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ON THE SPECTRUM OF SPHERICAL DIRAC-TYPE OPERATORS

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Dedicated to Professor Henri Moscovici on the occasion of his 65th birthday

ABSTRACT. We use polynomial Dirac spinors associated to Euclidean Dirac-type operators and separation of variables to investigate the spectral theory of certain spherical Dirac-type operators. While the spectral theories of our main examples, the spherical Dirac and Laplace-Beltrami operators, are known, this is the first time they are treated together, in a unified manner. In particular, the multiplicities of these spectra, a topic difficult to negotiate in many previous treatments, are presented in simple closed form.

0. Introduction. The spectral theory of the geometric (Dirac and Laplace-Beltrami) spherical operators has been addressed in the literature quite often, by a variety of methods pertaining to representation theory, complex analysis, spin geometry, harmonic analysis, etc. Here is a list of papers, by no means exhaustive, devoted to the topic: [**5, 22, 23**] for the Dirac operator, and [**11, 12, 16, 17, 20**] for the Laplace-Beltrami operator.

Interestingly enough, there is no simultaneous treatment of the spectral theories of spherical Dirac and Laplace-Beltrami operators, despite the fact that they belong to the same family of Dirac-type operators. In this paper we set out to accomplish this, by what was probably the first method used to tackle such problems, separation of variables in Euclidean spaces in the presence of spherical harmonics. We pay particular attention to the multiplicity of their spectra, an aspect that proved delicate in many of the previous references. While our paper is mainly expository, it does strive to make a point not known or not yet

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believed: when it comes to *any* spherical Dirac-type spectral problem, the oldest method of approach is still the best.

1. Euclidean Dirac operators and polynomial spinors. For an integer $n \geq 1$, consider a representation of the real Clifford algebra $\mathrm{Cl}_{n+1,0}$ on some finite-dimensional Hermitian vector space $\mathbf{V}$. This is equivalent to the prescription of $n+1$ skew-Hermitian endomorphisms of $\mathbf{V}$, $E_0, E_1, \dots, E_n$, which are Clifford in the sense that, for every i, $E_i^2 = -Id$, and $E_iE_j + E_jE_i = 0$ for every $i \neq j$. Then, in $\mathbf{R}^{n+1}$ with coordinates $(x_0, x_1, \dots, x_n)$ the Euclidean Dirac operator associated to $\mathbf{V}$ is the differential operator

$$\not{D} : C^\infty(U, \mathbf{V}) \longrightarrow C^\infty(U, \mathbf{V}), \quad U \subseteq \mathbf{R}^{n+1} \text{ open},$$

defined, for spinors $s \in C^\infty(U, \mathbf{V})$,

$$s = \sum_{\alpha=1}^{\dim \mathbf{V}} f_\alpha s_\alpha, \qquad (s_\alpha)_{\alpha=1}^{\dim \mathbf{V}}$$

some fixed basis of $\mathbf{V}$, $f_\alpha \in C^\infty(U, \mathbf{C})$, by

$$\not{D}s = \sum_{i=0}^{n} E_i \frac{\partial s}{\partial x_i}, \tag{1}$$

where $\partial s/\partial x_i$ represents ordinary component-wise differentiation of s with respect to x_i. It is easily seen from (1) that $\not{D}$ is a first order elliptic differential operator satisfying the following properties:

(2)
$$\not{D}(fs) = \operatorname{grad} f \cdot s + f\not{D}s, \quad f \in C^\infty(U, \mathbf{C}),\ \operatorname{grad} f \cdot s := \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} E_i s$$

(3)
$$\not{D}^2 = -\Delta, \text{ where } \Delta \text{ is the component-wise Laplacian on } C^\infty(U, \mathbf{V}).$$

Now denote by $P^k(\mathbf{V})$, $k = 0, 1, 2, \dots$, the subspace of $C^\infty(\mathbf{R}^{n+1}, \mathbf{V})$ consisting in spinors with polynomial components, homogeneous of degree k, in some (and therefore any) basis of $\mathbf{V}$, and by $H^k(\mathbf{V})$ the subspace of $P^k(\mathbf{V})$ consisting in *polynomial Dirac spinors*, i.e.,

$$H^k(\mathbf{V}) := \{h_k \in P^k(\mathbf{V}) \mid \not{D} h_k = 0\}.$$

Clearly, $\not{D}(P^k(\mathbf{V})) \subseteq P^{k-1}(\mathbf{V})$ $(P^{-1}(\mathbf{V}) = 0)$. If one denotes by $x\cdot$ Clifford multiplication in $\mathbf{V}$ by $x \in \mathbf{R}^{n+1}$, i.e.,

$$x \cdot v = \sum_{i=0}^{n} x_i E_i v, \quad v \in \mathbf{V},$$

then $x \cdot P^k(\mathbf{V}) \subseteq P^{k+1}(\mathbf{V})$. The following splitting, known as the *Fischer decomposition* [**2, 10**], holds true:

$$P^k(\mathbf{V}) = H^k(\mathbf{V}) \oplus x \cdot P^{k-1}(\mathbf{V}).$$

Since

$$\dim P^k(\mathbf{V}) = \dim(\mathbf{V}) \binom{n+k}{k},$$

we conclude that

$$\dim H^k(\mathbf{V}) = \dim(\mathbf{V}) \binom{n+k-1}{k},$$

and, therefore, by a dimension argument, $\not{D}(P^k(\mathbf{V})) = P^{k-1}(\mathbf{V})$.

Also, equation (1) implies that, if $p_k \in P^k(\mathbf{V})$, then

$$\not{D}(x \cdot p_k) + x \cdot \not{D} p_k = -(n+1+2k) p_k. \tag{4}$$

Consequently, $h_k \in P^k(\mathbf{V})$ is in $H^k(\mathbf{V})$ if and only if the components of $x \cdot h_k$ are harmonic polynomials.

For later use, we record here the structure of homogeneous Dirac spinors on the punctured Euclidean space $\mathbf{R}^{n+1} \setminus \{0\}$, that is of the elements $s \in C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V})$ such that $\not{D} s = 0$ and for which there is some $\alpha \in \mathbf{R}$ such that

$$s(x) = |x|^\alpha s\left(\frac{x}{|x|}\right), \quad x \in \mathbf{R}^{n+1} \setminus \{0\},\ |x| = \sqrt{\sum_{i=0}^{n} x_i^2}.$$

Proposition 1. *Let* $s \in C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V})$ *be a non-zero homogeneous Dirac spinor for some Euclidean Dirac operator* $\not{D}$*, with constant of homogeneity* $\alpha \in \mathbf{R}$*. Then* α *belongs to two families,* k *and* $-n-k$*,* $k = 0, 1, 2, \dots$ *.*

If $\alpha = k$*, then* $x = 0$ *is a removable singularity for* s *and* $s \in H^k(\mathbf{V})$*.*

If $\alpha = -n-k$*, then there is an* $h_k \in H^k(\mathbf{V})$ *such that*

$$s(x) = \frac{x\cdot h_k(x)}{|x|^{2k+n+1}}, \quad x \neq 0.$$

Proof. Since the components of s with respect to any basis of $\mathbf{V}$ are (homogeneous) harmonic functions, we can make use of the structure theorem for harmonic functions on punctured Euclidean spaces [**4**, page 209]. According to this theorem there is some non-negative integer m and some spinor $s_m \in C^\infty(\mathbf{R}^{n+1}, \mathbf{V})$ with harmonic components, homogeneous of degree m, such that either $\alpha = m$ and $s = s_{m_{|\mathbf{R}^{n+1}\setminus\{0\}}}$ or $\alpha = -n-m+1$ and $s(x) = (s_m(x))/|x|^{2m+n-1}$, $x \neq 0$.

In the first case, $x = 0$ is a removable singularity for s, and by elliptic regularity, $s_m \in H^m(\mathbf{V})$.

In the second case, we first represent $s_m(x)$ uniquely as $h_m(x) + x\cdot h_{m-1}(x)$, $h_m \in H^m(\mathbf{V})$, $h_{m-1} \in H^{m-1}(\mathbf{V})$. This can be accomplished by making use of a Fischer decomposition applied to homogeneous spinors with harmonic components [**2**, page 123]. Then a simple calculation relying on equations (2) and (4) implies that

$$s(x) = \frac{h_m(x) + x\cdot h_{m-1}(x)}{|x|^{2m+n-1}}$$

satisfies $\not{D}s = 0$ for $x \neq 0$ if and only if $h_m = 0$. Therefore,

$$s(x) = \frac{x\cdot h_{m-1}(x)}{|x|^{2m+n-1}}, \quad x \neq 0.$$

Notice that $m = 0$ cannot accommodate non-zero spinors s since $P^{-1}(\mathbf{V}) = 0$, and so $h_{-1} = 0$.

Proposition 1 now follows by letting $k = m$ if $\alpha = m$ and $k = m-1$ if $\alpha = -n-m+1$, $m \geq 1$. $\square$

Of interest to us will be the Euclidean Dirac operators associated to a particular type of *graded* actions of $\mathrm{Cl}_{n+1,0}$ on $\mathbf{V}$, in the sense that there is a vector space direct sum decomposition ($\mathbf{Z}$-grading)

$$\begin{gathered} \mathbf{V} = \oplus_{q=0}^{p+1} \mathbf{V}_q, \quad p \geq 0, \text{such that} \\ \mathbf{V}_q \neq 0, \quad E_i(\mathbf{V}_q) \subset \mathbf{V}_{q-1} \oplus \mathbf{V}_{q+1}, \\ i = 0, 1, \dots, n, \; q = 0, 1, \dots, p+1, \; (\mathbf{V}_{-1} = \mathbf{V}_{p+2} = 0), \end{gathered} \tag{5}$$

and such that the summands $\mathbf{V}_q$ are mutually orthogonal with respect to the Hermitian product of $\mathbf{V}$.

The spaces $H^k(\mathbf{V}_q) := C^\infty(\mathbf{R}^{n+1}, \mathbf{V}_q) \cap H^k(\mathbf{V})$ of $\mathbf{V}_q$-valued polynomial Dirac spinors of degree k will play a key role in our development. $H^0(\mathbf{V}_q) \simeq \mathbf{V}_q$, however, in general it is a non-trivial task to figure out even the dimensions of these vector spaces (in terms of n, k, and the dimension of $\mathbf{V}_q$). We will indicate here a derivation of these dimensions under additional hypotheses, satisfied by the two basic examples we have in mind.

To this end, we restrict the Dirac operator $\not{D}$ to $P(\mathbf{V}) := \oplus_{k=0}^{\infty} P^k(\mathbf{V})$. The $\mathbf{Z}$-grading of $\mathbf{V}$ allows one to further write

$$P(\mathbf{V}) = \oplus_{k,q} P^k(\mathbf{V}_q) = \oplus_{k,q} (P^k \otimes \mathbf{V}_q), \tag{6}$$

where, by P^k, we denote now the space of ordinary complex polynomials in variables $x_0, x_1, \dots, x_n$, homogeneous of degree k. If $E_i = E_i^+ + E_i^-$ is the natural splitting of Clifford endomorphisms E_i, $E_i^\pm(\mathbf{V}_*) \subset \mathbf{V}_{*\pm 1}$ (the signs correspond, here and further below), then clearly $(E_i^\pm)^* = -E_i^\mp$ and, for any $s \in P(\mathbf{V})$, we have $\not{D}s = \not{D}^+ s + \not{D}^- s$, where

$$\not{D}^\pm s := \sum_{i=0}^{n} E_i^\pm \frac{\partial s}{\partial x_i}.$$

Also, for $x \in \mathbf{R}^{n+1}$ and $v \in \mathbf{V}$, we can define $x^\pm \cdot v := \sum_{i=0}^n x_i E_i^\pm v$, and so $x \cdot v = x^+ \cdot v + x^- \cdot v$.

