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ON 2- AND 4-DISSECTIONS FOR SOME INFINITE PRODUCTS

ERNEST X.W. XIA AND X.M. YAO

ABSTRACT. The 2- and 4-dissections of some infinite products are established in this paper. As corollaries of our results, we derive the 4-dissections of some continued fractions appearing in Ramanujan's notebooks and their reciprocals.

1. Introduction and main results. Throughout this paper, we let $|q| < 1$. We use the standard notation

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$$

and often write

$$(a_1, a_2, \dots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

The Ramanujan theta function is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.1}$$

where $|ab| < 1$. The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [**5**]

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{1.2}$$

A special case of (1.1) is

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

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For n positive, we denote $f(-q^n)$ by f_n in this paper for convenience.

Recall that the Rogers-Ramanujan continued fraction is defined by

$$R(q) = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+\cdots} = \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}.$$

This identity was first established by Rogers [**16**]. Ramanujan [**14**] gave 2-dissections of this continued fraction and its reciprocal, and these were first proved by Andrews [**2**]. Ramanujan [**14**] also gave 5-dissections of $R(q)$ and its reciprocal, and these results were improved upon and proved by Hirschhorn [**8**]. In the same paper, Hirschhorn conjectured formulas for 4-dissections of $R(q)$ and its reciprocal, and these were first proved by Lewis and Liu [**12**]. Hirschhorn [**10**] also gave an elementary proof of his conjecture.

Gordon's continued fraction is

$$G(q) = 1 + q + \frac{q^2}{1+q^3+} \frac{q^4}{1+q^5+} \frac{q^6}{1+q^7+} \cdots = \frac{(q^3, q^5; q^8)_\infty}{(q, q^7; q^8)_\infty}.$$

This identity was established by Gordon [**7**]. Hirschhorn [**9**] established 8-dissections of $G(q)$ and its reciprocal, thereby demonstrating the periodicity of the sign of the coefficients in expansions of $G(q)$ and its reciprocal, and in particular that certain coefficients are zero, a phenomenon first observed and shown by Richmond and Szekeres [**15**]. Alladi and Gordon [**1**], Andrews and Bressoud [**3**] and Chan and Yesilyurt [**6**] generalized these themes. Recently, Xia and Yao [**20**] proved Hirschhorn's results by an iterative method.

Ramanujan's cubic continued fraction is defined by

$$RC(q) = \frac{1}{1+} \frac{q+q^2}{1+} \frac{q^2+q^4}{1+} \frac{q^3+q^6}{1+} \cdots = \frac{(q, q^5; q^6)_\infty}{(q^3, q^3; q^6)_\infty}. \tag{1.3}$$

This identity was first established by Ramanujan [**14**]. 2- and 4-dissections of $1/(RC(q))$ were first given by Srivastava [**18**]. Hirschhorn and Roselin [**11**] also obtained the 2-, 3-, 4- and 6-dissections of Ramanujan's cubic continued fraction and its reciprocal.

The objective of this paper is to establish 4-dissections of some infinite products. As corollaries of our results, we obtain 4-dissections of the reciprocal of Ramanujan-Selberg continued fraction, a continued fraction introduced by Vasuki, Bhaskar and Sharath [**19**], Ramanujan's cubic continued fraction and its reciprocal. We also discovered the periodicity of the sign of the coefficients in the expansion of the reciprocal of Ramanujan-Selberg continued fraction. Our main results can be stated as follows.

Theorem 1.1. *Let m be a positive number and u an odd number. If $m \equiv 0 \pmod 4$, we have*

$$\begin{aligned}(1.4)\quad & \frac{(q^{2u},q^{m-2u};q^m)_\infty}{(q^u,q^{m-u};q^m)_\infty}\\ &= \frac{(q^{2m+4u},q^{6m-4u};q^{8m})_\infty}{(q^m;q^m)_\infty}(-q^{6m-4u},-q^{10m+4u},q^{16m};q^{16m})_\infty\\ &\quad + q^u\frac{(q^{2m-4u},q^{6m+4u};q^{8m})_\infty}{(q^m;q^m)_\infty}(-q^{6m+4u},-q^{10m-4u},q^{16m};q^{16m})_\infty\\ &\quad - q^{m-2u}\frac{(q^{2m+4u},q^{6m-4u};q^{8m})_\infty}{(q^m;q^m)_\infty}(-q^{2m+4u},-q^{14m-4u},q^{16m};q^{16m})_\infty\\ &\quad - q^{m+3u}\frac{(q^{2m-4u},q^{6m+4u};q^{8m})_\infty}{(q^m;q^m)_\infty}(-q^{2m-4u},-q^{14m+4u},q^{16m};q^{16m})_\infty.\end{aligned}$$

