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TOPOLOGICAL PROPERTIES OF KERNELS OF PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. For a linear partial differential operator with constant coefficients on $\mathscr{D}'(\Omega)$, we investigate topological properties like barrelledness or bornologicity (which allow applications of fundamental principles like the Banach-Steinhaus or the open mapping theorem) of its kernel. Using recent functional analytic results inspired by homological algebra we prove that almost all barrelledness type conditions are equivalent in this situation and provide two distinct sufficient conditions which, in particular, are satisfied if the operator is surjective or hypoelliptic. This last case generalizes a classical result of Malgrange and Hörmander.

1. Introduction. Throughout this article, we consider a linear partial differential operator with constant coefficients

$$P(\partial) : \mathscr{D}'(\Omega) \longrightarrow \mathscr{D}'(\Omega)$$

on the space of Schwartz distributions over an open set $\Omega \subseteq \mathbf{R}^d$. For an open and relatively compact exhaustion $(\Omega_n)_{n\in\mathbf{N}}$ of Ω the space $\mathscr{D}'(\Omega)$ is the strong dual of the strict inductive limit $\mathscr{D}(\Omega) = \operatorname{Ind}_{n\in\mathbf{N}} \mathscr{D}(\overline{\Omega}_n)$ of the Fréchet-Schwartz spaces $\mathscr{D}(\overline{\Omega}_n) = \{f \in \mathscr{E}(\mathbf{R}^d) : \operatorname{supp}(f) \subseteq \overline{\Omega}_n\}$ endowed with the topology of uniform convergence of all partial derivatives. Since the inductive limit is strict, $\mathscr{D}(\Omega)$ is a complete Schwartz space and, by a theorem of Schwartz [**19,** page 43], see also [**13,** proposition 24.23], $\mathscr{D}'(\Omega)$ is ultrabornological. This is all we need to know, although, by a theorem of Valdivia [**21**] and (independently) Vogt [**22**], $\mathscr{D}'(\Omega)$ is even topologically isomorphic to $(s')^{\mathbf{N}}$ where s is the space of rapidly decreasing sequences and s' is its strong dual.

Being a closed subspace, the kernel $\mathscr{D}'_P(\Omega) = \operatorname{Kern} P(\partial)$ inherits all properties of $\mathscr{D}'(\Omega)$ which are stable with respect to closed subspaces, in particular, it is a complete nuclear locally convex space. However,

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many locally convex properties which are crucial for applications of basic functional analytic principles, like barrelledness to use the Banach-Steinhaus theorem (every pointwise bounded family of continuous linear operators into an arbitrary locally convex space is equicontinuous) or ultrabornologicity (by definition, this means that the space is the locally convex inductive limit of all continuously embedded Banach spaces) to use the closed graph and open mapping theorem, are not inherited by closed subspaces. Although quite implicitly, very concrete properties of a differential operator are encoded in such locally convex conditions of certain subspaces, for instance, $P(\partial)$ is surjective on $\mathscr{D}'(\Omega)$ if and only if the range of $P(\partial)^t = P(-\partial)$ is an ultrabornological subspace of $\mathscr{D}(\Omega)$ (in fact, then the closed graph theorem implies that $P(-\partial) : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is an isomorphism onto its range and hence its transposed $P(\partial) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective by the Hahn-Banach theorem).

Since properties of $\mathscr{D}'_P(\Omega)$ play an indispensable role in classical analysis (we only mention Runge's theorem for holomorphic functions–the kernel of the Cauchy-Riemann operator) and in the study of general partial differential operators, it is a most natural question whether the kernel is barrelled or ultrabornological so that, as we said above, general principles of functional analysis apply.

We thank Norbert Ortner from Innsbruck who posed this question explicitly and pointed out to us that, surprisingly, only very little can be found in the literature.

In Section 2 we recall and give a very simple proof of a result of Malgrange [**12**] and Hörmander [**9,** theorem 4.4.2] in the hypoelliptic case $\mathscr{D}'_P(\Omega) \subseteq \mathscr{E}(\Omega)$, stating that then the $\mathscr{D}'(\Omega)$- and $\mathscr{E}(\Omega)$-topologies coincide on $\operatorname{Kern} P(\partial)$, and hence $\mathscr{D}'_P(\Omega)$ is even a Fréchet space (for which the functional analytic principles had been proved originally). Moreover, we show that hypoellipticity is the only case where $\mathscr{D}'_P(\Omega)$ is Fréchet (which is neither difficult to show nor very surprising, but we did not find this result explicitly in the literature).

In the third section we apply results about acyclic LF-spaces due to Palamodov [**15**] and the author [**23**] to obtain that $\mathscr{D}'_P(\Omega)$ is ultrabornological whenever $P(\partial)$ is surjective on $\mathscr{D}'(\Omega)$ which is even a characterization if $P(\partial)$ is surjective on $\mathscr{E}(\Omega)$ (which is equivalent to Ω being P-convex for supports). These methods also allow proving that all kinds of barrelledness conditions are equivalent in our situation.

In Section 4 we finally explain recent results about the derived projective limit functor which allow proving two sufficient conditions which generalize the surjective case as well as the above-mentioned result of Malgrange.