$P(\mathbf{V})$ carries a useful positive definite Hermitian product $\langle \cdot, \cdot \rangle$, under which the decomposition (6) is orthogonal and which on $P^k(\mathbf{V}_q) = P^k \otimes \mathbf{V}_q$ equals the tensor product Hermitian product induced by those of P^k and $\mathbf{V}$. On P^k, we decree monomials corresponding to different multi-indices orthogonal, and for any multi-index $I = (i_0, i_1, \dots i_n)$, $|I| := i_0 + i_1 + \cdots + i_n = k$, and $x^I := x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$, we set $\langle x^I, x^I \rangle = i_0! i_1! \cdots i_n!$. For instance, in this Hermitian product,

$$(7) \qquad \left\langle \frac{\partial p_1}{\partial x_i}, p_2 \right\rangle = \langle p_1, x_i p_2 \rangle, \quad p_1, p_2 \in P(\mathbf{V}),\ i = 0, 1, \dots, n.$$

The following pairs of relations involving $\not{D}^{\pm}$ and $x^{\pm}\cdot$ are easily verified on $P(\mathbf{V})$:

$$(8) \qquad \left(\not{D}^{\pm}\right)^2 = 0, \qquad \left(x^{\pm}\cdot\right)^2 = 0, \qquad \not{D}^{\pm} x^{\pm}\cdot + \ x^{\pm}\cdot \not{D}^{\pm} = 0.$$

Clearly, with respect to the Hermitian product $\langle \cdot, \cdot \rangle$ of $P(\mathbf{V})$, we have

$$(9) \qquad \left\langle \not{D}^{\pm} p_1, p_2 \right\rangle = -\left\langle p_1, x^{\mp}\cdot p_2 \right\rangle, \quad p_1, p_2 \in P(\mathbf{V}).$$

With the notation $H^k_{\pm}(\mathbf{V}_q) := \ker(P^k(\mathbf{V}_q) \overset{\not{D}^{\pm}}{\to} P^{k-1}(\mathbf{V}_{q\pm 1}))$, we now have the following

Proposition 2. *If, for any fixed* $k = 0, 1, 2, \dots$ *and* $q = 0, 1, 2, \dots, p+1$*, the two complexes*

$$(10) \qquad 0 \to H^k_{\pm}(\mathbf{V}_q) \hookrightarrow P^k(\mathbf{V}_q) \overset{\not{D}^{\pm}}{\to} P^{k-1}(\mathbf{V}_{q\pm 1}) \overset{\not{D}^{\pm}}{\to} \cdots \overset{\not{D}^{\pm}}{\to} P^0(\mathbf{V}_{q\pm k}) \to 0$$

are exact $(\ker(P^{k-j}(\mathbf{V}_{q\pm j}) \overset{\not{D}^{\pm}}{\to} P^{k-j-1}(\mathbf{V}_{q\pm j\pm 1})) = \operatorname{im}(P^{k-j+1}(\mathbf{V}_{q\pm j\mp 1}) \overset{\not{D}^{\pm}}{\to} P^{k-j}(\mathbf{V}_{q\pm j}))$, $j = 1, 2, \dots, k$, *where* $\mathbf{V}_l = 0$ *for* $l \neq 0, 1, \dots, p+1)$, *then*

$$(11) \qquad \dim H^k(\mathbf{V}_q) = \sum_{j=-k}^{k} (-1)^j \dim\left(\mathbf{V}_{q+j}\right) \binom{n+k-|j|}{k-|j|}.$$

Proof. Obviously, $H^k(\mathbf{V}_q) = H^k_{+}(\mathbf{V}_q) \cap H^k_{-}(\mathbf{V}_q)$. The exactness of the complexes (10) implies that

$$(12) \qquad \dim H^k_{\pm}(\mathbf{V}_q) = \sum_{j=0}^{k} (-1)^j \dim\left(\mathbf{V}_{q\pm j}\right) \binom{n+k-j}{k-j}.$$

From equation (9), we also have $H^k_\pm(\mathbf{V}_q) = \operatorname{im}(P^{k-1}(\mathbf{V}_{q\pm1}) \overset{x^\mp\cdot}{\to} P^k(\mathbf{V}_q))^\perp$, where the orthogonal complements are taken inside $P^k(\mathbf{V}_q)$, so

$$H^k_\pm(\mathbf{V}_q)^\perp = \operatorname{im}(P^{k-1}(\mathbf{V}_{q\pm1}) \overset{x^\mp\cdot}{\to} P^k(\mathbf{V}_q)). \tag{13}$$

Consequently,

$$\begin{aligned} H^k(\mathbf{V}_q)^\perp &= \left(H^k_+(\mathbf{V}_q) \cap H^k_-(\mathbf{V}_q)\right)^\perp = H^k_+(\mathbf{V}_q)^\perp + H^k_-(\mathbf{V}_q)^\perp \\ &= \operatorname{im}(P^{k-1}(\mathbf{V}_{q+1}) \overset{x^-\cdot}{\to} P^k(\mathbf{V}_q)) \\ &\quad + \operatorname{im}(P^{k-1}(\mathbf{V}_{q-1}) \overset{x^+\cdot}{\to} P^k(\mathbf{V}_q)). \end{aligned}$$

We want to show that the sum $\operatorname{im}(P^{k-1}(\mathbf{V}_{q+1}) \overset{x^-\cdot}{\to} P^k(\mathbf{V}_q)) + \operatorname{im}(P^{k-1}(\mathbf{V}_{q-1}) \overset{x^+\cdot}{\to} P^k(\mathbf{V}_q))$ is in fact direct. Indeed, if $s_q \in \operatorname{im}(P^{k-1}(\mathbf{V}_{q+1}) \overset{x^-\cdot}{\to} P^k(\mathbf{V}_q)) \cap \operatorname{im}(P^{k-1}(\mathbf{V}_{q-1}) \overset{x^+\cdot}{\to} P^k(\mathbf{V}_q))$, then $s_q = x^-\cdot s_{q+1} = x^+\cdot s_{q-1}$, where $s_{q\pm1} \in P^{k-1}(\mathbf{V}_{q\pm1})$, and so $x\cdot s_q = x^+\cdot s_q + x^-\cdot s_q = (x^+\cdot)^2 s_{q-1} + (x^-\cdot)^2 s_{q+1} = 0$, which implies $s_q = 0$. Therefore,

$$\begin{aligned} P^k(\mathbf{V}_q) = H^k(\mathbf{V}_q) &\oplus \operatorname{im}(P^{k-1}(\mathbf{V}_{q+1}) \overset{x^-\cdot}{\to} P^k(\mathbf{V}_q)) \\ &\oplus \operatorname{im}(P^{k-1}(\mathbf{V}_{q-1}) \overset{x^+\cdot}{\to} P^k(\mathbf{V}_q)). \end{aligned} \tag{14}$$

Equations (13) and (14) finally give $\dim H^k(\mathbf{V}_q) = \dim H^k_+(\mathbf{V}_q) + \dim H^k_-(\mathbf{V}_q) - \dim P^k(\mathbf{V}_q)$, and from equation (12),

$$\begin{aligned} \dim H^k(\mathbf{V}_q) &= \dim(\mathbf{V}_q)\binom{n+k}{k} \\ &\quad + \sum_{j=1}^{k}(-1)^j\left(\dim(\mathbf{V}_{q+j}) + \dim(\mathbf{V}_{q-j})\right)\binom{n+k-j}{k-j}, \end{aligned}$$

which is the same as equation (11). □

Example 1. The classical Dirac-type operators. Hypothesis (10) of Proposition 2 holds for any Euclidean Dirac operator associated to a $\mathbf{Z}_2$-grading representation $\mathbf{V}$, that is, a $\mathbf{Z}$-grading such that $p = 0$.

Indeed, if $p = 0$, $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$, and so $E_i(\mathbf{V}_0) = \mathbf{V}_1$ and $E_i(\mathbf{V}_1) = \mathbf{V}_0$, $i = 0, 1, \dots, n$. Consequently, for $k = 0, 1, \dots$, $\not{D}(P^k(\mathbf{V}_0)) = P^{k-1}(\mathbf{V}_1)$, $\not{D}(P^k(\mathbf{V}_1)) = P^{k-1}(\mathbf{V}_0)$, $H^k_+(\mathbf{V}_0) = H^k(\mathbf{V}_0)$, $H^k_+(\mathbf{V}_1) = P^k(\mathbf{V}_1)$, $H^k_-(\mathbf{V}_0) = P^k(\mathbf{V}_0)$, $H^k_-(\mathbf{V}_1) = H^k(\mathbf{V}_1)$ and $H^k(\mathbf{V}) = H^k(\mathbf{V}_0) \oplus H^k(\mathbf{V}_1)$. Obviously, this is equivalent to the exactness of the complexes (10), and then Proposition 2 gives

$$\dim H^k(\mathbf{V}_0) = \dim H^k(\mathbf{V}_1) = \frac{1}{2}\dim(\mathbf{V})\binom{n+k-1}{k}.$$

Notice that any ungraded representation $\mathbf{V}$ can be viewed as a $\mathbf{Z}_2$-graded representation on $\mathbf{V}\oplus\mathbf{V}$ by setting $E_i(v_1 \oplus v_2) = E_i(v_2) \oplus E_i(v_1)$, $(\mathbf{V} \oplus \mathbf{V})_0 = \mathbf{V} \oplus 0$, and $(\mathbf{V} \oplus \mathbf{V})_1 = 0 \oplus \mathbf{V}$.

The most prominent specialization of Example 1 is the classical Dirac operator, corresponding to an irreducible ungraded representation $\mathbf{V}$ of $\mathrm{Cl}_{n+1,0}$. Here

$$\mathbf{V} = \mathbf{C}^{2^{[n/2]+1}}, \qquad \mathbf{V}_0 \simeq \mathbf{C}^{2^{[n/2]}} \times 0, \qquad \mathbf{V}_1 \simeq 0 \times \mathbf{C}^{2^{[n/2]}},$$

and the relevant Clifford endomorphisms $E_0, E_1, \dots, E_n$ are square matrices of size $2^{[n/2]+1}$ constructed [**7**, page 52] out of appropriate tensor products involving the basic complex 2×2 matrices

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix},$$
$$W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This construction is naturally $\mathbf{Z}_2$-graded if n is odd and ungraded but viewed as $\mathbf{Z}_2$-graded as explained above, if n is even.

Therefore, for the classical Euclidean Dirac operator, we have

$$\dim H^k(\mathbf{V}_0) = \dim H^k(\mathbf{V}_1) = 2^{[n/2]}\binom{n+k-1}{k}. \tag{15}$$

Another specialization is the signature operator $d + d^*$ defined on differential forms on $\mathbf{R}^{n+1}$, viewed as the Dirac operator associated to the Clifford representation on the exterior algebra $\mathbf{V} = \Lambda(\mathbf{R}^{n+1})$ [**8**] with grading $\mathbf{V}_0 = \Lambda^{\mathrm{even}}(\mathbf{R}^{n+1})$ and $\mathbf{V}_1 = \Lambda^{\mathrm{odd}}(\mathbf{R}^{n+1})$. However, a finer grading associated to this representation, taking into account the actual degree of a form, is more useful and will be specialized in the next example under the name of Gauss-Bonnet operator.

Example 2. The Gauss-Bonnet-type operators. Hypothesis (10) of Proposition 2 holds for an Euclidean Dirac operator associated to a $\mathbf{Z}$-graded representation $\mathbf{V}$ such that $E_i^+E_j^- + E_j^-E_i^+ = 0$ for every $i \neq j$.

Indeed, these extra anti-commutation relations imposed on the $\pm$-Clifford matrices $E_i^\pm$ allow us to add to the relations (8) another pair:

$$\not{D}^\pm x^\mp \cdot + \ x^\mp \cdot \not{D}^\pm = -(\Lambda + L^\pm), \tag{16}$$

where $\Lambda := \sum_{i=0}^n x_i(\partial/\partial x_i)$ is the 'polynomial degree operator,' i.e., $\Lambda = k$ on $P^k(\mathbf{V})$, and $L^\pm$ are '$\mathbf{Z}$-grading compatible positive $\mathbf{V}$-operators' induced by the (same name) endomorphisms of $\mathbf{V}$

$$L^\pm v = -\sum_{i=0}^n E_i^\pm E_i^\mp v, \quad v \in \mathbf{V}.$$

Notice that $L^\pm$ are Hermitian and positive with respect to the Hermitian product of $\mathbf{V}$, and leave the summands $\mathbf{V}_q$ invariant. As such, all eigenvalues of $L^\pm$ are non-negative and all the corresponding eigenspaces are subordinated to the decomposition (5). Since $L^+L^- = L^-L^+$, L^+ and L^- are simultaneously diagonalizable. In fact, the identity $L^+ + L^- = (n+1)Id$ shows that they carry identical information.

Notice also that, since $\not{D}^\pm L^\pm = (L^\pm - Id)\not{D}^\pm$ it suffices to test the exactness of the complexes (10) on polynomial spinors which are eigensections for $\Lambda + L^\pm$. Equations (16) then prove this exactness, with the possible exception of the slots $P^k(\mathbf{V}_q)$ where $\Lambda + L^\pm$ would have the eigenvalue 0. These slots could only be $P^0(\mathbf{V}_0)$ in a $\not{D}^+$-complex or $P^0(\mathbf{V}_{q+1})$ in a $\not{D}^-$-complex; however, the range of index j appearing in equation (10) excludes them. This proves that a representation as is Example 2 satisfies the conclusions of Proposition 2.

The notable specialization of Example 2 involves the Gauss-Bonnet operator, i.e., the Euclidean Dirac operator associated to the representation of $\mathrm{Cl}_{n+1,0}$ on itself ($\mathbf{V} = \mathrm{Cl}_{n+1,0} \otimes \mathbf{C}$) by left multiplication. If $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$, are generators of $\mathrm{Cl}_{n+1,0}$ such that $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = -2\delta_{ij}$, then an orthonormal basis of $\mathbf{V}_q$, $q = 0, 1, 2, \dots, n+1$, is

$$\{\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q} \mid 0 \le i_1 < i_2 < \cdots < i_q \le n\}, \quad \dim(\mathbf{V}_q) = \binom{n+1}{q},$$

$$\begin{aligned} E_i(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q}) &= \varepsilon_i\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q}, \\ E_i^+(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q}) &= \begin{cases} \varepsilon_i\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q} & \text{if } i \notin \{i_1, i_2, \dots, i_q\}, \\ 0 & \text{otherwise,} \end{cases} \\ E_i^-(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q}) &= \begin{cases} \varepsilon_i\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_q} & \text{if } i \in \{i_1, i_2, \dots, i_q\}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and $L^+_{|\mathbf{V}_q} = q$. (By default, $\mathbf{V}_0 = \mathbf{C}$, with basis $\{1\}$).