If $m \equiv 2 \pmod 4$, we have

$$\begin{aligned}(1.5)\quad & \frac{(q^{2u},q^{m-2u};q^m)_\infty}{(q^u,q^{m-u};q^m)_\infty}\\ &= \frac{(q^{m-2u},q^{m+2u};q^{2m})_\infty(-q^{m+2u},-q^{3m-2u};q^{4m})_\infty(-q^{6m},-q^{10m},q^{16m};q^{16m})_\infty}{(q^{2m},q^{2m},q^{4m};q^{4m})_\infty}\\ &\quad + q^u\frac{(q^{m-2u},q^{m+2u};q^{2m})_\infty(-q^{m-2u},-q^{3m+2u};q^{4m})_\infty(-q^{6m},-q^{10m},q^{16m};q^{16m})_\infty}{(q^{2m},q^{2m},q^{4m};q^{4m})_\infty}\\ &\quad + q^m\frac{(q^{m-2u},q^{m+2u};q^{2m})_\infty(-q^{m+2u},-q^{3m-2u};q^{4m})_\infty(-q^{2m},-q^{14m},q^{16m};q^{16m})_\infty}{(q^{2m},q^{2m},q^{4m};q^{4m})_\infty}\\ &\quad + q^{m+u}\frac{(q^{m-2u},q^{m+2u};q^{2m})_\infty(-q^{m-2u},-q^{3m+2u};q^{4m})_\infty(-q^{2m},-q^{14m},q^{16m};q^{16m})_\infty}{(q^{2m},q^{2m},q^{4m};q^{4m})_\infty}.\end{aligned}$$

The Ramanujan-Selberg continued fraction is

$$(1.6)\qquad S(q) = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+}\cdots = \frac{(q;q^2)_\infty}{(q^2,q^2;q^4)_\infty}.$$

Independently, Ramanujan [**13**] and Selberg [**17**] discovered this interesting continued fraction. Setting $u=1$ and $m=4$ in (1.4), we obtain the 4-dissection of $1/(S(q))$.

Corollary 1.2. *We have*

$$\begin{aligned}\frac{1}{S(q)} &= \frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty} \\ &= \frac{(-q^{20},-q^{44};q^{64})_\infty}{(q^4,q^8,q^{16},q^{24},q^{28};q^{32})_\infty(q^{32};q^{64})_\infty} \\ &\quad + \frac{q(-q^{28},-q^{36};q^{64})_\infty}{(q^8,q^{12},q^{16},q^{20},q^{24};q^{32})_\infty(q^{32};q^{64})_\infty} \\ &\quad - \frac{q^2(-q^{12},-q^{52};q^{64})_\infty}{(q^4,q^8,q^{16},q^{24},q^{28};q^{32})_\infty(q^{32};q^{64})_\infty} \\ &\quad - \frac{q^7(-q^{12},-q^{52};q^{64})_\infty}{(q^8,q^{12},q^{16},q^{20},q^{24};q^{32})_\infty(q^{32};q^{64})_\infty}.\end{aligned} \tag{1.7}$$

From (1.7), we immediately obtain the following corollary on the periodicity of the signs of the coefficients of $S^{-1}(q)$:

Corollary 1.3. *Let*

$$S^{-1}(q) = \frac{(q^2;q^4)_\infty^2}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} d_n q^n.$$

For $n \geq 0$, we have

$$d_{4n} \geq 0, \quad d_{4n+1} \geq 0, \quad d_{4n+2} \leq 0, \quad d_{4n+7} \leq 0. \tag{1.8}$$

Vasuki, Bhaskar and Sharath [**19**] introduced the following continued fraction

$$\begin{aligned}X(q) &= \frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty} \\ &= \frac{q^{1/2}(1-q^2)}{(1-q^{3/2})+}\,\frac{(1-q^{1/2})(1-q^{7/2})}{q^{1/2}(1-q^{3/2})(1+q^3)+}\,\frac{(1-q^{5/2})(1-q^{13/2})}{q^{3/2}(1-q^{3/2})(1+q^6)+}\cdots.\end{aligned}$$

Taking $u = 1$ and $m = 6$ in (1.5), we obtain the 4-dissection of $1/(X(q))$.

