_0}y_{\alpha_0}$, and $y_{\alpha_0} = b_{\alpha_0}x_{\alpha_0}$. Then $x_{\alpha_0} \sim y_{\alpha_0}$ in R_{α_0}, so there exist $r_{\alpha_0}, s_{\alpha_0} \in \operatorname{reg}(R_{\alpha_0})$ with $x_{\alpha_0} = r_{\alpha_0}y_{\alpha_0}$ and $y_{\alpha_0} = s_{\alpha_0}x_{\alpha_0}$. Let $r = \lambda_{\alpha_0}(r_{\alpha_0})$ and $s = \lambda_{\alpha_0}(s_{\alpha_0})$; so $x = ry$ and $y = sx$. Moreover, $r, s \in \operatorname{reg}(R)$. For, if say, $rt = 0$ in R, there exists a $\beta \geq \alpha_0$ and $t_\beta \in R_\beta$ with $\lambda_\beta(t_\beta) = t$ and $\lambda_\beta^{\alpha_0}(r_{\alpha_0})t_\beta = 0$. But $r_{\alpha_0} \in \operatorname{reg}(R_{\alpha_0})$ and $\lambda_\beta^{\alpha_0}$ preserve regular elements, so $\lambda_\beta^{\alpha_0}(r_{\alpha_0}) \in \operatorname{reg}(R_\beta)$. Hence, $t_\beta = 0$ and thus $t = 0$.

(3) The présimplifiable case is due to Bouvier, see [**1,** Theorem 3(3)]. The weakly présimplifiable case is similar.

(4) Each of the given four properties can be expressed in terms of a first-order sentence. The sentence for présimplifiable and strongly associate are given in the proof of [**1,** Theorem 3 (4)]. A sentence for strongly regular associate is $\sigma = \forall x \forall y \exists z \exists w \exists u \exists v \forall l \forall k \forall s \forall t\ [((xz = y) \wedge (yw = x)) \Rightarrow ((xu = y) \wedge (x = vy) \wedge ((ul = uk) \Rightarrow (l = k)) \wedge ((vs = vt) \Rightarrow (s = t)))]$ while a sentence for weakly présimplifiable is $\sigma = \forall x \forall y \exists w \exists v \forall z \forall t \forall u\ [(xy = x) \Rightarrow (((x = w) \wedge (wz = w)) \vee (((ty = v) \wedge (vz = v)) \Rightarrow (tu = u)))]$. Thus, (4) follows from Los's theorem. $\square$

Theorem 18. *Let R be a commutative ring and $\{X_\alpha\}$ a nonempty set of indeterminates over R.*

(1) *$R[\{X_\alpha\}]$ is présimplifiable if and only if* 0 *is a primary ideal of R* [**6**].

(2) *$R[\{X_\alpha\}]$ is weakly présimplifiable if and only if R is. Hence, if R is présimplifiable, $R[\{X_\alpha\}]$ is weakly présimplifiable.*

(3) *$R[\{X_\alpha\}]$ is always strongly regular associate. Hence if $a, b \in R$ with $a \sim b$ in R, then $a \approx_r b$ in $R[X]$.*

Proof. (1) This is given in **[6]**. Since $J(R[\{X_\alpha\}]) = \operatorname{nil}(R[\{X_\alpha\}])$, $Z(R[\{X_\alpha\}]) \subseteq J(R[\{X_\alpha\}]) \Leftrightarrow Z(R[\{X_\alpha\}]) \subseteq \operatorname{nil}(R[\{X_\alpha\}]) \Leftrightarrow Z(R[\{X_\alpha\}]) = \operatorname{nil}(R[\{X_\alpha\}]) \Leftrightarrow 0$ is a primary ideal of $R[\{X_\alpha\}] \Leftrightarrow 0$ is a primary ideal of R.

(2) ($\Rightarrow$). Suppose $x = xy$ in R. This also holds in $R[\{X_\alpha\}]$ so $x = 0$ or $y \in \operatorname{reg}(R[\{X_\alpha\}]) \cap R = \operatorname{reg}(R)$.

($\Leftarrow$). Since a polynomial only involves finitely many X_α, by induction it is enough to show that R weakly présimplifiable implies $R[X]$ is weakly présimplifiable. Let $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. If a_0 is regular, f is regular. If a_0 is not regular, $a_0 - 1$ is regular since R is weakly présimplifiable (Theorem 6). Thus $f - 1$ is regular. By Theorem 6, $R[X]$ is weakly présimplifiable.