2. The hypoelliptic case. By definition, the operator is hypoelliptic if $\mathscr{D}'_P(\Omega)$ is contained in $\mathscr{E}(\Omega)$. There are many characterisations of hypoellipticity due to Malgrange and Hörmander, in particular, this property is independent of Ω, and it holds if and only if $P(\partial)$ has a fundamental solution $E \in \mathscr{D}'(\mathbf{R}^d)$ with $E \in C^\infty(\mathbf{R}^d \setminus \{0\})$. In this case, $\operatorname{Kern} P(\partial)$ coincides with the kernel $\mathscr{E}_P(\Omega)$ of $P(\partial) : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ but, a priori, it is not evident that this coincidence holds topologically because the closed graph theorem (applied to the identity $\mathscr{D}'_P(\Omega) \to \mathscr{E}_P(\Omega)$) would require that $\mathscr{D}'_P(\Omega)$ be ultrabornological which is in question.

The topological equality $\mathscr{D}'_P(\Omega) = \mathscr{E}_P(\Omega)$ in the hypoelliptic case was proved by Malgrange [**12**] using duality theory and the fact that quotients of Fréchet-Schwartz spaces are again Fréchet-Schwartz and, independently and more directly, by Hörmander [**9,** Theorem 4.4.2] using a fundamental solution which is smooth outside the origin. We will present yet another proof using the fact that closed subspaces of LS-spaces (i.e., countable inductive limits of Banach spaces with compact inclusions which are also called DFS-spaces since they are precisely the strong duals of Fréchet-Schwartz spaces) are again LS and thus ultrabornological, see e.g., [**16,** Corollary 8.6.9] for a proof of this result of Sebastião e Silva [**20**]. To prepare the proof, let us fix some notation which will be used throughout this article.

For a fixed exhaustion $(\Omega_n)_{n\in\mathbf{N}}$ of Ω by open and relatively compact sets we write

$$X_n = \{u \in \mathscr{D}'(\overline{\Omega}_n) : P(\partial)u = 0\}$$

endowed with the relative topology of the strong dual of $\mathscr{D}(\overline{\Omega}_n)$ (by the preceding remark, X_n is thus an LS-space), $\varrho_m^n : X_m \to X_n$ denotes the restriction for $n \le m$, and

$$X_\infty = \mathscr{D}'_P(\Omega) = \{u \in \mathscr{D}'(\Omega) : P(\partial)u = 0\}$$

is the kernel endowed with the relative topology of $\mathscr{D}'(\Omega)$. X_∞ is thus the projective limit of the spaces X_n, so that the preimages $(\varrho_\infty^n)^{-1}(U)$ with 0-neighborhoods U in X_n and the restriction $\varrho_\infty^n : X_\infty \to X_n$ form a basis of the 0-neighborhoods in X_∞.

Theorem 2.1 ([**12**]). *If $P(\partial)$ is hypoelliptic, then the $\mathscr{D}'(\Omega)$-topology on $\mathscr{D}'_P(\Omega)$ coincides with the topology of uniform convergence of all partial derivatives on all compact sets.*

Before giving a simple proof inspired by [**4**] (this proof is a kind of dual version of Malgrange's arguments) let us remark that, for concrete operators like the Cauchy-Riemann, Laplace or heat operator, this theorem contains classical results of Vitali, Harnack and Täcklind, namely, if a sequence in the corresponding kernel converges uniformly on all compact sets, then the same is true for all partial derivatives (because the convergence in $\mathscr{C}(\Omega)$ implies the convergence in $\mathscr{D}'(\Omega)$ and thus in $\mathscr{E}(\Omega)$ by the theorem).

Proof. Since the topology induced by $\mathscr{E}(\Omega)$ is clearly finer than the $\mathscr{D}'(\Omega)$ topology we only have to prove the continuity of the inclusion $i : X_\infty \to \mathscr{E}(\Omega) = \operatorname{Proj}_{n\in\mathbf{N}}(\mathscr{E}(\Omega_n), \sigma_m^n)$ (where again, $\sigma_m^n : \mathscr{E}(\Omega_m) \to \mathscr{E}(\Omega_n)$ and $\sigma_\infty^n : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega_n)$ are the restrictions), and this is equivalent to the continuity of $\sigma_\infty^n \circ i : X_\infty \to \mathscr{E}(\Omega_n)$ for all $n \in \mathbf{N}$. If $r_{n+1} : X_{n+1} \to \mathscr{D}'(\Omega_n)$ again denotes the restriction, we have $r_{n+1} : X_{n+1} \to \mathscr{E}(\Omega_n)$ by hypoellipticity, and this map is continuous by the closed graph theorem since X_{n+1} is ultrabornological as a closed subspace of an LS-space. Since ϱ_∞^{n+1} is continuous, we obtain the continuity of $r_{n+1} \circ \varrho_\infty^{n+1} = \sigma_\infty^n \circ i$. □

As a supplement to this result, let us show that hypoellipticity is the only case where $\operatorname{Kern} P(\partial)$ is Fréchet:

Proposition 2.2. *If there is a Fréchet space topology on $\mathscr{D}'_P(\Omega)$ which is finer than the $\mathscr{D}'(\Omega)$-topology, then $P(\partial)$ is hypoelliptic and the given Fréchet topology coincides with the $\mathscr{E}(\Omega)$-topology.*

Consequently, the property that the kernel of $P(\partial) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is a Fréchet space is independent of Ω, i.e., if it holds for some open $\Omega \neq \emptyset$, then it holds for all such Ω.