Corollary 1.4. *We have*

$$
\begin{aligned}
(1.9)\quad & \frac{(q^2,q^4;q^6)_\infty}{(q,q^5;q^6)_\infty} \\
&= \frac{(q^4,q^8;q^{12})_\infty(-q^8,-q^{16};q^{24})_\infty(-q^{36},-q^{60},q^{96};q^{96})_\infty}{(q^{12},q^{12},q^{24};q^{24})_\infty} \\
&\quad + q\frac{(q^4,q^8;q^{12})_\infty(-q^4,-q^{20};q^{24})_\infty(-q^{36},-q^{60},q^{96};q^{96})_\infty}{(q^{12},q^{12},q^{24};q^{24})_\infty} \\
&\quad + q^6\frac{(q^4,q^8;q^{12})_\infty(-q^8,-q^{16};q^{24})_\infty(-q^{12},-q^{84},q^{96};q^{96})_\infty}{(q^{12},q^{12},q^{24};q^{24})_\infty} \\
&\quad + q^7\frac{(q^4,q^8;q^{12})_\infty(-q^4,-q^{20};q^{24})_\infty(-q^{12},-q^{84},q^{96};q^{96})_\infty}{(q^{12},q^{12},q^{24};q^{24})_\infty}.
\end{aligned}
$$

Theorem 1.5. *Let m and u be positive integers. If $m \equiv 2 \pmod 4$ and u is odd, we have*

$$
\begin{aligned}
(1.10)\quad & \frac{(q^{3u},q^{m-3u};q^m)_\infty}{(q^u,q^{m-u};q^m)_\infty} \\
&= \frac{(q^{2m-4u},q^{2m+4u};q^{4m})_\infty(q^{m-2u},q^{m+2u};q^{2m})_\infty}{(q^{2m};q^{4m})_\infty^4} \\
&\quad \times \left((-q^{m+2u},-q^{3m-2u};q^{4m})_\infty^2 + q^{2u}(-q^{m-2u},-q^{3m+2u};q^{4m})_\infty^2\right) \\
&\quad + q^u\frac{(q^{4m};q^{4m})_\infty^4}{(q^{2m};q^{2m})_\infty^4}\left((q^{2m-4u},q^{2m+4u};q^{4m})_\infty^2 - q^{m-4u}(q^{4u},q^{4m-4u};q^{4m})_\infty^2\right).
\end{aligned}
$$

Letting $u = 1$ and $m = 6$ in (1.10), we obtain the 4-dissection of the reciprocal of the Ramanujan cubic continued fraction.

Corollary 1.6. *We have*

$$
(1.11)\quad \frac{1}{RC(q)} = \frac{(q^3;q^6)_\infty^2}{(q,q^5;q^6)_\infty} = \frac{(q^4,q^{20};q^{24})_\infty(q^{16},q^{32};q^{48})_\infty^2}{(q^{12};q^{24})_\infty^4} + q\frac{(q^8,q^{16};q^{24})_\infty^2}{(q^{12};q^{24})_\infty^4}
$$

$$+ q^2 \frac{(q^8, q^{16}, -q^4, -q^{20}, -q^4, -q^{20}; q^{24})_\infty (q^4, q^8; q^{12})_\infty}{(q^{12}; q^{24})_\infty^4}$$
$$- q^3 \frac{(q^4, q^{20}; q^{24})_\infty^2}{(q^{12}; q^{24})_\infty^4}.$$

Theorem 1.7. *Let m and u be positive integers. If $m \equiv 2 \pmod 4$ and u is odd, we have*

$$\begin{aligned}
(1.12) \quad & \frac{(q^u, q^{m-u}; q^m)_\infty}{(q^{3u}, q^{m-3u}; q^m)_\infty} \\
&= \frac{(q^{4u}, q^{2m-4u}, q^{m+2u}, q^{m-2u}; q^{2m})_\infty (-q^{m+6u}, -q^{3m-6u}; q^{4m})_\infty^2}{(q^{2m}; q^{4m})_\infty^4 (q^{12u}, q^{4m-12u}; q^{4m})_\infty} \\
&\quad + q^{6u} \frac{(q^{4u}, q^{2m-4u}, q^{m+2u}, q^{m-2u}; q^{2m})_\infty (-q^{m-6u}, -q^{3m+6u}; q^{4m})_\infty^2}{(q^{2m}; q^{4m})_\infty^4 (q^{12u}, q^{4m-12u}; q^{4m})_\infty} \\
&\quad - q^u \Bigg(\frac{(-q^{m-2u}, -q^{3m+2u}; q^{4m})_\infty (-q^{m+6u}, -q^{3m-6u}; q^{4m})_\infty^3}{(q^{2m}; q^{4m})_\infty^6 (q^{12u}, q^{4m-12u}; q^{4m})_\infty} \\
&\quad - q^{8u} \frac{(-q^{m+2u}, -q^{3m-2u}; q^{4m})_\infty (-q^{m-6u}, -q^{3m+6u}; q^{4m})_\infty^3}{(q^{2m}; q^{4m})_\infty^6 (q^{12u}, q^{4m-12u}; q^{4m})_\infty} \Bigg) \\
&\quad + q^{3u} \frac{(-q^{m-6u}, -q^{m+6u}, q^{4u}, q^{2m-4u}, q^{m-2u}, q^{m+2u}; q^{2m})_\infty}{(q^{2m}; q^{4m})_\infty^4 (q^{12u}, q^{4m-12u}; q^{4m})_\infty}.
\end{aligned}$$