Clearly, $E_i^+E_j^- + E_j^-E_i^+ = 0$ for every $i \neq j$. By Proposition 2,

$$\dim H^k(\mathbf{V}_q) = \sum_{j=-k}^{k} (-1)^j \binom{n+1}{q+j}\binom{n+k-|j|}{k-|j|},$$

where $\binom{n+1}{r} = 0$, if $r < 0$ or $r > n+1$.

By induction on k, we can show now that, for $q = 0$ and $q = n+1$,

$$\dim H^k(\mathbf{V}_0) = \dim H^k(\mathbf{V}_{n+1}) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0 \end{cases},$$

a result that is obvious also without relying on Proposition 2, and by induction on q, we conclude [**15**] that

$$\dim H^k(\mathbf{V}_q) = \frac{q}{q+k}\frac{n+1+2k}{n+1-q+k}\binom{n+k}{k}\binom{n}{q}, \tag{17}$$
$$k = 0, 1, 2, \dots, \ q = 1, 2, \dots, n.$$

2. Spherical Dirac-type operators and separation of variables. We will now direct our attention to a class of *generalized Dirac* *operators* (in the sense of Gromov and Lawson, [**13**, page 103]), naturally induced on the unit sphere $\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} |\ |x|^2 = 1\}$ by any $\mathbf{Z}$-graded representation $\mathbf{V}$ of $\mathrm{Cl}_{n+1,0}$. For the convenience of the reader, and in order to emphasize that the Euclidean Dirac operators considered in Section 1 are in fact the simplest instances of such operators, we review here the concept of *generalized Dirac bundle* and its associated Dirac operator.

Let (M, g) be a complete Riemannian manifold of dimension d, and let $\mathrm{Cl}\,(M)$ be the real Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric g. There is a canonical embedding $TM \subset \mathrm{Cl}\,(M)$, and then the Riemannian metric and Levi-Cività connection extends from TM to $\mathrm{Cl}\,(M)$ in such a way that the connection ∇^{LC} of $\mathrm{Cl}(M)$ preserves the metric and acts as a derivation.

A complex bundle of left modules over the bundle of algebras $\mathrm{Cl}(M)$, say $S \to M$, will be called a (generalized) Dirac bundle if S is furnished with a Hermitian metric $\langle \cdot, \cdot \rangle$ and a metric connection ∇^S such that

i) The action on S by unit vectors in $TM \subset \mathrm{Cl}\,(M)$ is a pointwise isometry.

ii) The connection ∇^S is compatible with the Clifford multiplication, in the sense that for local sections e in TM, ϕ in $\mathrm{Cl}\,(M)$, and s in S, we have

$$\nabla_e^S(\phi{\cdot}s) = \left(\nabla_e^{LC}\phi\right){\cdot}s + \phi{\cdot}\left(\nabla_e^S s\right).$$

Above, the "$\cdot$" indicates the action of $\mathrm{Cl}\,(M)$ on S, while the multiplication in $\mathrm{Cl}\,(M)$ will be simply represented by juxtaposition.

There are two fundamental examples of Dirac bundles associated to M:

a) $S = \mathrm{Cl}\,(M) \otimes \mathbf{C}$. In this case $\mathrm{Cl}\,(M)$ acts on S by left algebra multiplication and ∇^S is the complexification of ∇^{LC}.

b) If M is a *spin* manifold [**18**, page 85], then S can be taken to be the spinor bundle $\Sigma(M)$ of M. To be more specific, for a spin manifold the principal $\mathrm{SO}\,(d)$-bundle $P_{\mathrm{SO}}(M)$ of oriented frames in TM lifts to a principal Spin-bundle $P_{\mathrm{Spin}}(M)$, equivariantly with respect to the 2-cover map $\mathrm{Spin}\,(d) \to \mathrm{SO}\,(d)$. The spinor bundle $\Sigma(M)$ is then the fiber product $\Sigma(M) := P_{\mathrm{Spin}}(M) \times_\mu \Delta$, where Δ is an irreducible representation of the algebra $\mathrm{Cl}_{\,d,0} \otimes \mathbf{C}$ and μ is the unitary representation $\mu : \mathrm{Spin}\,(d) \to U(\Delta)$ induced by the left multiplication with elements of $\mathrm{Spin}\,(d) \subset \mathrm{Cl}_{d,0} \otimes \mathbf{C}$. We then get the compatible connection ∇^{Spin} of $\Sigma(M)$ by lifting the Riemannian connection on $P_{\mathrm{SO}}(M)$ to $P_{\mathrm{Spin}}(M)$, via the Lie algebra isomorphism $\mathrm{so}\,(d) \simeq \mathrm{spin}\,(d)$ [**18**, page 108].

Any Dirac bundle S generates a distinguished differential operator $D : C^\infty(M,S) \to C^\infty(M,S)$, the generalized Dirac operator, defined as follows: If $m : TM \otimes S \to S$ denotes the restriction to TM of the Clifford action $\cdot$ of $\mathrm{Cl}\,(M)$ on S, then $D = m \circ \nabla^S$. Locally, D admits the representation

$$D = \sum_{i=1}^{d} v_i \cdot \nabla^S_{v_i},$$

where $(v_1, v_2, \ldots, v_d)$ is a local orthonormal frame in TM.

Since M is complete, D with domain $C^\infty_{\mathrm{cpt}}(M,S)$ is an essentially self-adjoint first order elliptic differential operator in $L^2(M,S)$ [**13**, page 106]. In fact, the principal symbol $\sigma_\xi(D)$, $\xi \in T^*(M)$ is the Clifford multiplication by the tangent vector field metric equivalent to ξ. This can be seen from the following obvious *symbol* formula:

$$D(fs) = \mathrm{grad}\, f \cdot s + fDs, \quad f \in C^\infty(M),\ s \in C^\infty(M,S).$$

If $W \subset\subset M$ is a relatively compact open subset of M with piecewise smooth boundary ∂W, and if $\mathbf{n}$ denotes the outward unit normal vector field to ∂W, then the following integration by parts formula holds for the Dirac operator D,

$$(Ds_1, s_2)_W = (s_1, Ds_2)_W + (\mathbf{n} \cdot s_1, s_2)_{\partial W}\,, \quad s_1, s_2 \in C^\infty(W,S),$$

where $(\cdot,\cdot)_W$ denotes the usual global (integrated) inner product in $C^\infty(W,S)$ and the integration on ∂W is carried out with respect to the measure induced from W. The above equation proves therefore the formal self-adjointness of D.

Finally, for the square of D, the following Bochner-Witzenböck formula holds true [**13**, page 111],

$$D^2 = \left(\nabla^S\right)^* \nabla^S + \mathcal{R},$$

where $\mathcal{R}$ is the Hermitian curvature bundle morphism acting on S according to the formula

$$\mathcal{R} = \sum_{i<j} v_i \cdot v_j \cdot R_{v_i,v_j}, \qquad R_{v_i,v_j} = [\nabla^S_{v_i}, \nabla^S_{v_j}] - \nabla^S_{[v_i,v_j]}.$$

The Dirac operator associated to a spin manifold M as in b) will be called the *classical* Dirac operator.

Generalized Dirac operators are used in this paper in conjunction with only two types of manifolds: The Euclidean space $M = \mathbf{R}^{n+1}$ and the n-dimensional unit sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$.

The ones on $\mathbf{R}^{n+1}$, which have already appeared in Section 1, are restricted to Dirac bundles of type $\mathbf{R}^{n+1} \times \mathbf{V}$, on which $\mathrm{Cl}(\mathbf{R}^{n+1}) = \mathbf{R}^{n+1} \times \mathrm{Cl}_{n+1,0}$ acts in the obvious way if $\mathbf{V}$ is a representation of $\mathrm{Cl}_{n+1,0}$. So, in the terminology of Section 1, $(\partial/\partial x_i)\cdot \equiv E_i$. Then the trivial connection on $\mathbf{R}^{n+1} \times \mathbf{V}$ is clearly compatible with Clifford multiplication, and the associated Dirac operator takes the form of equation (1) while the symbol formula, respectively the Bochner-Weitzenböck formula, becomes equation (2), respectively equation (3).

Sometimes, as in the proof of the separation of variables theorems below, the Euclidean Dirac operators will be restricted to $\mathbf{R}^{n+1} \setminus \{0\}$ and then locally represented in frames adapted to the polar coordinate representation $\mathbf{R}^{n+1} \setminus \{0\} \equiv (0, \infty) \times \mathbf{S}^n$.

The case $M = \mathbf{S}^n$ will only involve generalized Dirac operators induced, via a separation of variables, from Euclidean Dirac operators. There is a lot of work [**6, 9, 14, 23**] devoted to classical Dirac operators on hypersurfaces of spin manifolds. Typically, the hypersurface inherits in a canonical way a spin structure from the surrounding manifold, and with it the whole Dirac spinor bundle package. In this respect, the inclusion $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ does not present problems for $n \geq 2$, since the only (trivial) spin structure of $\mathbf{R}^{n+1}$ induces the only spin structure on $\mathbf{S}^n$. As for $n = 1$, it is elementary to see that the spin structure on $\mathbf{R}^2$ induces the nontrivial (of the two) spin structure on $\mathbf{S}^1$.

By contrast, there is no work on hypersurface generalized Dirac operators, and this will force us to present certain proofs below in more detail.

The Clifford bundle of algebras $\mathrm{Cl}\,(\mathbf{S}^n)$, associated to the tangent bundle $T(\mathbf{S}^n)$ equipped with the induced Euclidean metric $\langle\cdot,\cdot\rangle$, acts naturally by restriction on the trivial bundle $\mathbf{S}^n \times \mathbf{V} \subset \mathbf{R}^{n+1} \times \mathbf{V}$. However, the trivial connection on $\mathbf{S}^n \times \mathbf{V}$ is not compatible with the natural Levi-Cività connection ∇^{LC} of $\mathrm{Cl}\,(\mathbf{S}^n)$. This prevents us from directly constructing generalized Dirac operators on $\mathbf{S}^n \times \mathbf{V}$. Nonetheless, such operators do exist. We could invoke [**1**, page 88] to prove their existence, but prefer an explicit construction here, via the following lemma.

Lemma 1. *For any (ungraded) representation* $\mathbf{V}$ *of* $\mathrm{Cl}_{n+1,0}$ *there are metric connections* ∇^0 *on the trivial bundle* $\mathbf{S}^n \times \mathbf{V}$ *compatible with the Levi-Cività connection* ∇^{LC} *of* $\mathrm{Cl}(\mathbf{S}^n)$*, that is, if* e, ϕ*, and* σ *are, respectively, local sections in* $T(\mathbf{S}^n)$*,* $\mathrm{Cl}(\mathbf{S}^n)$ *and* $\mathbf{S}^n \times \mathbf{V}$*, then*

$$\nabla^0_e(\phi\cdot\sigma) = \left(\nabla^{LC}_e\phi\right)\cdot\sigma + \phi\cdot\left(\nabla^0_e\sigma\right). \tag{18}$$

Proof. We will construct a concrete connection ∇^0 satisfying equation (18). To this end, for e, σ local sections in $T(\mathbf{S}^n)$, respectively $\mathbf{S}^n \times \mathbf{V}$, define

$$\nabla^0_e\sigma := e(\sigma) + \frac{1}{2}e\cdot\mathbf{n}\cdot\sigma, \tag{19}$$

where $e(\sigma)$ represents ordinary component-wise differentiation of σ in the direction of e and $\mathbf{n}\cdot = \mathbf{n}(\omega)\cdot$ represents (pointwise) Clifford multiplication in $\{\omega\} \times \mathbf{V} \equiv \mathbf{V}$ by the position vector $\omega \in \mathbf{S}^n$, i.e., if $\omega = (\omega_0, \omega_1, \ldots, \omega_n) \in \mathbf{S}^n \subset \mathbf{R}^{n+1}$, then $\mathbf{n}(\omega)\cdot = \sum_{i=0}^n \omega_i E_i$, E_i as in Section 1.

To show that the assignment (19) satisfies equation (18), it suffices to verify (18) for local sections ϕ belonging to $T(\mathbf{S}^n)$.