Proof. With the notation introduced above we write $X_n = \mathrm{Ind}_{n\in\mathbf{N}} X_{n,N}$ where $X_{n,N}$ is the space of all $u \in X_n$ which are of order less than N (i.e., they are continuous with respect to the norm $\sup\{|\partial^\alpha f(x)| : x \in \mathbf{R}^d, |\alpha| \le N\}$ on $\mathscr{D}(\overline{\Omega}_n)$). Denoting the Fréchet topology of the proposition by $\mathcal{T}$ the continuity of $\varrho^n_\infty : (X_\infty, \mathcal{T}) \to X_n$ together with Grothendieck's factorization theorem [**13,** Theorem 24.33] gives $N \in \mathbf{N}$ such that $\varrho^n_\infty(X_\infty) \subseteq X_{n,N}$. In our concrete situation this means that for every $u \in \mathscr{D}'(\Omega)$ with $P(\partial)u = 0$ the restriction $u|_{\overline{\Omega}_n}$ has order less than N.

If Δ^k denotes a k-fold application of the Laplacian we have for $u \in X_\infty$ that $P(\partial)(\Delta^k u) = \Delta^k(P(\partial)u) = 0$ since $P(\partial)$ has constant coefficients. Therefore, $\Delta^k u|_{\overline{\Omega}_n}$ has order less than N for every $k \in \mathbf{N}$ which implies $u|_{\Omega_n} \in \mathscr{E}(\Omega_n)$. Since this is true for all $n \in \mathbf{N}$ we conclude $u \in \mathscr{E}(\Omega)$.

The coincidence of $\mathcal{T}$ and the $\mathscr{E}(\Omega)$-topology follows from the closed graph theorem. □

The proof used that $P(\partial)$ commutes with the Laplace operator which is the case for every convolution operator. Proposition 2.2 is false even for ordinary differential operators with non-constant coefficients. A very simple example is the multiplication operator $\mathscr{D}'(\mathbf{R}) \to \mathscr{D}'(\mathbf{R})$, $u \mapsto xu$ whose kernel is one-dimensional (and thus clearly Fréchet) but contains $\delta_0 \notin \mathscr{E}(\mathbf{R})$, see [**9,** Theorem 3.16]. Another example is the differential operator $u \mapsto u + xu'$ on $\mathscr{D}'(\mathbf{R})$ whose kernel is $\{\alpha\, pv(1/x) + \beta\delta_0 : \alpha, \beta \in \mathbf{C}\}$ where $pv(1/x)$ assigns to $\varphi \in \mathscr{D}(\mathbf{R})$ the principal value of the integral of $\varphi(x)/x$.

3. Consequences of the theory of LF-spaces. Beyond the case of hypoelliptic operators it seems that no results about topological properties of $\mathscr{D}'_P(\Omega)$ are explicitly stated in the literature, although a first important theorem can be obtained just by combining some classical results.

Theorem 3.1. *Let $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be a surjective, continuous linear operator. Then its kernel is ultrabornological.*

In particular, $\mathscr{D}'_P(\Omega)$ is ultrabornological for all polynomials if $\Omega \subseteq \mathbf{R}^d$ is convex or if $d = 1$ (in this trivial case it is finite dimensional). The case $d = 2$ is somehow special because a recent result **[10]** of Kalmes (confirming a conjecture of Trèves) says that $P(\partial)$ is already surjective on $\mathscr{D}'(\Omega)$ if it is surjective on $\mathscr{E}(\Omega)$.

Proof. Let $T^t : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ be the transposed operator which is a weak isomorphism onto $L = \operatorname{Range} T^t$ by the Hahn-Banach theorem. By a result of de Wilde **[2,** page 124] (a different proof is in **[23]**) T^t is then a strong isomorphism since $\mathscr{D}(\Omega)$ is an inductive limit of Fréchet-Schwartz spaces. A theorem of Palamodov **[15,** Section 7] then implies that the cokernel $\mathscr{D}(\Omega)/L$ of T^t is complete and that the quotient map $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)/L$ lifts bounded sets, that is, every bounded set in the quotient is contained in the image of a bounded set. Therefore, $\operatorname{Kern} T = (\mathscr{D}(\Omega)/L)'_\beta$ holds topologically, and the strong dual is ultrabornological by Schwartz's theorem **[19,** page 43] since $\mathscr{D}(\Omega)/L$ is a complete Schwartz space. $\square$

Using an earlier result from the author **[23]** one gets a kind of converse. A subspace of an LF-space is called a limit subspace if the relative topology is an LF-space topology. If the subspace is stepwise closed, which means that the intersection with every step of the LF-space is closed in that step, it is a limit subspace if and only if it is ultrabornological (this equivalence follows from the closed graph theorem for LF-spaces since there is a finer LF-space topology on the subspace).