Taking $u = 1$ and $m = 6$ in (1.12), we obtain the 4-dissection of $RC(q)$.

Corollary 1.8.

$$\begin{aligned}
(1.13) \quad \frac{(q, q^5; q^6)_\infty}{(q^3; q^6)_\infty^2} &= \frac{(q^4; q^4)_\infty^2 (q^{24}; q^{24})_\infty^{14}}{(q^{12}; q^{12})_\infty^{12} (q^{48}; q^{48})_\infty^4} \\
&\quad - q \Bigg(\frac{(-q^4, -q^{20}; q^{24})_\infty (-q^{12}; q^{24})_\infty^6}{(q^{12}; q^{24})_\infty^8} \\
&\quad - 8q^8 \frac{(-q^8, -q^{16}; q^{24})_\infty (-q^{24}; q^{24})_\infty^6}{(q^{12}; q^{24})_\infty^8} \Bigg)
\end{aligned}$$

$$+ 4q^6 \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^2 (q^{48};q^{48})_\infty^4}{(q^{12};q^{12})_\infty^8}$$
$$+ 2q^3 \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^8}{(q^{12};q^{12})_\infty^{10}}.$$

Note that Hirschhorn and Roselin [**11**] also derived the following 4-dissection of $RC(q)$:

(1.14)
$$\frac{(q,q^5;q^6)_\infty}{(q^3;q^6)_\infty^2} = \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^{14}}{(q^{12};q^{12})_\infty^{12} (q^{48};q^{48})_\infty^4}$$
$$- q\left(\frac{(q^8;q^8)_\infty^4 (q^{24};q^{24})_\infty^4}{(q^{12};q^{12})_\infty^8} + 4\frac{(q^4,;q^4)_\infty^4 (q^{24};q^{24})_\infty^4}{(q^8;q^8)_\infty^4 (q^{12};q^{12})_\infty^4} \right)$$
$$+ 4q^6 \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^2 (q^{48};q^{48})_\infty^4}{(q^{12};q^{12})_\infty^8}$$
$$+ 2q^3 \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty^8}{(q^{12};q^{12})_\infty^{10}},$$

which is different from (1.13).

2. Proof of Theorem 1.1. In order to prove our main theorems, we will require the following three identities, which were proved by Berndt [**4**, pages 45–47], respectively.

Lemma 2.1. *We have*

(2.1)
$$f(a,b) = f(a^3b, ab^3) + af(b/a, a^5b^3),$$

(2.2)
$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c, ac^2d)f(b/d, acd^2),$$

(2.3)
$$f(a,b)f(c,d)f(an,b/n)f(cn,d/n)$$
$$- f(-a,-b)f(-c,-d)f(-an,-b/n)f(-cn,-d/n)$$
$$= 2af(c/a,ad)f(d/(an),acn)f(n,ab/n)f(ab,a^3b^3),$$

where $ab = cd$.

Now we are ready to prove Theorem 1.1.

Proof. We first establish the 2-dissection of $((q^{2u},q^{m-2u};q^m)_\infty)/((q^u,q^{m-u};q^m)_\infty)$. Taking $a=q^u$ and $b=q^{m-u}$ in (2.1), we have

$$f(q^u,q^{m-u})=f(q^{m+2u},q^{3m-2u})+q^uf(q^{m-2u},q^{3m+2u}). \tag{2.4}$$

By (2.4), we can derive the 2-dissection of $(q^{2u},q^{m-2u};q^m)_\infty/(q^u,q^{m-u};q^m)_\infty$. It is a routine to verify that