Recall that, in polar coordinates, $\mathbf{R}^{n+1}\setminus\{0\} \equiv (0,\infty)\times\mathbf{S}^n$, $x \equiv (r,\omega)$, $r = |x|$, $\omega = x/|x|$, the Euclidean metric ds^2 on $\mathbf{R}^{n+1} \setminus \{0\}$ and the induced metric $d\sigma^2$ on $\mathbf{S}^n$ relate by $ds^2 = dr^2 + r^2 d\sigma^2$. Consequently, if $\{e_1, e_2, \ldots, e_n\}$ is an (oriented) local orthonormal frame in $T(\mathbf{S}^n)$, then $\{(\partial/\partial r), (e_1/r), \ldots, (e_n/r)\}$ is a local orthonormal frame in $T(\mathbf{R}^{n+1} \setminus \{0\})$ and [**19**, page 206] the Levi-Cività connection ∇ in $T(\mathbf{R}^{n+1} \setminus \{0\})$ and the standard lift of the Levi-Cività connection ∇^{LC} in $T(\mathbf{S}^n)$ to $T(\mathbf{R}^{n+1} \setminus \{0\})$ are related by the equations

$$\nabla_{\partial/\partial r}\frac{\partial}{\partial r} = 0, \quad \nabla_{\partial/\partial r}e_i = \nabla_{e_i}\frac{\partial}{\partial r} = \frac{e_i}{r}, \quad \nabla_{e_i}e_j = -r\delta_{ij}\frac{\partial}{\partial r} + \nabla^{LC}_{e_i}e_j. \tag{20}$$

Since, obviously, $(\partial/\partial r) = \sum_{i=0}^n \omega_i(\partial/\partial x_i)$ at $x = r\omega$, Clifford multiplication by $(\partial/\partial r)$ in $\{r\omega\} \times \mathbf{V} \equiv \mathbf{V}$ equals precisely $\mathbf{n}\cdot$. As a result, for

e, ϕ local sections in $T(\mathbf{S}^n)$, σ local section in $\mathbf{S}^n \times \mathbf{V}$, $\mathbf{n}{\cdot}\phi{\cdot} + \phi{\cdot}\mathbf{n}{\cdot} = 0$, and

$$\begin{aligned}
\nabla^0_e(\phi{\cdot}\sigma) &= e(\phi{\cdot}\sigma) + \frac{1}{2} e{\cdot}\mathbf{n}{\cdot}\phi{\cdot}\sigma \\
&= (\nabla_e \phi)\,{\cdot}\sigma + \phi{\cdot}e(\sigma) - \frac{1}{2}(e\phi){\cdot}\mathbf{n}{\cdot}\sigma \\
&= -\langle e, \phi\rangle \mathbf{n}{\cdot}\sigma + \left(\nabla^{LC}_e \phi\right){\cdot}\sigma + \phi{\cdot}e(\sigma) - \frac{1}{2}(e\phi){\cdot}\mathbf{n}{\cdot}\sigma \\
&= \left(\nabla^{LC}_e \phi\right){\cdot}\sigma + \phi{\cdot}\nabla^0_e\sigma - \frac{1}{2}(e\phi + \phi e){\cdot}\mathbf{n}{\cdot}\sigma - \langle e, \phi\rangle \mathbf{n}{\cdot}\sigma \\
&= \left(\nabla^{LC}_e \phi\right){\cdot}\sigma + \phi{\cdot}\nabla^0_e\sigma,
\end{aligned}$$

since Clifford multiplication in $\mathrm{Cl}\,(\mathbf{S}^n)$ satisfies $e\phi + \phi e = -2\langle e, \phi\rangle$. □

Remark. The metric connection ∇^0 on $\mathbf{S}^n \times \mathbf{V}$ constructed in the Lemma 1 also satisfies the commutation relation

(21)
$$\nabla^0_e(\mathbf{n}{\cdot}\sigma) = \mathbf{n}{\cdot}\nabla^0_e\sigma, \quad e, \sigma \text{ local sections in } T(\mathbf{S}^n), \text{ respectively, } \mathbf{S}^n \times \mathbf{V}.$$

The remark is an immediate consequence of one of equations (20) and the very definition of ∇^0.

The following theorem now gives a standard separation of variables for Euclidean Dirac operators on $\mathbf{R}^{n+1}$ associated to (ungraded) representations $\mathbf{V}$ of $\mathrm{Cl}_{n+1,0}$ and their $\mathrm{Cl}\,(\mathbf{S}^n)$-compatible connections.

Separation of variables–Ungraded version. *Let $\not{D}$ be an Euclidean Dirac operator on $C^\infty(\mathbf{R}^{n+1}, \mathbf{V})$, where $\mathbf{V}$ is an ungraded representation of $\mathrm{Cl}_{n+1,0}$. Assume that $\mathbf{S}^n \times \mathbf{V}$, viewed as a $\mathrm{Cl}(\mathbf{S}^n)$-bundle in the obvious way, admits a $\mathrm{Cl}(\mathbf{S}^n)$-compatible metric connection ∇^0 (cf. equation (18)) which also satisfies equation (21). Then, via the identification $C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V}) \equiv C^\infty((0, \infty), C^\infty(\mathbf{S}^n, \mathbf{V}))$, the following separation of variables formula holds*

$$\not{D}|_{\mathbf{R}^{n+1}\setminus\{0\}} \equiv \mathbf{n}{\cdot}\frac{\partial}{\partial r} + \frac{1}{r}\not{\partial} + \frac{1}{r}A, \tag{22}$$

where $\not{\partial}$ is the generalized Dirac operator on $C^\infty(\mathbf{S}^n, \mathbf{V})$, associated to the connection ∇^0 and the Clifford multiplication on $\mathbf{S}^n \times \mathbf{V}$, i.e., *if* $\{e_1, e_2, \dots, e_n\}$ *is a local orthonormal frame in* $\mathbf{S}^n$ *and* σ *is a local section of* $\mathbf{S}^n \times \mathbf{V}$, *then*

$$\not\partial\sigma = \sum_{i=1}^{n} e_i\cdot\nabla^0_{e_i}\sigma, \tag{23}$$

and where $A \in \mathrm{End}\,(\mathbf{S}^n \times \mathbf{V})$ *is a suitable bundle morphism. Moreover,* $\not\partial, \mathbf{n}\cdot$ *and* A *satisfy the following commutation relations*:

$$\not\partial\mathbf{n}\cdot + \mathbf{n}\cdot\not\partial = 0 \tag{24}$$

$$A\mathbf{n}\cdot + \mathbf{n}\cdot A = -n\mathrm{Id}_{\mathbf{S}^n\times\mathbf{V}}. \tag{25}$$

Proof. Representing $\not{D}$ in the local orthonormal frame

$$\left\{\frac{\partial}{\partial r}, \frac{e_1}{r}, \dots, \frac{e_n}{r}\right\} \quad \text{in } T(\mathbf{R}^{n+1}\setminus\{0\})$$

associated to a local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ in $T(\mathbf{S}^n)$ yields, for $s \in C^\infty(\mathbf{R}^{n+1}\setminus\{0\}, \mathbf{V}) \equiv C^\infty((0,\infty), C^\infty(\mathbf{S}^n, \mathbf{V}))$,

$$\not{D}s = \left(\frac{\partial}{\partial r}\right)\cdot\frac{\partial s}{\partial r} + \sum_{i=1}^{n}\left(\frac{e_i}{r}\right)\cdot\frac{e_i}{r}(s). \tag{26}$$

As before, $(\partial/\partial r)\cdot = \mathbf{n}\cdot$. Also,

$$\frac{e_i}{r} = \frac{1}{r}\sum_{j=0}^{n} e_i(r\omega_j)\frac{\partial}{\partial x_j} = \sum_{j=0}^{n} e_i(\omega_j)\frac{\partial}{\partial x_j}$$

implies that

$$\left(\frac{e_i}{r}\right)\cdot = \sum_{j=0}^{n} e_i(\omega_j)E_j = e_i\cdot,$$

and so $(e_i/r)\cdot$ does not depend on r. Consequently, the representation (26) becomes

$$\not{D}s = \mathbf{n}\cdot\frac{\partial s}{\partial r} + \frac{1}{r}\sum_{i=1}^{n} e_i\cdot e_i(s). \tag{27}$$

Since the differential operator appearing in equation (27),

$$\delta\sigma := \sum_{i=1}^{n} e_i{\cdot}e_i(\sigma), \quad \sigma \in C^\infty(\mathbf{S}^n, \mathbf{V}), \tag{28}$$

does not engage the radial dependence of s, the separation of variables formula (22) follows now if we set, for $\sigma \in C^\infty(\mathbf{S}^n, \mathbf{V})$,

$$A\sigma := \sum_{i=1}^{n} e_i{\cdot} \left(e_i(\sigma) - \nabla^0_{e_i}\sigma\right). \tag{29}$$

Clearly, A is $C^\infty(\mathbf{S}^n)$-linear so it belongs to End$\,(\mathbf{S}^n \times \mathbf{V})$.

The commutation relation (24) then follows immediately by applying the remark (equation (21)) to equation (23), given that $\mathbf{n}{\cdot}e{\cdot} + e{\cdot}\mathbf{n}{\cdot} = 0$, for local sections e in $T(\mathbf{S}^n)$.

Since $A = \delta - \partial\!\!\!/$, equation (25) is equivalent to $\delta\mathbf{n}{\cdot} + \mathbf{n}{\cdot}\delta = -nId_{\mathbf{S}^n\times\mathbf{V}}$, which in turn, follows from equation (28), giving δ, by using that $\nabla_{e_i}(\partial/\partial r) = (e_i/r)$ (cf. (20)). □

The chief purpose of this paper is the investigation of the spectrum of the Dirac-type operator $\partial\!\!\!/$ on $C^\infty(\mathbf{S}^n, \mathbf{V})$. Namely, we seek a relationship between the eigenvalues and eigensections of $\partial\!\!\!/$, on one hand, and the polynomial Dirac spinor spaces associated to $D\!\!\!\!/$, on the other hand. The main vehicle in this analysis, inspired by Shubin's derivation of the spectral decomposition of the ordinary Laplace operator on Euclidean spheres [**21**, page 160], is of course the separation of variables formula (22). Such a task would be hopeless if one did not know more about the bundle morphism A appearing in (22). Guided by what happens in the case of our motivating examples, the classical Dirac and Gauss-Bonnet operators on $\mathbf{S}^n$, we will refine the separation of variables formula (22) by making use of graded representations $\mathbf{V}$ of $\mathrm{Cl}_{n+1,0}$.

To this end let, $\mathbf{V}$ be a $\mathbf{Z}$-graded representation of $\mathrm{Cl}_{n+1,0}$, satisfying the requirements of equation (5). It is then reasonable to request that a $\mathrm{Cl}(\mathbf{S}^n)$-compatible connection ∇^0 on $\mathbf{S}^n \times \mathbf{V}$ leave invariant the subbundles $\mathbf{S}^n \times \mathbf{V}_q$, in the sense that if σ_q is a $\mathbf{V}_q$-valued local section of $\mathbf{S}^n \times \mathbf{V}$ and e is a local section of $T(\mathbf{S}^n)$, then $\nabla^0_e(\sigma_q)$ is $\mathbf{V}_q$-valued. This is not the case for the connection constructed in Lemma 1, unless the representation is, as in Example 1, $\mathbf{Z}_2$-graded. Nor is it obvious how to prove their existence, in general; however, since they exist in the case of the main examples we have in mind we will always require them.

The following lemma takes a different look at the $\mathrm{Cl}(\mathbf{S}^n)$-compatible connections with respect to the $\mathbf{Z}$-grading of $\mathbf{V}$. This look will be relevant for the spectral analysis of $\not{\partial}$.

Lemma 2. *Let* $\mathbf{V} = \oplus_{q=0}^{p+1}\mathbf{V}_q$ *be a* $\mathbf{Z}$*-graded representation of* $\mathrm{Cl}_{n+1,0}$*, satisfying the requirements of equation* (5)*, and let* ∇^0 *be a* $\mathrm{Cl}\,(\mathbf{S}^n)$*-compatible connection on* $\mathbf{S}^n \times \mathbf{V}$ *which also satisfies the provisions of equation* (21)*. Then there is a subbundle* Ω^0 *of* $\mathbf{S}^n \times \mathbf{V}$ *and an orthogonal bundle decomposition*

$$\Omega^0 = \sum_{q=0}^{p} \Omega_q^0,$$

such that

$$\mathbf{S}^n \times \mathbf{V}_q = \Omega_q^0 \oplus \mathbf{n}\cdot\Omega_{q-1}^0, \quad q = 0, 1, \dots, p+1, \ (\Omega_{-1}^0 = \Omega_{p+1}^0 = 0). \tag{30}$$

Moreover, $\mathbf{S}^n \times \mathbf{V}_q$*,* $q = 0, 1, \dots, p+1$*, are* ∇^0*-invariant if and only if* Ω_q^0*,* $q = 0, 1, \dots, p$*, are* ∇^0*-invariant.*

Proof. For any $q = 0, 1, \dots, p$ and any $\omega \in \mathbf{S}^n$, define

$$\Omega_{q,\omega}^0 := \{(\omega, v) | v \in \mathbf{V}_q \quad \text{and} \quad \mathbf{n}(\omega)\cdot v \in \mathbf{V}_{q+1}\}. \tag{31}$$

$\Omega_{q,\omega}^0$ has a natural structure of Hermitian vector spaces, induced by that of $\mathbf{V}$. Since $\mathbf{n}\cdot\mathbf{V}_0 \subset \mathbf{V}_1$, we have that $\Omega_0^0 = \mathbf{S}^n \times \mathbf{V}_0$ and so Ω_0^0 is a (trivial) subbundle of $\mathbf{S}^n \times \mathbf{V}$.