(3) It is enough to show that $R[X]$ is strongly regular associate. For $l \in R[X]$, $c(l)$ denotes the ideal of R generated by the coefficients of l. Suppose $f \sim g$ for $f, g \in R[X]$; say $fh = g$ and $gk = f$ for $h, k \in R[X]$. Then $c(g) = c(fh) \subseteq c(f)c(h) \subseteq c(f)$ and $c(f) = c(gk) \subseteq c(g)c(k) \subseteq c(g)$. So $c(f) = c(g)$, and thus $c(f) = c(f)c(h)$. Hence, there exists $a \in c(h)$ with $(1-a)c(f) = 0$; so $(1-a)f = 0$. Put $\overline{h} = h + (1-a)X^{n+1}$ where $n = \deg h$. So $c(\overline{h}) = c(h) + R(1-a) = R$, and hence $\overline{h}$ is regular. Now $f\overline{h} = f(h + (1-a)X^{n+1}) = fh + (1-a)fX^{n+1} = fh = g$. Likewise, there is a regular $\overline{k} \in R[X]$ with $f = \overline{k}g$. So $f \approx_r g$. □

Example 19. Let R be a présimplifiable ring in which 0 is not primary, e.g., $R = K[[S,T]]/(S^2, ST)$ (K a field). Then $R[X]$ is weakly présimplifiable, but not présimplifiable.

Certainly if $R[\{X_\alpha\}]$ is strongly associate, then so is R. But R strongly associate (even présimplifiable) does not imply that $R[X]$ is strongly associate [**1,** Example 19]. We next do the power series case.

Theorem 20. *Let R be a commutative ring.*

(1) *$R[[X_1, \ldots, X_n]]$ is (weakly) présimplifiable if and only if R is (weakly) présimplifiable.*
(2) *If R is Noetherian, then $R[[X_1, \ldots, X_n]]$ is strongly regular associate.*

Proof. (1) ($\Rightarrow$). Suppose $x = xy$ for $x, y \in R$. Then $x = xy$ in $R[[X_1, \ldots, X_n]]$ so $x = 0$ or y is (regular) a unit in $R[[X_1, \ldots, X_n]]$ and hence in R.

($\Leftarrow$). Let $f \in Z(R[[X_1, \ldots, X_n]])$. Then the constant term a of f lies in $Z(R)$. So $1 - a \in U(R)$ ($\operatorname{reg}(R)$). Then the constant term of $1 - f$ is a unit (regular). Thus $1 - f$ is a unit (regular).

(2) The proof is similar to the proof of Theorem 18 (3). We sketch the modification. Since $c(f)$ is finitely generated, there is an $a \in c(h)$ with $(1 - a)c(f) = 0$. Now if $h = a_0 + a_1X + a_2X^2 + \cdots$, then $c(h) = (a_0, \ldots, a_n)$ for some n. Put $\overline{h} = h + (1 - a)X^{n+1}$; so $c(\overline{h}) = R$ and $f\overline{h} = g$. As R is Noetherian and $c(\overline{h}) = R$, $\overline{h}$ is regular. □

We make the belated remark that a subring of a weakly présimplifiable ring is again weakly présimplifiable. Since $R[[X]]$ may be présimplifiable while $R[X]$ is not (Example 19), a subring of a présimplifiable ring need not be présimplifiable. Also, for any commutative ring R, R embeds into $\prod_{M \in \operatorname{Max}(R)} R_M$ which is strongly associate; thus, a subring of a strongly (regular) associate ring need not inherit the property. As any commutative ring is a homomorphic image of $\mathbf{Z}[\{X_\alpha\}]$ for some set $\{X_\alpha\}$ of indeterminants, it follows that none of the four properties is preserved by homomorphic image. If R is weakly présimplifiable or strongly regular associate, so is $R[X]$. However, if R is présimplifiable or strongly associate, $R[X]$ need not be. Example 19 gives an example of a présimplifiable ring R with $R[X]$ not présimplifiable, while [**1,** Example 19] shows that $R = \mathbf{Z}_{(2)}(+)\mathbf{Z}_4$ is strongly associate while $R[X]$ is not. If R is (weakly) présimplifiable, so is $R[[X]]$. We do not know if the property of being strongly (regular) associate is preserved by power series adjunction. Example 20 [**1**] gives an example of a local ring with a regular ring of quotients that is not strongly associate. Thus, a regular quotient ring of a présimplifiable (strongly associate) ring need not be présimplifiable (strongly associate). Now a total quotient ring is présimplifiable (or, equivalently, weakly présimplifiable) if and only if it is quasilocal. Thus, if R is a ring with total quotient ring $T(R)$ (weakly) présimplifiable, $Z(R)$ is a prime ideal. Hence, the ring $R = \mathbf{Z}[X, Y, Z]/(X - XYZ)$ given in the first paragraph is weakly présimplifiable, but $T(R)$ is not.

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