$$\begin{aligned}\frac{(q^{2u},q^{m-2u};q^m)_\infty}{(q^u,q^{m-u};q^m)_\infty}&=\frac{f_{2m}f(-q^{2u},-q^{m-2u})f(q^u,q^{m-u})}{f_m^2f(-q^{2u},-q^{2m-2u})}\\&=\frac{f_{2m}f(-q^{2u},-q^{m-2u})}{f_m^2f(-q^{2u},-q^{2m-2u})}\big(f(q^{m+2u},q^{3m-2u})\\&\qquad+q^uf(q^{m-2u},q^{3m+2u})\big)\\&=A(q^2)+q^uB(q^2),\end{aligned} \tag{2.5}$$

where

$$A(q)=\frac{(q^{m/2-u},q^{m/2+u};q^m)_\infty(-q^{m/2+u},-q^{3m/2-u},q^{2m};q^{2m})_\infty}{(q^{m/2};q^{m/2})_\infty}$$

and

$$B(q)=\frac{(q^{m/2-u},q^{m/2+u};q^m)_\infty(-q^{m/2-u},-q^{3m/2+u},q^{2m};q^{2m})_\infty}{(q^{m/2};q^{m/2})_\infty}.$$

We first consider the case $m\equiv 0 \pmod 4$. In (2.1), let $a=-q^{m/2-u}$ and $b=-q^{3m/2+u}$, we see that

$$f(-q^{m/2-u},-q^{3m/2+u})=f(q^{3m-2u},q^{5m+2u})-q^{m/2-u}f(q^{m+2u},q^{7m-2u}). \tag{2.6}$$

From (2.6), it is easy to verify that

$$\begin{aligned}A(q)&=\frac{f(-q^{m+2u},-q^{3m-2u})}{f_{4m}f_{m/2}}f(-q^{m/2-u},-q^{3m/2+u})\\&=\frac{f(-q^{m+2u},-q^{3m-2u})}{f_{4m}f_{m/2}}\Big(f(q^{3m-2u},q^{5m+2u})\\&\qquad-q^{m/2-u}f(q^{m+2u},q^{7m-2u})\Big).\end{aligned} \tag{2.7}$$

On the other hand, in (2.6), replacing u by $-u$, we have

(2.8)
$$\begin{aligned} B(q) &= \frac{f(-q^{m-2u},-q^{3m+2u})}{f_{4m}f_{m/2}} f(-q^{m/2+u},-q^{3m/2-u}) \\ &= \frac{f(-q^{m-2u},-q^{3m+2u})}{f_{4m}f_{m/2}} \Big(f(q^{3m+2u},q^{5m-2u}) \\ &\qquad - q^{m/2+u} f(q^{m-2u},q^{7m+2u}) \Big). \end{aligned}$$

In view of (2.5), (2.7) and (2.8), we obtain (1.4).

It remains to consider the case $m \equiv 2 \pmod 4$. In (2.1), taking $a = q^{m/2}$ and $b = q^{3m/2}$, we have

$$f(q^{m/2},q^{3m/2}) = f(q^{3m},q^{5m}) + q^{m/2} f(q^m,q^{7m}). \tag{2.9}$$

It follows from (2.9) that

(2.10)
$$\begin{aligned} A(q) &= \frac{f(-q^{m/2-u},-q^{m/2+u}) f(q^{m/2+u},q^{3m/2-u})}{f_m^3} f(q^{m/2},q^{3m/2}) \\ &= \frac{f(-q^{m/2-u},-q^{m/2+u}) f(q^{m/2+u},q^{3m/2-u})}{f_m^3} \\ &\quad \times \Big(f(q^{3m},q^{5m}) + q^{m/2} f(q^m,q^{7m}) \Big). \end{aligned}$$

Similarly, by (2.9), we have

(2.11)
$$\begin{aligned} B(q) &= \frac{f(-q^{m/2-u},-q^{m/2+u}) f(q^{m/2-u},q^{3m/2+u})}{f_m^3} f(q^{m/2},q^{3m/2}) \\ &= \frac{f(-q^{m/2-u},-q^{m/2+u}) f(q^{m/2-u},q^{3m/2+u})}{f_m^3} \\ &\quad \times \Big(f(q^{3m},q^{5m}) + q^{m/2} f(q^m,q^{7m}) \Big). \end{aligned}$$