If we prove (30), then an induction on q shows that Ω_q^0 will be a subbundle of $\mathbf{S}^n \times \mathbf{V}$, since $\mathbf{S}^n \times \mathbf{V}_q$ and $\mathbf{n}\cdot\Omega_{q-1}^0$ are subbundles, assuming that Ω_{q-1}^0 is a subbundle. Clearly, $\Omega_{q,\omega}^0 \perp \mathbf{n}(\omega)\cdot\Omega_{q-1,\omega}^0$, since $\mathbf{n}(\omega)\cdot$ is a skew-Hermitian operator on $\mathbf{V}$ and $\mathbf{V}_{q-1} \perp \mathbf{V}_{q+1}$. As a result, $\Omega_q^0 \oplus \mathbf{n}\cdot\Omega_{q-1}^0 \subseteq \mathbf{S}^n \times \mathbf{V}_q$.

Consider now $(\omega, v_q) \in \mathbf{S}^n \times \mathbf{V}_q$. Then the $\mathbf{Z}$-grading property of $\mathbf{V}$ implies that $\mathbf{n}(\omega)\cdot v_q = \sigma_{q-1}(\omega) + \sigma_{q+1}(\omega)$, where $\sigma_{q\mp 1} \in \mathbf{V}_{q\mp 1}$.

Thus, $-v_q = \mathbf{n}(\omega)\cdot\sigma_{q-1}(\omega) + \mathbf{n}(\omega)\cdot\sigma_{q+1}(\omega)$. This last equation shows that both $\mathbf{n}(\omega)\cdot\sigma_{q\mp1}$ belong to $\mathbf{V}_q$, i.e., $(\omega, \sigma_{q-1}(\omega)) \in \Omega^0_{q-1,\omega}$ and $(\omega, \mathbf{n}(\omega)\cdot\sigma_{q+1}(\omega)) \in \Omega^0_{q,\omega}$. This proves that $\mathbf{S}^n \times \mathbf{V}_q \subseteq \Omega^0_q \oplus \mathbf{n}\cdot\Omega^0_{q-1}$.

It is worth noting that equations (30) uniquely determine the subbundles Ω^0_q. Formally, Ω^0_{p+1} is also defined, but $\Omega^0_{p+1} = 0$. Equations (30) also make it clear that

$$\mathbf{S}^n \times \mathbf{V} = \Omega^0 \oplus \mathbf{n}\cdot\Omega^0. \tag{32}$$

Since $\dim \mathbf{V}_q = \operatorname{rank} \Omega^0_q + \operatorname{rank} \Omega^0_{q-1}$ and $\dim \mathbf{V}_0 = \operatorname{rank} \Omega^0_0$ we see that

$$\operatorname{rank} \Omega^0_q = \dim \mathbf{V}_q - \dim \mathbf{V}_{q-1} + \dim \mathbf{V}_{q-2} - \cdots . \tag{33}$$

Finally, the simultaneous ∇^0-invariance of $\mathbf{S}^n \times \mathbf{V}_q$, $q = 0, 1, \dots, p+1$, and Ω^0_q, $q = 0, 1, \dots, p$, follows immediately from the definition (31) of Ω^0_q and equation (21). ◻

Yet again motivated by what happens in the case of the spherical classical Dirac and Gauss-Bonnet operators and in order to get more accurate results later we make one last assumption. The set-up being that described in the ungraded version of the separation of variables theorem and in Lemma 2 we assume that, for $q = 0, 1, \dots, p$, there is $\lambda_q \in \mathbf{R}$, $0 \le \lambda_q \le n$, such that

$$A_{|\Omega^0_q} = \lambda_q \mathbf{n}\cdot. \tag{34}$$

Then A is completely determined on $\mathbf{S}^n \times \mathbf{V}$ since, by equation (25),

$$A_{|\mathbf{n}\cdot\Omega^0_q} = (n - \lambda_q)\mathbf{n}\cdot, \quad q = 0, 1, \dots, p. \tag{35}$$

Separation of variables–Graded version. *Let* $\not{D}$ *be an Euclidean Dirac operator on* $C^\infty(\mathbf{R}^{n+1}, \mathbf{V})$*, where* $\mathbf{V} = \oplus_{q=0}^{p+1} \mathbf{V}_q$ *is a graded representation of* $\mathrm{Cl}_{n+1,0}$ *(cf. equation* (5)*). Assume that* $\mathbf{S}^n \times \mathbf{V}$ *admits a* $\mathrm{Cl}\,(\mathbf{S}^n)$*-compatible metric connection* ∇^0 *(cf. equation* (18)*) which leaves* $\mathbf{S}^n \times \mathbf{V}_q$*,* $q = 0, 1, \dots, p+1$*, invariant, and in addition satisfies equation* (21)*. Then, via the identification* $C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V}) \equiv$

$C^\infty((0,\infty), C^\infty(\mathbf{S}^n, \mathbf{V}))$ *the following separation of variables formula holds*:

$$\not{D}|_{\mathbf{R}^{n+1}\setminus\{0\}} \equiv \mathbf{n}\cdot\frac{\partial}{\partial r} + \frac{1}{r}\not{\partial} + \frac{1}{r}A, \tag{36}$$

where $\not{\partial}$ *is the generalized Dirac operator on* $C^\infty(\mathbf{S}^n, \mathbf{V})$*, associated to the connection* ∇^0 *and the induced Clifford multiplication on* $\mathbf{S}^n \times \mathbf{V}$*, and where* $A \in \mathrm{End}\,(\mathbf{S}^n \times \mathbf{V})$ *is a suitable bundle morphism.*

There is also a subbundle Ω^0 *of* $\mathbf{S}^n \times \mathbf{V}$*, and a* ∇^0*-invariant orthogonal bundle decomposition* $\Omega^0 = \oplus_{q=0}^{p}\Omega_q^0$*, where* $\mathbf{S}^n \times \mathbf{V}_q = \Omega_q^0 \oplus \mathbf{n}\cdot\Omega_{q-1}^0$*,* $q = 0, 1, \ldots, p+1$*, such that, if* $\Gamma_q := C^\infty(\mathbf{S}^n, \Omega_q^0)$*,* $q = 0, 1, \ldots, p$*, then*

$$\not{\partial}(\Gamma_q) \subset \Gamma_{q-1} \oplus \Gamma_{q+1} \oplus \mathbf{n}\cdot\Gamma_q. \tag{37}$$

If, in addition, there is a $\lambda_q \in \mathbf{R}$ *such that* $A_{|\Omega_q^0} = \lambda_q \mathbf{n}\cdot$*,* $q = 0, 1, \ldots, p$*, then*

$$\not{\partial}^2(\Gamma_q) \subset \Gamma_q. \tag{38}$$

Proof. A good portion of the theorem is embedded in the ungraded version of it or in Lemma 2. The only things that need to be addressed are the contents of equations (37) and (38).

If $\sigma_q \in \Gamma_q$, then $\not{\partial}\sigma_q \in C^\infty(\mathbf{S}^n, \mathbf{V}_{q-1}) \oplus C^\infty(\mathbf{S}^n, \mathbf{V}_{q+1})$, given the definition (23) of $\not{\partial}$, the ∇^0-invariance of $\sigma_q \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$ and the $\mathbf{Z}$-grading property (5) of the Clifford multiplication. If $\not{\partial}\sigma_q = s_{q-1} + s_{q+1}$, $s_{q\mp1} \in C^\infty(\mathbf{S}^n, \mathbf{V}_{q\mp1})$, we claim that $s_{q-1} \in \Gamma_{q-1}$ and $s_{q+1} \in \Gamma_{q+1} \oplus \mathbf{n}\cdot\Gamma_q$, which claim proves equation (37).

$s_{q-1} \in \Gamma_{q-1}$ is equivalent to $\mathbf{n}\cdot s_{q-1} \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, i.e., $\mathbf{n}\cdot s_{q-1}$ has no $\mathbf{V}_{q-2}$-valued component. Now, $\mathbf{n}\cdot s_{q-1} = \mathbf{n}\cdot\not{\partial}\sigma_q - \mathbf{n}\cdot s_{q+1} = -\not{\partial}(\mathbf{n}\cdot\sigma_q) - \mathbf{n}\cdot s_{q+1}$. Since, by definition, $\mathbf{n}\cdot\sigma_q \in C^\infty(\mathbf{S}^n, \mathbf{V}_{q+1})$, $-\not{\partial}(\mathbf{n}\cdot\sigma_q) \in C^\infty(\mathbf{S}^n, \mathbf{V}_q) \oplus C^\infty(\mathbf{S}^n, \mathbf{V}_{q+2})$. Also, $-\mathbf{n}\cdot s_{q+1} \in C^\infty(\mathbf{S}^n, \mathbf{V}_q) \oplus C^\infty (\mathbf{S}^n, \mathbf{V}_{q+2})$, and so $s_{q-1} \in \Gamma_{q-1}$.

Obviously, $s_{q+1} \in \Gamma_{q+1} \oplus \mathbf{n}\cdot\Gamma_q$, since $C^\infty(\mathbf{S}^n, \mathbf{V}_{q+1}) = \Gamma_{q+1} \oplus \mathbf{n}\cdot\Gamma_q$. This completes the proof of statement (37).

Towards the proof of equation (38), we will show first that $\not{\partial}^2$ leaves invariant $C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, $q = 0, 1, \ldots, p+1$. Since $\mathbf{S}^n \times \mathbf{V}_q$ is a trivial bundle there are constant sections $\varepsilon_\alpha : \mathbf{S}^n \to V_q$, $\alpha = 1, 2, \ldots, N_q$, $N_q = \dim \mathbf{V}_q$, such that every element of $C^\infty(\mathbf{S}^n, \mathbf{V}_q)$ is representable as $\sum_{\alpha=1}^{N_q} f_\alpha \varepsilon_\alpha$, $f_\alpha \in C^\infty(\mathbf{S}^n)$. Consequently, it suffices to show that $\not{\partial}^2(f_\alpha \varepsilon_\alpha) \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$ for any $\alpha = 1, 2, \ldots, N_q$.

Since ε_α is a constant section, equation (29) gives $\delta\varepsilon_\alpha = 0$ and since $\not{\partial} = \delta - A$, we have $\not{\partial}\varepsilon_\alpha = -A\varepsilon_\alpha$. Now $\varepsilon_\alpha \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$ is uniquely representable as $\varepsilon_\alpha = \sigma_q + \mathbf{n}{\cdot}\sigma_{q-1}$, $\sigma_q \in \Gamma_q$, $\sigma_{q-1} \in \Gamma_{q-1}$. As a result, hypotheses (25) and (34) yield

$$\not{\partial}\varepsilon_\alpha = \lambda_q \mathbf{n}{\cdot}\sigma_q - (n - \lambda_{q-1})\sigma_{q-1} = \lambda_q \mathbf{n}{\cdot}\varepsilon_\alpha + (\lambda_q + \lambda_{q-1} - n)\sigma_{q-1}.$$

Consequently,

(39)

$$\begin{aligned} \not{\partial}^2\varepsilon_\alpha &= -\lambda_q \mathbf{n}{\cdot}\not{\partial}\varepsilon_\alpha + (\lambda_q + \lambda_{q-1} - n)\not{\partial}\sigma_{q-1} \\ &= \lambda_q^2\varepsilon_\alpha - \lambda_q(\lambda_q + \lambda_{q-1} - n)\mathbf{n}{\cdot}\sigma_{q-1} + (\lambda_q + \lambda_{q-1} - n)\not{\partial}\sigma_{q-1}. \end{aligned}$$

Equations (37) and (39) imply that

$$\not{\partial}^2\epsilon_\alpha \in C^\infty(\mathbf{S}^n, \mathbf{V}_q) \oplus \Gamma_{q-2}. \tag{40}$$

Similar algebraic manipulations also yield

$$\not{\partial}^2\varepsilon_\alpha = (n - \lambda_{q-1})^2\varepsilon_\alpha + (n - \lambda_{q-1})(\lambda_q + \lambda_{q-1} - n)\sigma_q - (\lambda_q + \lambda_{q-1} - n)\mathbf{n}{\cdot}\not{\partial}\sigma_q,$$

which amounts to

$$\not{\partial}^2\varepsilon_\alpha \in C^\infty(\mathbf{S}^n, \mathbf{V}_q) \oplus \mathbf{n}{\cdot}\Gamma_{q+1}. \tag{41}$$

It is now clear from equations (40) and (41) that $\not{\partial}^2\epsilon_\alpha \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$.

An iteration of the symbol formula for the Dirac operator $\not{\partial}$ gives

$$\not{\partial}^2(f_\alpha\varepsilon_\alpha) = -\left(\Delta^0 f_\alpha\right)\varepsilon_\alpha + 2\nabla^0_{\operatorname{grad} f_\alpha}\varepsilon_\alpha + f_\alpha\not{\partial}^2\varepsilon_\alpha, \tag{42}$$

where Δ^0 is the Laplace operator on $\mathbf{S}^n$. Since ∇^0 leaves $\mathbf{S}^n \times \mathbf{V}_q$ invariant and $\not{\partial}^2\varepsilon_\alpha \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, we conclude that $\not{\partial}^2(f_\alpha\varepsilon_\alpha) \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, as stated.