Proposition 3.2. *Let $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be a continuous linear operator. Then* $\operatorname{Kern} T$ *is ultrabornological if and only if* $\overline{\operatorname{Range}(T^t)}$ *is a limit subspace of* $\mathscr{D}(\Omega)$.

Proof. Let us denote $L = \operatorname{Kern} T$, so that

$$\overline{\operatorname{Range}(T^t)} = \operatorname{Range}(T^t)^{\circ\circ} = (\operatorname{Kern} T)^\circ = L^\circ.$$

The dual sequence of $0 \to L \to \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)/L \to 0$ is

$$0 \longrightarrow L^\circ \longrightarrow \mathscr{D}(\Omega) \longrightarrow L' \longrightarrow 0,$$

which is algebraically exact by the Hahn-Banach theorem, and the restriction map $\mathscr{D}(\Omega) \to L'$ is continuous for the strong topologies. By Schwartz's theorem for the complete Schwartz space L, the strong dual

L'_β is ultrabornological and, by the open mapping theorem, $\mathscr{D}(\Omega) \to L'_\beta$ is open, so that L'_β is an inductive limit of Fréchet-Schwartz spaces.

If now L is ultrabornological, then L'_β is complete and, by **[23]**, L'_β is acyclic which implies that the kernel L° of $\mathscr{D}(\Omega) \to L'$ is a limit subspace.

Vice versa, if L° is a limit subspace, then $L'_\beta = \mathscr{D}(\Omega)/L^\circ$ is acyclic and hence complete by Palamodov's result already used in the proof of Theorem 3.1, so that $\operatorname{Kern} T = (\mathscr{D}(\Omega)/L^\circ)'_\beta$ is ultrabornological again by Schwartz's theorem. □

Corollary 3.3. *If $P(\partial) : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ is surjective, then $\mathscr{D}'_P(\Omega)$ is ultrabornological if and only if $P(\partial) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective.*

Note that a theorem of Malgrange **[12]** characterizes the surjectivity on $\mathscr{E}(\Omega)$ by P-convexity for supports, i.e., for every compact set $K \subseteq \Omega$ there is another compact set $M \subseteq \Omega$ such that every $u \in \mathscr{E}'(\Omega)$ has support in M whenever $P(-\partial)u$ has support in K (it is enough to verify this for $u \in \mathscr{D}(\Omega)$).

Proof. By a result of Floret **[3]**, Ω is P-convex for supports if and only if $\operatorname{Range}(P(\partial)^t)$ is closed in $\mathscr{D}(\Omega)$. If $\operatorname{Kern} P(\partial)$ is ultrabornological, the range is thus a limit subspace, and hence $P(\partial)^t$ is an isomorphism onto its range by the open mapping theorem. Hence, $P(\partial) = P(\partial)^{tt}$ is surjective on $\mathscr{D}'(\Omega)$. □

The main argument in the proof of Theorem 3.2 was that complete inductive limits of Fréchet-Schwartz spaces are acyclic. Again, by **[23]**, the same is true for locally complete limits. Thus, one can replace ultrabornologicity of $\operatorname{Kern} T$ by any condition which implies local completeness of the strong dual (which is in a certain sense, the weakest barrelledness condition called c_0-quasibarrelledness, see the book of Bonet and Pérez Carreras **[16,** Chapter 8.2]).

Theorem 3.4. *The following are equivalent for the kernel of a continuous linear operator $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$:*

(a) $\operatorname{Kern} T$ *is (ultra-) bornological,*
(b) $\operatorname{Kern} T$ *is (c_0-quasi-) barrelled.*

If Ω is not P-convex for supports, the characterization in Theorem 3.2 is not easily evaluable since no concrete description of the closure of Range(T^t) is known. In the next section we therefore abandon duality theory and directly investigate the projective limits $\mathscr{D}'_P(\Omega) = X_\infty = \operatorname{Proj} X_n$. The general theory is, in principle, dual to the theory of LF-spaces, but some aspects are easier to see on the projective side.

4. Consequences of the theory of the derived projective limit functor. Derived functors in functional analysis and, in particular, the derived projective limit functor were introduced in the late 1960's by Palamodov [**14, 15**] and from the late 1980's on further developed by many authors, we refer to [**25**] for a much broader introduction to this subject. Here, we will give only an ad hoc definition of Proj^1 and explain the connection with our problem. The following definition is for an arbitrary countable projective spectrum, but the reader should always have our concrete case with the notation introduced in Section 2 in mind.

For a spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ where the linear spectral maps $\varrho_m^n : X_m \to X_n$ satisfy $\varrho_m^n \circ \varrho_k^m = \varrho_k^n$ for $n \leq m \leq k$ and $\varrho_n^n = id_{X_n}$, the projective limit

$$X_\infty = \left\{ (x_n)_{n \in \mathbf{N}} \in \prod_{n \in \mathbf{N}} X_n : x_n = \varrho_m^n(x_m) \quad \text{for all } n \leq m \right\}$$

is the kernel of the difference map

$$\Delta_{\mathscr{X}} : \prod_{n \in \mathbf{N}} X_n \longrightarrow \prod_{n \in \mathbf{N}} X_n, (x_n)_{n \in \mathbf{N}} \longmapsto (x_n - \varrho_{n+1}^n(x_{n+1}))_{n \in \mathbf{N}},$$