Formula (1.5) follows from (2.5), (2.10) and (2.11). The proof is complete. □

3. Proof of Theorem 1.5. In this section, we turn to prove Theorem 1.5.

Proof. We also first derive the 2-dissection of $(q^{3u}, q^{m-3u}; q^m)_\infty / (q^u, q^{m-u}; q^m)_\infty$. In (2.2), letting $a = q^u$, $b = q^{m-u}$, $c = -q^{3u}$ and $d = -q^{m-3u}$, we have

$$\begin{aligned}(3.1)\quad & f(q^u, q^{m-u}) f(-q^{3u}, -q^{m-3u}) \\ &= f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u}) \\ &\quad + q^u f(-q^{m-4u}, q^{m+4u}) f(-q^{2u}, -q^{2m-2u}).\end{aligned}$$

By (3.1), it is easy to verify that

(3.2)

$$\begin{aligned}\frac{(q^{3u}, q^{m-3u}; q^m)_\infty}{(q^u, q^{m-u}; q^m)_\infty} &= \frac{f_{2m} f(-q^{3u}, -q^{m-3u}) f(q^u, q^{m-u})}{f_m^2 f(-q^{2u}, -q^{2m-2u})} \\ &= \frac{f_{2m}}{f_m^2} \Bigg(\frac{f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u})}{f(-q^{2u}, -q^{2m-2u})} \\ &\qquad + q^u f(-q^{m-4u}, -q^{m+4u}) \Bigg) \\ &= C(q^2) + q^u D(q^2),\end{aligned}$$

where

$$C(q) = \frac{(q^{2u}, q^{m-2u}, q^{m/2-u}, q^{m/2+u}; q^m)_\infty}{(q^{m/2}, q^{m/2}, q^u, q^{m-u}; q^m)_\infty}$$

and

$$D(q) = \frac{(q^{m/2-2u}, q^{m/2+2u}; q^m)_\infty}{(q^{m/2}, q^{m/2}; q^m)_\infty}.$$

Taking $a = q^u$, $b = q^{m-u}$, $c = q^{m/2}$ and $d = q^{m/2}$ in (2.2), we obtain

$$\begin{aligned}(3.3)\quad & f(q^u, q^{m-u}) f(q^{m/2}, q^{m/2}) \\ &= f^2(q^{m/2+u}, q^{3m/2-u}) + q^u f^2(q^{m/2-u}, q^{3m/2+u}).\end{aligned}$$

Therefore, by (3.3), we have

(3.4)

$$\begin{aligned}C(q) &= \frac{(q^{2u}, q^{m-2u}, q^{m/2-u}, q^{m/2+u}, -q^{m/2}, -q^{m/2}, -q^u, -q^{m-u}; q^m)_\infty}{(q^m, q^m, q^{2u}, q^{2m-2u}; q^{2m})_\infty} \\ &= \frac{f_{2m}^2 f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-u}, -q^{m/2+u})}{f_m^4 f(-q^m, -q^m) f(-q^{2u}, -q^{2m-2u})} \\ &\quad \times f(q^{m/2}, q^{m/2}) f(q^u, q^{m-u}) \\ &= \frac{f_{2m}^2 f(-q^{2u}, -q^{m-2u}) f(-q^{(m/2)-u}, -q^{(m/2)+u})}{f_m^4 f(-q^m, -q^m) f(-q^{2u}, -q^{2m-2u})} \\ &\quad \times \left(f^2(q^{(m/2)+u}, q^{(3m/2)-u}) + q^u f^2(q^{(m/2)-u}, q^{(3m/2)+u}) \right).\end{aligned}$$

Similarly, letting $a = -q^{m/2-2u}$, $b = -q^{m/2+2u}$, $c = q^{m/2}$ and $d = q^{m/2}$ in (2.2), we see that

$$\begin{aligned} (3.5)\quad & f(-q^{(m/2)-2u}, -q^{(m/2)+2u}) f(q^{m/2}, q^{m/2}) \\ & = f^2(-q^{m-2u}, -q^{m+2u}) - q^{(m/2)-2u} f^2(-q^{2u}, -q^{2m-2u}), \end{aligned}$$

which implies that

(3.6)

$$\begin{aligned} D(q) &= \frac{(q^{m/2-2u}, q^{m/2+2u}, -q^{m/2}, -q^{m/2}; q^m)_\infty}{(q^m, q^m; q^{2m})_\infty} \\ &= \frac{f_{2m}^2}{f_m^4} f(q^{m/2}, q^{m/2}) f(-q^{m/2-2u}, -q^{m/2+2u}) \\ &= \frac{f_{2m}^2}{f_m^4} \left(f^2(-q^{m-2u}, -q^{m+2u}) - q^{m/2-2u} f^2(-q^{2u}, -q^{2m-2u}) \right). \end{aligned}$$

Thus, combining (3.2), (3.4) and (3.6), we obtain (1.10). This completes the proof. □