We will prove that $\not{\partial}^2(\Gamma_q) \subset \Gamma_q$ by induction on q. For $q = 0$, the statement follows from the above discussion, since $\Gamma_0 = C^\infty(\mathbf{S}^n, \mathbf{V}_0)$.

Assume $\not\partial^2(\Gamma_{q-1}) \subset \Gamma_{q-1}$, and let $\sigma_q \in \Gamma_q$. Since $\Gamma_q \subset C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, we have $\not\partial^2\sigma_q \in C^\infty(\mathbf{S}^n, \mathbf{V}_q)$, or $\not\partial^2\sigma_q = s_q + \mathbf{n}\cdot s_{q-1}$, $s_q \in \Gamma_q$, $s_{q-1} \in \Gamma_{q-1}$. Using the global (integrated) product $(\cdot,\ \cdot)$ and the associated norm $||\cdot||$ on $L^2(\mathbf{S}^n, \mathbf{V})$, we have

$$||\mathbf{n}\cdot s_{q-1}||^2 = (\not\partial^2\sigma_q, \mathbf{n}\cdot s_{q-1}) = (\sigma_q, \not\partial^2(\mathbf{n}\cdot\sigma_{q-1})) = (\sigma_q, \mathbf{n}\cdot\not\partial^2\sigma_{q-1}) = 0,$$

since by the inductive hypothesis $\mathbf{n}\cdot\not\partial^2\sigma_{q-1} \in \mathbf{n}\cdot\Gamma_{q-1}$, and $\Gamma_q \perp \mathbf{n}\cdot\Gamma_{q-1}$. Consequently, $\mathbf{n}\cdot s_{q-1} = 0$, or $\not\partial^2\sigma_q \in \Gamma_q$. The proof of the theorem is complete. ☐

Equations (37) and (38) of the previous theorem suggest that looking at the spectrum of $\not\partial^2$ rather than $\not\partial$ might be a simpler endeavor. It is easy to relate the spectra of the two operators, as the following proposition shows.

Proposition 3. *Let $\not\partial$ be the Dirac operator which appears in the separation of variables theorems and $\not\partial^2$ its square, both with domains $C^\infty(\mathbf{S}^n, \mathbf{V})$.*

a) *$\lambda = 0$ is an eigenvalue of $\not\partial$ if and only if it is an eigenvalue of $\not\partial^2$, with the same multiplicity. In fact, the 0-eigenspaces of the two operators are equal.*

b) *A real number $\lambda \neq 0$ is an eigenvalue of $\not\partial$ with multiplicity m_λ if and only if λ^2 is an eigenvalue of $\not\partial^2$ with multiplicity $2m_\lambda$.*

Therefore, if $\not\partial^2(\Gamma_q) \subset \Gamma_q$, for the spectral analysis of $\not\partial$, it suffices to study the spectral decomposition of the restrictions of $\not\partial^2$ to Γ_q, $q = 0, 1, \dots, p$.

Proof. a) Denote by $E_\lambda(\not\partial)$ the λ-eigenspace of $\not\partial$ and similarly by $E_\lambda(\not\partial^2)$ the λ^2-eigenspace of $\not\partial^2$. Clearly, $E_0(\not\partial) \subseteq E_0(\not\partial^2)$. If $\not\partial^2\sigma = 0$, $\sigma \in C^\infty(\mathbf{S}^n, \mathbf{V})$, then $0 = (\not\partial^2\sigma, \sigma) = (\not\partial\sigma, \not\partial\sigma)$, i.e., $\not\partial\sigma = 0$. Thus, $E_0(\not\partial^2) \subseteq E_0(\not\partial)$.

b) For $\lambda \neq 0$ real, the mapping $\sigma \to \mathbf{n}\cdot\sigma$ is a linear isomorphism from $E_\lambda(\not\partial)$ onto $E_{-\lambda}(\not\partial)$. Moreover, $E_\lambda(\not\partial)$ and $E_{-\lambda}(\not\partial)$ are orthogonal and

$$E_\lambda(\not\partial) \oplus E_{-\lambda}(\not\partial) \subseteq E_{\lambda^2}(\not\partial^2).$$

Now define the linear map $L : E_{\lambda^2}(\not\partial^2) \to E_\lambda(\not\partial)$ by

$$L\sigma = \frac{1}{2}\left(\sigma + \frac{1}{\lambda}\not\partial\sigma\right).$$

Since the restriction of L to the subspace $E_\lambda(\not\partial)$ of $E_{\lambda^2}(\not\partial^2)$ is the identity, we see that L is an onto mapping. Thus, $\dim(E_{\lambda^2}(\not\partial^2)) - \dim(\ker L) = \dim(E_\lambda(\not\partial))$. However, $\ker L = E_{-\lambda}(\not\partial)$.

The last statement in Proposition 3 follows from (24) and the graded version of the separation of variables theorem. $\square$

We now define two important classes of spherical Dirac operators $\not\partial$ on which our main result, a complete spectral decomposition theorem, will rest.

Definition. Assume that $(\mathbf{V}, \not{D}, \nabla^0, \not\partial, A, \Omega^0)$ satisfies all the hypotheses set forth in the graded version of the separation of variables theorem, including equation (34).

a) $\not\partial$ is said to be a spherical classical Dirac-type operator if

$$\not\partial(\Gamma_q) \subset \mathbf{n}\cdot\Gamma_q, \quad q = 0, 1, 2, \ldots, p.$$

b) $\not\partial$ is said to be a spherical Gauss-Bonnet-type operator if

$$\not\partial(\Gamma_q) \subset \Gamma_{q-1} \oplus \Gamma_{q+1}, \quad q = 0, 1, 2, \ldots, p.$$

In case b), $\not\partial^2_{|\Omega^0}$, which has the property $\not\partial^2(\Gamma_q) \subset \Gamma_q$, is said to be a spherical Laplace-Beltrami-type operator, by analogy with the Laplace-Beltrami operator on differential forms.

Notice that, in the case of a spherical classical Dirac-type operator, Ω^0 becomes a $\mathrm{Cl}\,(\mathbf{S}^n)$-module when Clifford multiplication by e, e local section in $T(\mathbf{S}^n)$ is replaced by $\mathbf{n}\cdot e\cdot$. Therefore, $\mathbf{n}\cdot\not\partial$ is also a Dirac-type operator, $(\mathbf{n}\cdot\not\partial)^2 = \not\partial^2$, $(\mathbf{n}\cdot\not\partial)(\Gamma_q) \subset \Gamma_q$, and so all the spectral information of $\not\partial$ is captured by $(\mathbf{n}\cdot\not\partial)_{|\Omega^0}$. For this reason, it is more appropriate to call $(\mathbf{n}\cdot\not\partial)_{|\Omega_0}$ a spherical classical Dirac-type operator.

The separation of variables formula (36) now becomes

$$\not{D}|_{\mathbf{R}^{n+1}\setminus\{0\}}s \equiv \mathbf{n}\cdot\left(\frac{\partial}{\partial r} + \frac{1}{r}(-\mathbf{n}\cdot\not\partial) + \frac{1}{r}\lambda_q\right)s, \tag{43}$$
$$s \in C^{\infty}((0,\infty),\Gamma_q) \subset C^{\infty}(\mathbf{R}^{n+1}\setminus\{0\},\mathbf{V}_q).$$

For an example of a spherical classical Dirac-type operator, take $\not{D}$ to be the classical Euclidean Dirac operator ($\dim\mathbf{V} = 2^{[n/2]+1}$, cf., Example 1), with the connection ∇^0 given by equation (19). Then $\Omega^0 = \Omega^0_0 \simeq \mathbf{C}^{[n/2]}$ and $A_{|\Omega^0_0} = (n/2)\mathbf{n}\cdot$. $\mathbf{n}\cdot\not\partial$ is then associated to a Dirac bundle isomorphic to that generating the spherical classical Dirac operator. In such a case, Ω^0 identifies naturally with the spinor bundle $\Sigma(\mathbf{S}^n)$ associated to the spin manifold $\mathbf{S}^n$ inheriting its spin structure from $\mathbf{R}^{n+1}$.

Indeed, $P_{\mathrm{Spin}}(\mathbf{S}^n)$ is the reduction of $P_{\mathrm{Spin}}(\mathbf{R}^{n+1})$ via the inclusion maps

$$\begin{tikzcd}
\mathrm{Spin}\,(n+1) \arrow[r] & \mathrm{SO}\,(n+1) \\
\cup & \cup \\
\mathrm{Spin}\,(n) \arrow[r] & \mathrm{SO}\,(n)
\end{tikzcd}$$

and taking into consideration the structure of the irreducible representations of $\mathrm{Cl}_{n,0}\otimes\mathbf{C}$ and $\mathrm{Cl}_{n+1,0}\otimes\mathbf{C}$, [**3**, page 12], we conclude that $\Sigma(\mathbf{R}^{n+1})\,|_{\mathbf{S}^n} \equiv \Sigma(\mathbf{S}^n)$, when n is *even*, and more generally $\Sigma(\mathbf{S}^n) \equiv \Omega^0$. For the identification of the $\nabla^0_{|\Omega^0}$ with the spinor connection ∇^{Spin} of $\Sigma(\mathbf{S}^n)$, see [**9**, page 10].

Naturally, an example of spherical Gauss-Bonnet-type operator is linked to Example 2. We elaborate here on *the* spherical Gauss-Bonnet operator, associated to the Euclidean Gauss-Bonnet operator specialized in Example 2. When $\mathbf{V} = \mathrm{Cl}_{n+1,0}\times\mathbf{C}$, it follows that a local basis of Ω^0_q, $q = 0, 1, \dots, n$, is given by $\{e_{i_1}e_{i_2}\cdots e_{i_q}\}_{1\le i_1<i_2<\cdots<i_q\le n}$, where $\{e_1, e_2, \dots, e_n\}$ is as usual a local orthonormal frame in $T(\mathbf{S}^n)$, and juxtaposition means multiplication in $\mathrm{Cl}_{n+1,0}\otimes\mathbf{C}$ via the pointwise identification

$$T_\omega(\mathbf{S}^n) \ni e_i = \sum_{j=0}^{n} c_{ij}\frac{\partial}{\partial x_j} \longleftrightarrow \sum_{j=0}^{n} c_{ij}\varepsilon_j \in \mathrm{Cl}_{n+1,0}\otimes\mathbf{C}.$$

This can be seen by induction on q, as in the proof of Lemma 2. Obviously,

$$T(\mathbf{S}^n)\cdot\Omega^0_q \subset \Omega^0_{q-1}\oplus\Omega^0_{q+1}, \quad q = 0, 1, \dots, n. \tag{44}$$

There is at most one $\mathrm{Cl}(\mathbf{S}^n)$-compatible connection ∇^0 on Ω^0 which on Ω^0_q would clearly satisfy

$$\nabla^0_{e_i}(e_{i_1}e_{i_2}\cdots e_{i_q})=\sum_{k=1}^{q} e_{i_1}e_{i_2}\cdots e_{i_{k-1}}\left(\nabla^{LC}_{e_i}e_{i_k}\right)e_{i_{k+1}}\cdots e_{i_q},\ (\nabla^0_{e_i}(1)=0). \tag{45}$$

Since, for any point of $\mathbf{S}^n$, there are local orthonormal frames $\{e_1, e_2, \dots, e_n\}$ in $T(\mathbf{S}^n)$ such that $\nabla^{LC}_{e_i}e_j = 0$ at that point, one can see that equation (45) indeed defines a connection which, moreover, leaves Ω^0_q invariant. Also, equations (20) and (29) prove that

$$A_{|\Omega^0_q} = q\mathbf{n}\cdot, \quad q = 0, 1, 2, \dots, n. \tag{46}$$

For a different way of defining the connection ∇^0, based on Lie algebra representations, see [**18**, page 107].

Finally, equations (23) and (44) show that

$$\not{\partial}(\Gamma_q) \subset \Gamma_{q-1} \oplus \Gamma_{q+1}, \quad q = 0, 1, 2, \dots, n. \tag{47}$$

We are now in a position to state and prove the main result of this paper.