and the cokernel $\prod_{n \in \mathbf{N}} X_n / \operatorname{Range}(\Delta_{\mathscr{X}})$ is denoted by $\operatorname{Proj}^1 \mathscr{X}$, so that $\operatorname{Proj}^1 \mathscr{X} = 0$ precisely means that $\Delta_{\mathscr{X}}$ is surjective. This simple definition somehow hides that Proj^1 is the derivative of the Proj-functor (which can be defined by injective resolutions). The following result recovers this origin and indicates the main types of applications. A morphism $f : \mathscr{X} \to \mathscr{Y}$ between two spectra is a sequence of linear maps $f_n : X_n \to Y_n$ commuting with the spectral maps, i.e., $\sigma_m^n \circ f_m = f_n \circ \varrho_m^n$ for all $n \leq m$. Then $f_\infty : X_\infty \to Y_\infty, (x_n)_{n \in \mathbf{N}} \mapsto (f_n(x_n))_{n \in \mathbf{N}}$ is well defined and, in fact, typically f_∞ is given in concrete situations and the morphism $f : \mathscr{X} \to \mathscr{Y}$ has to be constructed.

Proposition 4.1. *For every short exact sequence* $0 \to \mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z} \to 0$ *of projective spectra (i.e.,* $g_n : Y_n \to Z_n$ *is surjective and* f_n *is an isomorphism onto* $\operatorname{Kern} g_n$*) there is an exact sequence of vector spaces.*

$$0 \to X_\infty \xrightarrow{f_\infty} Y_\infty \xrightarrow{g_\infty} Z_\infty \xrightarrow{\delta^*} \operatorname{Proj}^1 \mathscr{X} \xrightarrow{f^*} \operatorname{Proj}^1 \mathscr{Y} \xrightarrow{g^*} \operatorname{Proj}^1 \mathscr{Z} \to 0.$$

The maps f^* and g^* are canonical (i.e., the class generated by $(x_n)_{n\in\mathbf{N}}$ is mapped to the class generated by $(f_n(x_n))_{n\in\mathbf{N}}$), and only the definition of δ^* needs a diagram chase.

Recall that exactness of the sequence of vector spaces means that the kernel of every outgoing arrow is the range of the corresponding incoming arrow. In particular, g_∞ is surjective whenever f^* is injective which is automatically the case if $\operatorname{Proj}^1 \mathscr{X} = 0$ (this is the most important case for applications). One might object that the question whether $\operatorname{Proj}^1 \mathscr{X} = 0$ is also a surjectivity problem which, in general, is as hard as a proof of the surjectivity of g^* (which in fact is equivalent whenever $\operatorname{Proj}^1 \mathscr{Y} = 0$). This, however, is not the case since there are highly developed criteria for $\operatorname{Proj}^1 \mathscr{X} = 0$ which are rather impossible to see in a concrete situation (for instance, the necessity in the theorem below is proved by applying Baire's theorem to the complete metric group $\prod_{n\in\mathbf{N}} X_n$ where each factor is endowed with the discrete metric, the sufficiency needs very abstract convexity, compactness and completeness considerations). The following characterization of $\operatorname{Proj}^1 \mathscr{X} = 0$ for LS-spectra is a conglomerate of results from Palamodov [**15**], Retah [**17**], Wengenroth [**23**], Frerick and Wengenroth [**5**], Braun and Vogt [**1**], and Langenbruch [**11**], we refer to [**25,** theorem 3.2.18] for the proof.

Theorem 4.2. *Let* $\mathscr{X} = (X_n, \varrho^n_m)$ *be a projective spectrum of* LS-*spaces* $X_n = \operatorname{Ind} X_{n,N}$ *with continuous spectral maps. Then* $\operatorname{Proj}^1 \mathscr{X} = 0$ *holds if and only if for all* $n \in \mathbf{N}$ *there exists* $m \geq n$ *such that, for all* $k \geq m$ *there exists* $N \in \mathbf{N}$ *such that*

$$\text{(R) } \varrho^n_m X_m \subseteq \overline{\varrho^n_k X_k} \quad \textit{and} \quad \text{(P3) } \varrho^n_m X_m \subseteq \varrho^n_k X_k + X_{n,N}.$$

The first condition is called *reducedness.* In our concrete situation it requires an approximation in $\mathscr{D}'(\overline{\Omega}_n)$ of all zero solutions on $\overline{\Omega}_m$ by zero solutions in $\overline{\Omega}_k$. The second condition means that the restriction to $\overline{\Omega}_n$ of every zero solution in $\overline{\Omega}_m$ can be decomposed into a zero solution on $\overline{\Omega}_k$ and a distribution on $\overline{\Omega}_n$ with fixed bound on the order. The notation (P3) is for historical reasons since (stronger) predecessors of the condition had been called (P1) and (P2) by Vogt.

As sufficient conditions (R) and (P3) are weakest possible but $\mathrm{Proj}^1 \mathscr{X} = 0$ implies stronger necessary conditions where $k \in \mathbf{N}$ is replaced by $k = \infty$, we call these conditions *strong reducedness* and $(\overline{P3})$, respectively (for the necessity part it is enough to assume that X_n are separated LB-spaces, no compactness is thus needed, see [**25,** Theorem 3.2.8]).