4. Proof of Theorem 1.7. Now we turn to prove Theorem 1.7.

Proof. We first establish 2-dissection of $(q^u, q^{m-u}; q^m)_\infty/(q^{3u}, q^{m-3u}; q^m)_\infty$. Taking $a = -q^u$, $b = -q^{m-u}$, $c = q^{3u}$ and $d = q^{m-3u}$ in (2.2), we have

$$\begin{aligned} (4.1)\quad & f(-q^u, -q^{m-u}) f(q^{3u}, q^{m-3u}) \\ & = f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u}) \\ & \quad - q^u f(-q^{m-4u}, -q^{m+4u}) f(-q^{2u}, -q^{2m-2u}). \end{aligned}$$

In view of (4.1), we see that

(4.2)

$$\begin{aligned} \frac{(q^u, q^{m-u}; q^m)_\infty}{(q^{3u}, q^{m-3u}; q^m)_\infty} &= \frac{f_{2m} f(-q^u, -q^{m-u}) f(q^{3u}, q^{m-3u})}{f_m^2 f(-q^{6u}, -q^{2m-6u})} \\ &= \frac{f_{2m} \left(f(-q^{4u}, -q^{2m-4u}) f(-q^{m-2u}, -q^{m+2u}) \right.}{f_m^2 f(-q^{6u}, -q^{2m-6u})} \\ & \quad - \frac{\left. q^u f(-q^{m-4u}, -q^{m+4u}) f(-q^{2u}, -q^{2m-2u}) \right)}{f_m^2 f(-q^{6u}, -q^{2m-6u})}. \end{aligned}$$

Let

$$E(q) = \frac{f_m f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-u}, -q^{m/2+u})}{f_{m/2}^2 f(-q^{3u}, -q^{m-3u})}$$

and

$$F(q) = \frac{f_m f(-q^{m/2-2u}, -q^{m/2+2u}) f(-q^{u}, -q^{m-u})}{f_{m/2}^2 f(-q^{3u}, -q^{m-3u})}.$$

Hence, by (4.2), we see that

$$\frac{(q^u, q^{m-u}; q^m)_\infty}{(q^{3u}, q^{m-3u}; q^m)_\infty} = E(q^2) - q^u F(q^2). \tag{4.3}$$

Taking $a = q^{3u}$, $b = q^{m-3u}$, $c = q^{m/2}$ and $d = q^{m/2}$ in (2.2), we have

$$\begin{aligned} f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) \\ = f^2(q^{m/2+3u}, q^{3m/2-3u}) + q^{3u} f^2(q^{m/2-3u}, q^{3m/2+3u}). \end{aligned} \tag{4.4}$$

Therefore, we have

(4.5)

$$\begin{aligned} E(q) &= \frac{f_{2m}^3 f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-u}, -q^{m/2+u})}{f_m^6 f(-q^{6u}, -q^{2m-6u})} \\ &\quad \times f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) \\ &= \frac{f_{2m}^3 f(-q^{2u}, -q^{m-2u}) f(-q^{m/2-u}, -q^{m/2+u})}{f_m^6 f(-q^{6u}, -q^{2m-6u})} \\ &\quad \times \left(f^2(q^{m/2+3u}, q^{3m/2-3u}) + q^{3u} f^2(q^{m/2-3u}, q^{3m/2+3u}) \right). \end{aligned}$$

Setting $a = -q^u$, $b = -q^{m-u}$, $c = -q^{m/2-2u}$, $d = -q^{m/2+2u}$ and $n = -q^{m/2-u}$ in (2.3), we have

(4.6)

$$\begin{aligned} &f(-q^u, -q^{m-u}) f(-q^{m/2-2u}, -q^{m/2+2u}) f(q^{m/2}, q^{m/2}) f(q^{3u}, q^{m-3u}) \\ &\quad - f(q^u, q^{m-u}) f(q^{m/2-2u}, q^{m/2+2u}) f(-q^{m/2}, -q^{m/2}) f(-q^{3u}, -q^{m-3u}) \\ &= -2q^u f(q^{m/2-3u}, q^{m/2+3u}) f(-q^{2u}, -q^{m-2u}) \\ &\quad \times f(-q^{m/2-u}, -q^{m/2+u}) f(q^m, q^{3m}). \end{aligned}$$