The spectral decomposition for spherical Dirac-type operators. *Assume that* $\not{D}$ *is an Euclidean Dirac operator on* $C^\infty(\mathbf{R}^{n+1}, \mathbf{V})$, *where* $\mathbf{V} = \oplus_{q=0}^{p+1}\mathbf{V}_q$ *is a graded representation of* $\mathrm{Cl}_{n+1,0}$ (*cf. equation* (5)). *Assume that* $\mathbf{S}^n \times \mathbf{V}$ *admits a* $\mathrm{Cl}(\mathbf{S}^n)$-*compatible metric connection* ∇^0 (*cf. equation* (18)) *which leaves* $\mathbf{S}^n \times \mathbf{V}_q$, $q = 0, 1, \dots, p+1$, *and therefore the subbundles* Ω^0_q *of* $\mathbf{S}^n \times \mathbf{V}_q$, $q = 0, 1, \dots, p$ (*cf. Lemma* 2), *invariant, and in addition, satisfies equation* (21). *Assume that the bundle morphism* A *appearing in the separation of variables formula* (36) *satisfies the provisions of equation* (34), *for* $0 \le \lambda_q \le n$, $q = 0, 1, \dots, p$. *Then, for the spherical Dirac operator* $\not{\partial}$ *of equation* (36), *we have the following spectral decomposition* (*see also Proposition* 3):

a) *If* $\not{\partial}$ *is a spherical classical Dirac-type operator* (*cf. Definition,* a)), *then for any* $q = 0, 1, \dots, p$, *the spectrum of* $\mathbf{n}\cdot\not{\partial}_{|\Omega^0_q}$, *belongs to two disjoint families,* $\lambda_q + k$, *and* $\lambda_q - n - k$, $k = 0, 1, 2, \dots$. *Moreover, if*

$E_\lambda(\mathbf{n}\cdot\not{\partial}_q)$ *is the eigenspace of* $\mathbf{n}\cdot\not{\partial}_{|\Omega_q^0}$ *corresponding to the eigenvalue* λ*, and* m_λ^q *is the associated multiplicity, then* $E_\lambda(\mathbf{n}\cdot\not{\partial}_q)$ *embeds naturally in* $H^k(\mathbf{V}_q)$ *if* $\lambda=\lambda_q+k$*, and in* $H^k(\mathbf{V}_{q+1})$ *if* $\lambda=\lambda_q-n-k$*, and*

$$m^q_{\lambda_q+k}+m^{q-1}_{\lambda_{q-1}-n-k}=\dim H^k(\mathbf{V}_q),\quad k=0,1,2,\dots\,. \tag{48}$$

In particular,

$$m^0_{\lambda_0+k}=\dim H^k(\mathbf{V}_0)\quad \textit{and}\quad m^p_{\lambda_p-n-k}=\dim H^k(\mathbf{V}_{p+1}),$$
$$k=0,1,2,\dots\,.$$

Therefore, if $p=0$*, the spherical classical Dirac-type operator* $\mathbf{n}\cdot\not{\partial}$ *has only one component,* $\mathbf{n}\cdot\not{\partial}:\Gamma_0\to\Gamma_0$ *whose spectrum is precisely* λ_0+k*,* λ_0-n-k*,* $k=0,1,2,\dots$*, with multiplicities*

$$m^0_{\lambda_0+k}=m^0_{\lambda_0-n-k}=\frac{1}{2}\dim H^k(\mathbf{V}).$$

b) *If* $\not{\partial}$ *is a spherical Gauss-Bonnet-type operator* (*cf. Definition,* b))*, then for any* $q=0,1,\dots,p$*, the positive spectrum of the q-Laplace-Beltrami-type operator* $\not{\partial}^2_{|\Omega_q^0}$*, belongs to the non-zero elements of two families,* $(\lambda_{q+1}+k)(n+k-\lambda_q)$ *and* $(\lambda_q+k)(n+k-\lambda_{q-1})$*,* $k=0,1,2,\dots$ *. Moreover, if* m_λ^q *is the multiplicity of* $\lambda>0$ *as an eigenvalue of* $\not{\partial}^2_{|\Omega_q^0}$*, then*

$$\begin{aligned} &m^q_{(\lambda_{q+1}+k)(n+k-\lambda_q)}=\dim H^k(\mathbf{V}_{q+1}),\\ &m^q_{(\lambda_q+k)(n+k-\lambda_{q-1})}=\dim H^k(\mathbf{V}_q),\quad k=0,1,2,\dots\,, \end{aligned} \tag{49}$$

unless there are non-negative integers k *and* l *such that* $(\lambda_{q+1}+k)(n+k-\lambda_q)=(\lambda_q+l)(n+l-\lambda_{q-1})$*, in which case*

$$m^q_{(\lambda_{q+1}+k)(n+k-\lambda_q)}=m^q_{(\lambda_q+l)(n+l-\lambda_{q-1})}=\dim H^k(\mathbf{V}_{q+1})+\dim H^l(\mathbf{V}_q). \tag{50}$$

$\lambda=0$ *may be an eigenvalue of* $\not{\partial}^2_{|\Omega_q^0}$ *only if either* $\lambda_q=0$ *or* $\lambda_q=n$*. If* $\lambda_q=0$*, then the corresponding eigenspace embeds naturally in* $H^0(\mathbf{V}_q)$*, while if* $\lambda_q=n$*, it embeds in* $H^0(\mathbf{V}_{q+1})$*. If* $q=0$ *and* $\lambda_0=0$*,* *then* $m_0^0 = \dim H^0(\mathbf{V}_q) = \dim \mathbf{V}_q$, *and if* $q = p$ *and* $\lambda_p = n$, *then* $m_0^p = \dim H^0(\mathbf{V}_{p+1}) = \dim \mathbf{V}_{p+1}$. *These two last cases also fit the description provided by equation* (49), *for the zero values of the families of eigenvalues indicated there.*

Proof. a) For a fixed $q = 0, 1, \dots, p$, let $\lambda \in \mathbf{R}$ be an eigenvalue of $\mathbf{n}\cdot\not\partial_{|\Omega_q^0}$, with non-zero eigensection $\sigma_q \in \Gamma_q$. Then equation (43) shows that $r^{\lambda-\lambda_q}\sigma_q(\omega) \in C^\infty((0,\infty), \Gamma_q) \subset C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V}_q)$ is a 0-eigenspinor for $\not{D}|_{\mathbf{R}^{n+1}\setminus\{0\}}$. By Proposition 1, there is a non-negative integer k such that either $\lambda - \lambda_q = k$ or $\lambda - \lambda_q = -n-k$. If $\lambda = \lambda_q + k$, $x = 0$ is a removable singularity of $r^{\lambda-\lambda_q}\sigma_q(\omega) = r^k\sigma_q(\omega)$, which then belongs to $H^k(\mathbf{V}_q)$. If $\lambda = \lambda_q - n - k$, then, again by Proposition 1, there is an $h_k \in H^k(\mathbf{V})$ such that

$$r^{\lambda-\lambda_q}\sigma_q(\omega) = r^{-n-k}\sigma_q(\omega) = \frac{x\cdot h_k(x)}{|x|^{2k+n+1}}, \quad x = r\omega,\ |x| = r,\ x \neq 0.$$

Since $x\cdot h_k(x) = r^{k+1}\mathbf{n}\cdot h_k(\omega)$, we conclude that $h_k(\omega) = -\mathbf{n}\cdot\sigma_q(\omega) \in \mathbf{V}_{q+1}$, and so $h_k \in H^k(\mathbf{V}_{q+1})$.

Conversely, if $p_k \in H^k(\mathbf{V}_q)$ for some $k = 0, 1, 2, \dots$, then $p_k(x) = r^k(\sigma_q(\omega) + \mathbf{n}\cdot\sigma_{q-1}(\omega))$, $\sigma_q \in \Gamma_q$, $\sigma_{q-1} \in \Gamma_{q-1}$, and then $\not{D}(p_k) = 0$ is equivalent to $\mathbf{n}\cdot\not\partial\sigma_q = (\lambda_q + k)\sigma_q$ and $\mathbf{n}\cdot\not\partial\sigma_{q-1} = (\lambda_{q-1} - n - k)\sigma_{q-1}$. The rest of the claims in a) are then obvious.

b) Assume now that $\not\partial$ is a spherical Gauss-Bonnet-type operator, and for $\lambda \geq 0$ and $q = 0, 1, \dots, p$, define

$$\not\partial_q^2 := \not\partial^2_{|\Omega_q^0}, \qquad E_\lambda(\not\partial_q^2) := \{\sigma_q \in \Gamma_q \mid \not\partial^2\sigma_q = \lambda\sigma_q\},$$
$$E_\lambda^\pm(\not\partial_q^2) := \{\sigma_q \in \Gamma_q \mid \not\partial^2\sigma_q = \lambda\sigma_q,\ \not\partial\sigma_q \in \Gamma_{q\pm1}\}.$$

In preparation for proving b), we first show that, if $\lambda > 0$ is an eigenvalue of $\not\partial_q^2$, then there is an isomorphism

$$E_\lambda(\not\partial_q^2) \simeq E_\lambda^+(\not\partial_{q-1}^2) \oplus E_\lambda^-(\not\partial_{q+1}^2),$$

implemented by the mapping

$$E_\lambda(\not\partial_q^2) \ni \sigma_q \longmapsto \not\partial\sigma_q = \sigma_{q-1} + \sigma_{q+1} \in \Gamma_{q-1} \oplus \Gamma_{q+1}.$$

This mapping is well defined in the sense that, if $\sigma_q \in E_\lambda(\not\partial^2_q)$ then $\sigma_{q\mp1} \in E^\pm_\lambda(\not\partial^2_{q\mp1})$. Indeed, $\not\partial^2\sigma_q = \lambda\sigma_q$ implies that $\not\partial\sigma_{q-1} + \not\partial\sigma_{q+1} = \lambda\sigma_q$, and since $\not\partial\sigma_{q\mp1} \in \Gamma_{q\mp2} \oplus \Gamma_q$, we have $\not\partial\sigma_{q\mp1} \in \Gamma_q$. Also, from $\not\partial^3\sigma_q = \lambda\not\partial\sigma_q$, we see that $\not\partial^2\sigma_{q-1} + \not\partial^2\sigma_{q+1} = \lambda(\sigma_{q-1} + \sigma_{q+1})$, and since $\not\partial^2(\Gamma_{q\mp1}) \subset \Gamma_{q\mp1}$, we have $\not\partial^2\sigma_{q\mp1} = \lambda\sigma_{q\mp1}$. Consequently, if $\sigma_q \in E_\lambda(\not\partial^2_q)$, then $\not\partial\sigma_q = \sigma_{q-1} + \sigma_{q+1} \in E^+_\lambda(\not\partial^2_{q-1}) \oplus E^-_\lambda(\not\partial^2_{q+1})$.

The mapping $\sigma_q \mapsto \not\partial\sigma_q = \sigma_{q-1} + \sigma_{q+1}$ is also one-to-one. Indeed, if $\not\partial\sigma_q = 0$, then $\lambda\sigma_q = \not\partial^2\sigma_q = 0$, and since $\lambda \neq 0$, $\sigma_q = 0$.

Finally, to the end of proving that the mapping $\sigma_q \mapsto \not\partial\sigma_q = \sigma_{q-1} + \sigma_{q+1}$ is onto, we infer that the mapping $\not\partial_{|\Omega^0} : C^\infty(\mathbf{S}^n, \Omega^0) \to C^\infty(\mathbf{S}^n, \Omega^0)$ splits $C^\infty(\mathbf{S}^n, \Omega^0)$ as $C^\infty(\mathbf{S}^n, \Omega^0) = \ker\not\partial_{|\Omega^0} \oplus (\ker\not\partial_{|\Omega^0})^\perp$ and $(\ker\not\partial_{|\Omega^0})^\perp = \operatorname{im}\not\partial_{|\Omega^0}$. This is a general property of elliptic self-adjoint differential operators on compact manifolds. Here the orthogonal complement is taken with respect to the global inner product $(\cdot,\cdot)$ of $C^\infty(\mathbf{S}^n, \Omega^0) \subset L^2(\mathbf{S}^n, \mathbf{V})$. We claim that

$$E^+_\lambda(\not\partial^2_{q-1}) \oplus E^-_\lambda(\not\partial^2_{q+1}) \subset \left(\ker\not\partial_{|\Omega^0}\right)^\perp.$$

Indeed, if $\tau_{q-1} + \tau_{q+1} \in E^+_\lambda(\not\partial^2_{q-1}) \oplus E^-_\lambda(\not\partial^2_{q+1})$ and if $\alpha \in \ker\not\partial_{|\Omega^0}$, then

$$\begin{aligned}\lambda(\tau_{q-1} + \tau_{q+1}, \alpha) &= (\lambda\tau_{q-1} + \lambda\tau_{q+1}, \alpha)\\ &= (\not\partial^2\tau_{q-1} + \not\partial^2\tau_{q+1}, \alpha)\\ &= (\not\partial\tau_{q-1} + \not\partial\tau_{q+1}, \not\partial\alpha) = 0,\end{aligned}$$

and so $\tau_{q-1} + \tau_{q+1} \perp \alpha$. There is then $\alpha \in C^\infty(\mathbf{S}^n, \Omega^0)$ such that $\not\partial\alpha = \tau_{q-1} + \tau_{q+1}$. If $\alpha = \sum_{r=0}^p \alpha_r$, $\alpha_r \in \Gamma_r$, then $\not\partial^2\alpha = \not\partial\tau_{q-1} + \not\partial\tau_{q+1} \in \Gamma_q$ implies $\sum_{r=0}^p \not\partial^2\alpha_r \in \Gamma_q$, and so we have $\not\partial^2\alpha_r = 0$ if $r \neq q$, since $\not\partial^2\alpha_r \in \Gamma_r$. Thus, $\not\partial^2\alpha_q = \not\partial\tau_{q-1} + \not\partial\tau_{q+1} \in \Gamma_q$ implies

$$\not\partial^3\alpha_q = \not\partial^2\tau_{q-1} + \not\partial^2\tau_{q+1} = \lambda(\tau_{q-1} + \tau_{q+1}),$$

which also gives $\not\partial(\not\partial^2\alpha_q/\lambda) = \tau_{q-1} + \tau_{q+1}$. We just proved that there is a $\beta_q \in \Gamma_q$ such that $\not\partial\beta_q = \tau_{q-1} + \tau_{q+1}$. Without loss of generality, we can choose this β_q such that $\beta_q \perp (\ker\not\partial_{|\Omega^0}) \cap \Gamma_q$. Indeed, $\Gamma_q = \ker\not\partial^2_q \oplus (\ker\not\partial^2_q)^\perp$; however, $\ker\not\partial^2_q = \ker\not\partial_{|\Omega^0_q}$. Now $\not\partial\beta_q = \tau_{q-1} + \tau_{q+1}$ implies $\not\partial^3\beta_q = \lambda(\tau_{q-1} + \tau_{q+1}) = \lambda\not\partial\beta_q$, and so

