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SHARP INEQUALITIES INVOLVING THE POWER MEAN AND COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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ABSTRACT. In this paper, we prove that $M_p(\mathcal{K}(r),\mathcal{K}(r')) \geq \mathcal{K}(\sqrt{2}/2)$ and $M_q(\mathcal{K}(r),\mathcal{K}(r')) \leq \mathcal{K}(\sqrt{2}/2)$ for all $r \in (0,1)$ if and only if $p \geq 1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\ldots$ and $q \leq (\log 2)/[\log(\pi/2)-\log\mathcal{K}(\sqrt{2}/2)] = -4.180\ldots$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta$ is the complete elliptic integral of the first kind, $r'=\sqrt{1-r^2}$, and $M_p(x,y)$ is the power mean of order p of two positive numbers x and y.

1. Introduction. Throughout this paper, we denote $r'=\sqrt{1-r^2}$ for $0<r<1$. The well-known complete elliptic integrals of the first and second kinds [**13, 15**] are defined by

$$\begin{cases} \mathcal{K}=\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathcal{K}'=\mathcal{K}'(r)=\mathcal{K}(r'), \\ \mathcal{K}(0)=\pi/2, \quad \mathcal{K}(1)=\infty \end{cases}$$

and

$$\begin{cases} \mathcal{E}=\mathcal{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \\ \mathcal{E}'=\mathcal{E}'(r)=\mathcal{E}(r'), \\ \mathcal{E}(0)=\pi/2, \quad \mathcal{E}(1)=1, \end{cases}$$

respectively.

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory,

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quasiconformal analysis, theory of mean values, number theory and other related fields [**4, 5, 8, 9, 15, 17–20**].

Recently, complete elliptic integrals have been the subject of intensive research. In particular, many remarkable properties and inequalities can be found in the literature [**1–4, 6, 7, 10–12, 16, 19**].

For $p \in \mathbf{R}$, the power mean $M_p(x,y)$ of order p of two positive numbers x and y is defined by

$$M_p(x,y) = \begin{cases} (x^p + y^p/2)^{1/p} & p \neq 0, \\ \sqrt{xy} & p = 0. \end{cases}$$

The main properties of the power mean are given in [**14**].

In [**8**, Lemma 3.32 (1), (3)], Anderson, Vamanamurthy and Vuorinen studied the monotonicity of $\mathcal{K}(r)\mathcal{K}(r')$ and $\mathcal{K}(r)^p + \mathcal{K}(r')^p$ for $p \in [-3,0)$ and $r \in (0,1)$ and established the following inequalities:

$$M_0(\mathcal{K}(r), \mathcal{K}(r')) \geq \mathcal{K}\Big(\sqrt{2}/2\Big) \tag{1.1}$$

and

$$\mathcal{K}\Big(\sqrt{2}/2\Big) \leq M_p(\mathcal{K}(r), \mathcal{K}(r')) < \pi/2^{1+1/p},$$

for all $r \in (0,1)$ and $p \in [-3,0)$.

It is natural to ask what are the least value p and the greatest value q such that $M_p(\mathcal{K}(r), \mathcal{K}(r')) \geq \mathcal{K}(\sqrt{2}/2)$ and $M_q(\mathcal{K}(r), \mathcal{K}(r')) \leq \mathcal{K}(\sqrt{2}/2)$ for all $r \in (0,1)$. The main purpose of this paper is to answer this question. Our main result is the following Theorem 1.1.

Theorem 1.1. *Inequalities*

$$M_p(\mathcal{K}(r), \mathcal{K}(r')) \geq \mathcal{K}\Big(\sqrt{2}/2\Big) \tag{1.2}$$

and

$$M_q(\mathcal{K}(r), \mathcal{K}(r')) \leq \mathcal{K}(\sqrt{2}/2) \tag{1.3}$$

hold for all $r \in (0,1)$ *if and only if* $p \geq 1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\ldots$ *and* $q \leq (\log 2)/[\log(\pi/2) - \log \mathcal{K}(\sqrt{2}/2)] = -4.180\ldots$.

2. Lemmas. In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$, the following formulas were presented in [**8**, Appendix E, pages 474–475]:

$$d\mathcal{K}/dr = (\mathcal{E} - r'^2\mathcal{K})/(rr'^2), \qquad d\mathcal{E}/dr = (\mathcal{E} - \mathcal{K})/r,$$

$$d(\mathcal{E} - r'^2\mathcal{K})/dr = r\mathcal{K}, \qquad d(\mathcal{K} - \mathcal{E})/dr = r\mathcal{E}/r'^2,$$

$$\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}' = \pi/2. \tag{2.1}$$

The following Lemma 2.1 can be found in [**8**, Theorem 3.21 (1) and (7), and Exercise 3.43 (16) and (46)].