Taking $a=-q^u$, $b=-q^{m-u}$, $c=-q^{m/2-2u}$ and $d=-q^{m/2+2u}$ in (2.2), we have

$$\begin{aligned}(4.7)\quad f(-q^u,-q^{m-u})f(-q^{m/2-2u},-q^{m/2+2u}) &= f(q^{m/2-u},q^{3m/2+u})f(q^{m/2+3u},q^{3m/2-3u}) \\ &\quad - q^u f(q^{m/2+u},q^{3m/2-u})f(q^{m/2-3u},q^{3m/2+3u}).\end{aligned}$$

In view of (4.4) and (4.7), we have

(4.8)
$$\begin{aligned}f(-q^u,&-q^{m-u})f(-q^{m/2-2u},-q^{m/2+2u})f(q^{m/2},q^{m/2})f(q^{3u},q^{m-3u}) \\ &+ f(q^u,q^{m-u})f(q^{m/2-2u},q^{m/2+2u}) \\ &\times f(-q^{m/2},-q^{m/2})f(-q^{3u},-q^{m-3u}) \\ &= 2f(q^{(m/2)-u},q^{(3m/2)+u})f^3(q^{(m/2)+3u},q^{(3m/2)-3u}) \\ &- 2q^{4u}f(q^{(m/2)+u},q^{(3m/2)-u})f^3(q^{(m/2)-3u},q^{(3m/2)+3u}).\end{aligned}$$

It follows from (4.6) and (4.8) that

(4.9)
$$\begin{aligned}f(-q^u,&-q^{m-u})f(-q^{m/2-2u},-q^{m/2+2u})f(q^{m/2},q^{m/2})f(q^{3u},q^{m-3u}) \\ &= f(q^{(m/2)-u},q^{(3m/2)+u})f^3(q^{(m/2)+3u},q^{(3m/2)-3u}) \\ &- q^{4u}f(q^{(m/2)+u},q^{(3m/2)-u})f^3(q^{(m/2)-3u},q^{(3m/2)+3u}) \\ &- q^u f(q^{m/2-3u},q^{m/2+3u})f(-q^{2u},-q^{m-2u}) \\ &\times f(-q^{m/2-u},-q^{m/2+u})f(q^m,q^{3m}).\end{aligned}$$

Therefore, we have

(4.10)
$$\begin{aligned}F(q) &= \frac{f_{2m}^3 f(-q^u,-q^{m-u})f(-q^{m/2-2u},-q^{m/2+2u})f(q^{m/2},q^{m/2})f(q^{3u},q^{m-3u})}{f_m^6 f(-q^{6u},-q^{2m-6u})} \\ &= \frac{f_{2m}^3}{f_m^6 f(-q^{6u},-q^{2m-6u})}\Big(f(q^{(m/2)-u},q^{(3m/2)+u})f^3(q^{(m/2)+3u},q^{(3m/2)-3u}) \\ &- q^{4u}f(q^{(m/2)+u},q^{(3m/2)-u})f^3(q^{(m/2)-3u},q^{(3m/2)+3u}) \\ &- q^u f(q^{m/2-3u},q^{m/2+3u})f(-q^{2u},-q^{m-2u}) \\ &\times f(-q^{m/2-u},-q^{m/2+u})f(q^m,q^{3m})\Big).\end{aligned}$$

It follows from (4.2), (4.5) and (4.10) that

$$\begin{aligned}\frac{(q^u,q^{m-u};q^m)_\infty}{(q^{3u},q^{m-3u};q^m)_\infty} &= \frac{f_{4m}^3 f(-q^{4u},-q^{2m-4u})f(-q^{m-2u},-q^{m+2u})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})} f^2(q^{m+6u},q^{3m-6u}) \\ &+ q^{6u}\frac{f_{4m}^3 f(-q^{4u},-q^{2m-4u})f(-q^{m-2u},-q^{m+2u})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})} \\ &\times f^2(q^{m-6u},q^{3m+6u}) \\ &- q^u\left(\frac{f_{4m}^3 f(q^{m-2u},q^{3m+2u})f^3(q^{m+6u},q^{3m-6u})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})}\right. \\ &\left. - q^{8u}\frac{f(q^{m+2u},q^{3m-2u})f^3(q^{m-6u},q^{3m+6u})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})}\right) \\ &+ q^{3u}\frac{f_{4m}^3 f(q^{m-6u},q^{m+6u})f(-q^{4u},-q^{2m-4u})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})} \\ &\times \frac{f(-q^{m-2u},-q^{m+2u})f(q^{2m},q^{6m})}{f_{2m}^6 f(-q^{12u},-q^{4m-12u})},\end{aligned}$$

which is nothing but (1.12). The proof is complete. $\square$

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Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China
Email address: Ernestxwxia@163.com

Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China
Email address: yaoxiangmei@163.com