$\partial\!\!\!/(\partial\!\!\!/^2\beta_q - \lambda\beta_q) = 0$. Since $\partial\!\!\!/^2\beta_q - \lambda\beta_q \in \ker \partial\!\!\!/_q^2 \cap (\ker \partial\!\!\!/_q^2)^\perp$, $\partial\!\!\!/^2\beta_q = \lambda\beta_q$. The proof of the claim that

$$E_\lambda(\partial\!\!\!/_q^2) \ni \sigma_q \longmapsto \partial\!\!\!/(\sigma_q) = \sigma_{q-1} + \sigma_{q+1} \in E_\lambda^+(\partial\!\!\!/_{q-1}^2) \oplus E_\lambda^-(\partial\!\!\!/_{q+1}^2)$$

is an isomorphism is now complete.

However, for $\lambda \neq 0$, the mappings $\sigma_{q\mp1} \mapsto \partial\!\!\!/\sigma_q$ trivially implement isomorphisms $E_\lambda^\pm(\partial\!\!\!/_{q\mp1}^2) \simeq E_\lambda^\mp(\partial\!\!\!/_q^2)$, and, since $E_\lambda^+(\partial\!\!\!/_q^2) \oplus E_\lambda^-(\partial\!\!\!/_q^2) \subset E_\lambda(\partial\!\!\!/_q^2)$, we just proved that

$$E_\lambda(\partial\!\!\!/_q^2) = E_\lambda^+(\partial\!\!\!/_q^2) \oplus E_\lambda^-(\partial\!\!\!/_q^2). \tag{51}$$

We claim now that if, for $\lambda > 0$, $E_\lambda^+(\partial\!\!\!/_q^2) \neq 0$, then necessarily λ belongs to the family $(\lambda_{q+1} + k)(n + k - \lambda_q)$, $k = 0, 1, \dots$, and then $E^+_{(\lambda_{q+1}+k)(n+k-\lambda_q)}(\partial\!\!\!/_q^2)$ is naturally isomorphic to $H^k(\mathbf{V}_{q+1})$. Likewise, if $E_\lambda^-(\partial\!\!\!/_q^2) \neq 0$, then necessarily λ belongs to the family $(\lambda_q + k)(n + k - \lambda_{q-1})$, $k = 0, 1, \dots$, and then $E^-_{(\lambda_q+k)(n+k-\lambda_{q-1})}(\partial\!\!\!/_q^2)$ is naturally isomorphic to $H^k(\mathbf{V}_q)$.

To this end, let $\sigma_q \neq 0$ be an element of $E_\lambda^+(\partial\!\!\!/_q^2)$. Then the separation of variables formula (36) shows that there are real numbers α and c such that, for

$$C^\infty(\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{V}_{q+1}) \ni s \equiv r^\alpha(c\partial\!\!\!/\sigma_q + \mathbf{n}\cdot\sigma_q) \in C^\infty((0,\infty), C^\infty(\mathbf{S}^n, \mathbf{V}_{q+1})),$$

$D\!\!\!/ s = 0$ if and only if α and c satisfy

$$(\alpha + \lambda_{q+1})(\alpha + n - \lambda_q) = \lambda, \quad \text{and} \quad \lambda c = \alpha + n - \lambda_q. \tag{52}$$

Since $0 \leq \lambda_q, \lambda_{q+1} \leq n$, for fixed $\lambda > 0$ and q there are real numbers α and c, α unique subject to the inequality $\alpha > \max(-\lambda_{q+1}, \lambda_q - n)$, such that (52) holds. Also, since $s(x) = r^\alpha(c\partial\!\!\!/\sigma_q(\omega) + \mathbf{n}\cdot\sigma_q(\omega))$ is homogeneous of degree α and $D\!\!\!/ s = 0$, we must have that α is an integer such that $\alpha \geq 0$ or $\alpha \leq -n$, one more time invoking Proposition 1. Now the values of α in the range $\alpha \leq -n$ are excluded by the inequality $\alpha > \max(-\lambda_{q+1}, \lambda_q - n)$, so we must have $\alpha \geq 0$. Therefore, $x = 0$ is a removable singularity of s and, moreover, $s \in H^k(\mathbf{V}_{q+1})$ for some non-negative integer k.

Since, conversely, the elements of $H^k(\mathbf{V}_{q+1})$ can be written on $\mathbf{R}^{n+1}\setminus\{0\}$ as $r^k(\sigma_{q+1}(\omega)+\mathbf{n}\boldsymbol{\cdot}\sigma_q(\omega))$, where $\sigma_q\in C^\infty(\mathbf{S}^n,\Omega^0_q)$, $\sigma_{q+1}\in C^\infty(\mathbf{S}^n,\Omega^0_{q+1})$, $\not{\partial}^2\sigma_q=(\lambda_{q+1}+k)(n+k-\lambda_q)\sigma_q$, and $\not{\partial}\sigma_q=(\lambda_{q+1}+k)\sigma_{q+1}$, the claimed isomorphism between $E^+_{(\lambda_{q+1}+k)(n+k-\lambda_q)}(\not{\partial}^2_q)$ and $H^k(\mathbf{V}_{q+1})$ also follows.

The statement about $E^-_\lambda(\not{\partial}^2_q)$ can be proved similarly, by considering 0-eigenspinors of $\not{D}$ on $\mathbf{R}^{n+1}\setminus\{0\}$ of type $r^\alpha(\sigma_q(\omega)+c\mathbf{n}\boldsymbol{\cdot}\not{\partial}\sigma_q(\omega))$, associated to elements $\sigma_q\in E^-_\lambda(\not{\partial}^2_q)$.

The precise description of $E^\pm_\lambda(\not{\partial}^2_q)$ in terms of polynomial Dirac spinors from $H^k(\mathbf{V}_{q+1})$ or $H^k(\mathbf{V}_q)$ together with equation (51) now yield the multiplicity (49) and (50) of $\lambda>0$ as an eigenvalue of $\not{\partial}^2_q$.

Finally, the possible case of the eigenvalue $\lambda=0$ is implicit in the analysis provided at a). □

Corollary. a) *For the spherical classical Dirac operator* $\mathbf{n}\boldsymbol{\cdot}\not{\partial}_{|\Omega^0}$ *induced by the Euclidean classical Dirac operator* $\not{D}$ (*cf. Definition,* a), *and Example* 1) *the spectrum is* $\pm((n/2)+k)$, $k=0,1,2,\ldots$, *and the multiplicity of* $\pm((n/2)+k)$ *is*

$$2^{[n/2]}\binom{n+k-1}{k}.$$

b) *For the spherical Laplace-Beltrami operator* $\not{\partial}^2_{|\Omega^0_q}$, $q=0,1,\ldots,n$, *associated to the Euclidean Gauss-Bonnet operator* $\not{D}$ (*cf. Definition,* b), *and Example* 2) *the spectrum belongs to the families* $(q+1+k)(n+k-q)$ *and* $(q+k)(n+k-q+1)$, $k=0,1,2,\ldots$. *The multiplicity of* $(q+1+k)(n+k-q)$ *equals*

$$\begin{cases}\frac{q+1}{q+1+k}\frac{n+1+2k}{n-q+k}\binom{n+k}{k}\binom{n}{q+1} & \text{if } 0\le q\le n-1,\ k=0,1,2,\ldots\\ 1 & \text{if } q=n, k=0,\end{cases}$$

and the multiplicity of $(q+k)(n+k-q+1)$ *equals*

$$\begin{cases}\frac{q}{q+k}\frac{n+1+2k}{n+1-q+k}\binom{n+k}{k}\binom{n}{q} & \text{if } 1\le q\le n,\ k=0,1,2,\ldots\\ 1 & \text{if } q=0, k=0,\end{cases}$$

unless n is an even integer and $q = n/2$, in which case the multiplicity of

$$\left(\frac{n}{2} + 1 + k\right)\left(\frac{n}{2} + k\right)$$

equals

$$4\frac{n}{n+2k}\frac{n+1+2k}{n+2+2k}\binom{n+k}{k}\binom{n}{n/2}.$$

Proof. The proof of a) is an obvious consequence of the spectral decomposition theorem for spherical classical Dirac-type operators, since $p = 0$, $\lambda_0 = n/2$ and

$$\dim H^k(\mathbf{V}) = 2^{[n/2]+1}\binom{n+k-1}{k}$$

(cf. Definition, a), and Example 1).

The proof of b) is an obvious consequence of the spectral decomposition theorem for spherical Laplace-Beltrami-type operators, since $p = n$, $\lambda_q = q$, $q = 0, 1, \ldots, n$ (cf. Definition, b), Example 2 and equation (17)). Notice that the two families of eigenvalues are disjoint, unless n is an even integer and $q = n/2$. Notice also that the elements in the two families corresponding to $q = 0$, n and $k > 0$ are not eigenvalues, while the eigenvalue $\lambda = 0$, treated separately in the spectral decomposition for the spherical Laplace-Beltrami-type operators, incorporates nicely in the two-family eigenvalue description. □

REFERENCES

1. N. Anghel, *L^2-index formulae for perturbed Dirac operators*, Comm. Math. Phys. **128** (1990), 77–97.

2. ———, *Projecting on polynomial Dirac spinors*, Geom., Integr. Quantiz. **8** (2007), 121–126.

3. M. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.

4. S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer, New York, 2001.

5. C. Bär, *The Dirac operator on space forms of positive curvature*, J. Math. Soc. Japan **48** (1996), 69–83.

6. ———, *Extrinsic bounds for eigenvalues of the Dirac operator*, Ann. Global Anal. Geom. **16** (1998), 573–596.

7. H. Baum, *Spin-Strukturen und Dirac-Operatoren über Pseudoriemannschen Mannigfaltigkeiten*, Teub.-Text. Math. **41** (1981).

8. F. Brackx, R. Delanghe and F. Sommen, *Differential forms and/or multi-vector functions*, Cubo **7** (2005), 139–169.

9. A. Chow, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. **289** (1985), 1–40.

10. F. Colombo, I. Sabadini, F. Sommen and D. Struppa, *Analysis of Dirac systems and computational algebra*, Birkhäuser, Boston, 2004.

11. G. Folland, *Harmonic analysis of the De Rham complex on the sphere*, J. reine angew. Math. **398** (1989), 130–143.

12. S. Gallot and D. Meyer, *Opérateur de Courbure et Laplacien des Formes Différentielles d'une Variété Riemannienne*, J. Math. Pure. Appl. **54** (1975), 259–284.

13. M. Gromov and B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IHES **58** (1983), 83–196.

14. O. Hijazi and X. Zhang, *Lower bounds for the eigenvalues of the Dirac operator*: Part I. *The hypersurface Dirac operator*, Ann. Global Anal. Geom. **19** (2001), 355–376.

15. Y. Homma, *Spinor-valued and Clifford algebra-valued harmonic polynomials*, J. Geom. Phys. **37** (2001), 201–215.

16. A. Ikeda and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on* S^n *and* $P^n(C)$, Osaka J. Math. **15** (1978), 515–546.

17. I. Iwasaki and K. Katase, *On the spectra of Laplace operator on* $\Lambda^*(S^n)$, Proc. Japan Acad. **55** (1979), 141–145.

18. B. Lawson and M-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, NJ, 1989.

19. B. O'Neil, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

20. L. Paquet, *Méthode de Séparation des Variables et Calcul du Spectre d'Operateurs sur les Formes Différentielles* , Bull. Sci. Math. **105** (1981), 85–112.

21. M. Shubin, *Pseudodifferential operators and spectral theory*, Springer, New York, 1978.

22. S. Sulanke, *Berechnung des Spektrums des Quadrates des Dirac-Operators auf der Sphäre*, Ph.D. thesis, Humboldt-Universität, Berlin, 1981.

23. A. Trautman, *The Dirac operator on hypersurfaces*, Acta Phys. **26** (1995), 1283–1310.

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