In order to demonstrate that conditions (R) and (P3) are indeed evaluable let us insert a simple approximation result for our concrete situation which provides a handy description of the "local spaces" X_n one has to work with.

Proposition 4.3. *Let $E \in \mathscr{D}'(\mathbf{R}^d)$ be a fixed fundamental solution of $P(\partial)$. Then the linear span of all restrictions to $\overline{\Omega}_n$ of translates $\delta_\xi * E$ with $\xi \notin \overline{\Omega}_n$ is dense in $X_n = \{u \in \mathscr{D}'(\overline{\Omega}_n) : P(\partial)u = 0\}$.*

For the Cauchy-Riemann operator $\partial_x + i\partial_y$ and $E = 1/(2\pi(x+iy))$, the proposition is a very simple instance of Runge's theorem.

Proof. By the Hahn-Banach theorem we have to show that every $\varphi \in \mathscr{D}(\overline{\Omega}_n)$ vanishing in all $\delta_\xi * E$ vanishes in X_n. Let $\check{E}$ be the symmetric distribution of E and $f = \check{E} * \varphi \in \mathscr{E}(\mathbf{R}^d)$. Since $\langle \delta_\xi * E, \varphi\rangle = 0$ for all $\xi \notin \overline{\Omega}_n$ we get $f(\xi) = 0$ for $\xi \notin \overline{\Omega}_n$, and hence $f \in \mathscr{D}(\overline{\Omega}_n)$ satisfies $P(-\partial)f = \varphi$. For $u \in X_n$, we thus get $\langle u, \varphi\rangle = \langle u, P(-\partial)f\rangle = \langle P(\partial)u, f\rangle = 0$. □

The connection between Proj^1 and our problem is the following theorem of Vogt, see [**25,** Theorem 3.3.4] where Vogt's unpublished proof is reproduced.

Theorem 4.4. *Let $\mathscr{X}$ be a projective spectrum of Hausdorff* LB*-spaces satisfying* $\mathrm{Proj}^1 \mathscr{X} = 0$*. Then $X_\infty = \mathrm{Proj}\, \mathscr{X}$ is ultrabornological.*

This theorem gives a different proof of Theorem 3.10. because, by Proposition 4.1, the surjectivity of T on $\mathscr{D}'(\Omega)$ implies $\operatorname{Proj}^1 \mathscr{X} = 0$ for the spectrum of the kernels.

A kind of converse of Vogt's theorem (obtained by duality similarly as in Proposition 3.2) is [**25,** Corollary 3.3.10]:

Theorem 4.5. *If $\mathscr{X}$ is a strongly reduced spectrum of* LS*-spaces, then* $\operatorname{Proj}^1 \mathscr{X} = 0$ *if and only if* $X_\infty = \operatorname{Proj} \mathscr{X}$ *is ultrabornological.*

At first sight, this theorem does not give a better result than Corollary 3.3 because of the following result:

Proposition 4.6. *For the spectrum $\mathscr{X}$ with $X_n = \{u \in \mathscr{D}'(\overline{\Omega}_n) : P(\partial)u = 0\}$, the following are equivalent*:

(a) *$\mathscr{X}$ is reduced,*
(b) *$\mathscr{X}$ is strongly reduced,*
(c) *Ω is P-convex for supports.*

Proof. We first show (a) implies (c). For every $n \in \mathbf{N}$, we thus have to find $m \in \mathbf{N}$ such that every $\varphi \in \mathscr{D}(\Omega)$ has support in $\overline{\Omega}_m$ whenever $P(-\partial)(\varphi)$ has support in $\overline{\Omega}_n$.

We take $m \geq n$ such that $\varrho^n_m X_m \subseteq \overline{\varrho^n_k X_k}$ for all $k \geq m$. Fix $\varphi \in \mathscr{D}(\Omega)$ such that $P(-\partial)(\varphi)$ has support in $\overline{\Omega}_n$, and take $k \geq m$ such that $\varphi \in \mathscr{D}(\overline{\Omega}_k)$. Then we have $P(-\partial)\varphi \in (\varrho^n_k X_k)^\circ$, the polar taken in $\mathscr{D}(\overline{\Omega}_n)$, since for $u \in X_k$ we have $\langle \varrho^n_k u, P(-\partial)\varphi\rangle = \langle u, P(-\partial)\varphi\rangle = \langle P(\partial)u, \varphi\rangle = 0$. We thus get $P(-\partial)\varphi \in (\varrho^n_m X_m)^\circ$. For $\xi \notin \overline{\Omega}_m$ and a fundamental solution E we have $E * \delta_\xi \in X_m$, and thus

$$0 = \langle E * \delta_\xi, P(-\partial)\varphi\rangle = \langle \delta_\xi, \varphi\rangle = \varphi(\xi)$$

so that $\operatorname{supp}\varphi \subseteq \overline{\Omega}_m$.

On the other hand, P-convexity for supports implies that the spectrum $\{u \in \mathscr{E}(\Omega_n) : P(\partial)u = 0\}$ is reduced (by the same duality as above since, by regularization, the convexity condition is the same for $\varphi \in \mathscr{E}'(\Omega)$ instead of $\varphi \in \mathscr{D}(\Omega)$), and it is therefore strongly reduced; this is a version of the abstract Mittag-Leffler procedure, see [**25,** Chapter 3]. For $u \in X_{m+1}$, a cut off function $\psi \in \mathscr{D}(\Omega_{m+1})$ which is 1 near

$\overline{\Omega}_m$ and an approximate identity e_k we can approximate $\varrho^n_{m+1}(u)$ by $\varrho^n_{m+1}(e_k * \psi u)$ which itself can be approximated by some $\varrho^n_\infty v$ with $v \in Y_\infty \subseteq X_\infty$. □

Although Theorems 4.4 and 4.5 do not give more information than Corollary 3.3 when applied to $\mathscr{D}'_P(\Omega)$, we will nevertheless get an improvement. We call $P(\partial)$ surjective modulo $\mathscr{E}$ on Ω if $T : \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$, $(u, f) \mapsto P(\partial)(u) - f$ is surjective. Hörmander **[8,** Theorems 10.7.6 and 10.7.8] proved that this condition is equivalent to P-convexity for *singular* supports. Trivially, Corollary 3.3 may be reformulated as follows:

> If $P(\partial)$ is surjective on $\mathscr{E}(\Omega)$, then $\mathscr{D}'_P(\Omega)$ is ultrabornological if and only if $P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω.

We can now relax the hypothesis of surjectivity on $\mathscr{E}(\Omega)$ by the weaker condition that every smooth right hand side of the equation $P(\partial)u = f$ is smoothly solvable whenever it is solvable with a distribution, that is, $\mathscr{E}(\Omega) \cap P(\partial)(\mathscr{D}'(\Omega)) = P(\partial)(\mathscr{E}(\Omega))$. Clearly, this condition is also satisfied by hypoelliptic operators. Unfortunately, we do not know much about this property. According to a result of Hörmander, $P(\partial)$ is surjective on $\mathscr{E}(\Omega)$ whenever *every* smooth function is distributionally solvable, but this property does not always hold individually (an example is the first partial derivative on $\mathbf{R}^2 \setminus \{0\}$ and $f(x) = \phi(x)/\|x\|$ with a test function ϕ which equals 1 in a neighborhood of the origin, since f defines a distribution on all $\mathbf{R}^2$ it is distributionally solvable).

Theorem 4.7. *Assume that $\mathscr{E}(\Omega) \cap P(\partial)(\mathscr{D}'(\Omega)) = P(\partial)(\mathscr{E}(\Omega))$ and that $P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω. Then $\mathscr{D}'_P(\Omega)$ is ultrabornological.*

Proof. The advantage of $T : \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$, $(u, f) \mapsto P(\partial)(u) - f$ compared to $P(\partial) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is that the canonical spectrum of $\operatorname{Kern} T = \{(u, f) \in D'(\Omega) \times \mathscr{E}(\Omega) : P(\partial)u = f\}$ is always strongly reduced: if $P(\partial)u = f$ holds in Ω_{n+1}, $\chi \in \mathscr{D}(\Omega_{n+1})$ is equal to 1 in a neighborhood of $\overline{\Omega}_n$, $e_n \in \mathscr{D}(\mathbf{R}^d)$ is an approximative unit and E is a fundamental solution of $P(\partial)$, then $(u_n, f_n) = (E * e_n * (\chi u), e_n * (\chi u))$ is a sequence in $\operatorname{Kern} T$ whose restrictions to $\overline{\Omega}_n$

converge to (u, f). Applying Proposition 4.1 and Theorem 4.5 to T we thus obtain that $P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω if and only if $\operatorname{Kern} T$ is ultrabornological, which is thus the case by our assumption. We now consider the short exact sequence

$$0 \longrightarrow \mathscr{E}_P(\Omega) \longrightarrow \mathscr{D}'_P(\Omega) \times \mathscr{E}(\Omega) \longrightarrow \operatorname{Kern} T \longrightarrow 0$$

defined by the maps $f \mapsto (f, f)$ and $(v, g) \mapsto (g - v, P(\partial)g)$. The surjectivity of the latter is implied by (and even equivalent to) the given assumption $\mathscr{E}(\Omega) \cap P(\partial)(\mathscr{D}'(\Omega)) = P(\partial)(\mathscr{E}(\Omega))$ (indeed, if $P(\partial)u = f$ and $P(\partial)g = f$ just put $v = g - u \in \mathscr{D}'_P(\Omega)$), and thus, the sequence is algebraically exact. It is even topologically exact: the first map is obviously an isomorphism onto its range, the second is clearly continuous and it is open by de Wilde's open mapping theorem for webbed spaces (see, e.g., [**13,** 24.30]) since $\operatorname{Kern} T$ is ultrabornological.

Since $\mathscr{E}_P(\Omega)$ is Fréchet and thus barrelled, the three space theorem for barrelled locally convex spaces of Dierolf and Roelcke [**18,** Theorem 2.6] implies that $\mathscr{D}'_P(\Omega) \times \mathscr{E}(\Omega)$ is also barrelled which implies barrelledness of $\mathscr{D}'_P(\Omega)$ and thus its ultrabornologicity by Theorem 3.4. □

It should be noted that being ultrabornological is *not* a three space property in the category of locally convex spaces and it is thus necessary to make the "detour" via barrelledness (this detour proves a general three space property for the category of PLS-spaces).

We will now introduce an object related to PLS-spaces which is not so easily seen on the inductive side. For a projective limit $X_\infty = \operatorname{Proj} \mathscr{X}$ of LS-spaces we define the *associated strongly reduced spectrum* $\widetilde{X}_n = \overline{\varrho^n_\infty X_\infty}$ which again consists of LS-spaces since this class is stable with respect to closed subspaces. Then $\operatorname{Proj} \mathscr{X} = \operatorname{Proj} \widetilde{\mathscr{X}}$, and since $\widetilde{\mathscr{X}}$ is strongly reduced, Theorems 4.2 and 4.4 imply that X_∞ is ultrabornological if and only if $\widetilde{\mathscr{X}}$ satisfies (P3).

Unfortunately, this is not very handy since, apparently, there is no concrete description similar to the one in Proposition 4.3. Nevertheless, we obtain the following sufficient condition:

Proposition 4.8. *If an* LS-*spectrum* $\mathscr{X}$ *satisfies* $(\overline{\text{P3}})$, *i.e., for all* $n \in \mathbf{N}$ *there exist* $m \geq n$, $N \in \mathbf{N}$, *such that* $\varrho^n_m(X_m) \subseteq \varrho^n_\infty(X_\infty) + X_{n,N}$, *then* X_∞ *is ultrabornological.*

Proof. We have to show (P3) for the associated strongly reduced spectrum. Since $\widetilde{X}_m \subseteq X_m$ and $\varrho^n_m(\widetilde{X}_m) \subseteq \widetilde{X}_n$, we get

$$\varrho^n_m(\widetilde{X}_m) \subseteq \varrho^n_m(X_m) \cap \widetilde{X}_n \subseteq (\varrho^n_\infty(X_\infty) + X_{n,N}) \cap \widetilde{X}_n.$$

For any element $x = \varrho^n_\infty(a) + b \in \widetilde{X}_n$ of the right hand side we have $b = x - \varrho^n_\infty(a) \in X_{n,N} \cap \widetilde{X}_n$ since $\varrho^n_\infty(a) \in \widetilde{X}_n$. Hence, $\varrho^n_m(\widetilde{X}_m) \subseteq \varrho^n_\infty X_\infty + (X_{n,N} \cap \widetilde{X}_n)$.

It remains to note that $X_{n,N} \cap \widetilde{X}_n$ is contained in some step of the LS-space $\widetilde{X}_n = \operatorname{Ind}(\widetilde{X}_n \cap X_{n,N})$ which is true for any closed subspace of the LS-space X_n. □

Let us note that Proposition 4.8 contains both previous results Theorems 2.1 and 3.1 as special cases (in the hypoelliptic case one can even drop the summand $\varrho^n_\infty X_\infty$ and in the surjective case we have $\operatorname{Proj}^1 \mathscr{X} = 0$ which implies ($\overline{\text{P3}}$) as mentioned after Theorem 4.2).

We do not know whether (P3) and ($\overline{\text{P3}}$) are the same conditions in our situation. This, of course, would be a very desirable result because (P3) is equivalent to a very classical condition. Indeed, using [**7**], it is not hard to show the following characterization:

Theorem 4.9. *For $P(\partial) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ the following conditions are equivalent*:

(a) *The kernel spectrum satisfies* (P3),
(b) *$P(\partial)$ has good fundamental solutions in the following sense*: *for all $n \in \mathbf{N}$, there exists $m \geq n$ such that, for all $k \geq m$ and $\xi \notin \overline{\Omega}_m$, there is $E \in \mathscr{D}'(\Omega)$ with*

$$P(\partial)E = \delta_\xi \quad \text{in } \Omega_k \text{ and } E|_{\Omega_n} \in \mathscr{C}^k(\Omega_n),$$

(c) *$P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω,*
(d) *Ω is P-convex for singular supports.*

As mentioned above, the equivalence of (c) and (d) is due to Hörmander [**8**]. A functional analytic proof of (c) ⇔ (d) based on LF-space theory is in [**6**].

In a similar manner, one can characterize ($\overline{\text{P3}}$) by almost the same condition as (b) but requiring that $P(\partial)E = \delta_\xi$ in all of Ω and not only in Ω_k. For more information about "good fundamental solutions" as in (b) we refer to **[7, 24]**.

The similarity between (P3) and ($\overline{\text{P3}}$) together with Proposition 4.8 support the conjecture that $\mathscr{D}'_P(\Omega)$ might be ultrabornological whenever $P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω. Even the converse of this might be true but beyond the situation in Corollary 3.3 where Ω is assumed to be P-convex for supports, but we do not have further evidence.

We believe that a better understanding of the condition $\mathscr{E}(\Omega) \cap P(\partial)(\mathscr{D}'(\Omega)) = P(\partial)(\mathscr{E}(\Omega))$ should be useful also beyond our topological problem. A concrete question is whether it is always satisfied if $P(\partial)$ is surjective modulo $\mathscr{E}$ on Ω.

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