Lemma 2.1. (1) $(\mathcal{E} - r'^2\mathcal{K})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$;

(2) *For* $c \in [1/2, \infty)$, $r'^c\mathcal{K}$ *is strictly decreasing from* $[0,1)$ *onto* $(0, \pi/2]$;

(3) $[\mathcal{E}^2 - (r'\mathcal{K})^2]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/32, 1)$;

(4) $(\mathcal{E} - r^2\mathcal{K})/(r^2\mathcal{K})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, 1/2)$.

Lemma 2.2. *Let* $r \in (0,1)$. *Then the function* $f(r) = (\mathcal{E} - r'^2\mathcal{K})(\mathcal{E}' - r^2\mathcal{K}')/(r^2r'^2\mathcal{K}\mathcal{K}')$ *is strictly increasing from* $(0, \sqrt{2}/2)$ (*or strictly decreasing from* $(\sqrt{2}/2, 1)$, *respectively*) *onto* $(0, \pi^2/\{4[\mathcal{K}(\sqrt{2}/2)]^4\})$.

Proof. By differentiation, we have

$$\begin{aligned} f'(r) &= \frac{r\mathcal{K}(r^2\mathcal{K}) - (\mathcal{E} - r'^2\mathcal{K})[2r\mathcal{K} + r^2(\mathcal{E} - r'^2\mathcal{K})/(rr'^2)]}{r^4\mathcal{K}^2} \\ &\quad \times \left(\frac{\mathcal{E}' - r^2\mathcal{K}'}{r'^2\mathcal{K}'}\right) + \left(\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2\mathcal{K}}\right) \\ &\quad \times \frac{-r\mathcal{K}'(r'^2\mathcal{K}') - (\mathcal{E}' - r^2\mathcal{K}')[-2r\mathcal{K}' - r'^2(\mathcal{E}' - r^2\mathcal{K}')/(rr'^2)]}{r'^4\mathcal{K}'^2}, \\ &= r[f_1(r) - f_1(r')], \end{aligned} \tag{2.2}$$

where

$$f_1(r) = \frac{\mathcal{E} - r'^2\mathcal{K}}{r^2\mathcal{K}} \frac{(\mathcal{E}')^2 - (r\mathcal{K}')^2}{r'^4} \frac{1}{(r\mathcal{K}')^2}. \tag{2.3}$$

It follows from (2.3) and Lemma 2.1 (2)–(4) that $f_1(r)$ is strictly decreasing in $(0,1)$. Then (2.2) leads to the conclusion that $f'(r) > 0$ for $r \in (0, \sqrt{2}/2)$ and $f'(r) < 0$ for $r \in (\sqrt{2}/2, 1)$. Hence, $f(r)$ is strictly increasing in $(0, \sqrt{2}/2)$ and strictly decreasing in $(\sqrt{2}/2, 1)$. Moreover, making use of Lemma 2.1 (4) and (2.1) we clearly see that $f(0^+) = f(1^-) = 0$ and

$$f(\sqrt{2}/2) = \frac{4\left[\mathcal{E}(\sqrt{2}/2) - (1/2)\mathcal{K}(\sqrt{2}/2)\right]^2}{\mathcal{K}(\sqrt{2}/2)^2} = \frac{\pi^2}{4[\mathcal{K}(\sqrt{2}/2)]^4}. \qquad \square$$

Lemma 2.3. *Let $p \in \mathbf{R}$ and $g(r) = (\mathcal{K}/\mathcal{K}')^{p-1}(\mathcal{E} - r'^2\mathcal{K})/(\mathcal{E}' - r^2\mathcal{K}')$. Then $g(r)$ is strictly increasing in $(0,1)$ if and only if $p \geq 1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\ldots$, and $g(r) < 1$ for $r \in (0, \sqrt{2}/2)$ and $g(r) > 1$ for $r \in (\sqrt{2}/2, 1)$ if $p \geq 1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$. Moreover, if $p < 1 - 4\mathcal{K}(\sqrt{2}/2)^4/\pi^2$, then there exists an $r_0 = r_0(p) \in (0, \sqrt{2}/2)$, such that $g(r_0) = g(r_0') = 1$, $g(r) < 1$ for $r \in (0, r_0) \cup (\sqrt{2}/2, r_0')$, and $g(r) > 1$ for $r \in (r_0, \sqrt{2}/2) \cup (r_0', 1)$.*

Proof. Simple computations lead to

$$g\big(\sqrt{2}/2\big) = 1 \tag{2.4}$$

and

$$\begin{aligned} \frac{g'(r)}{g(r)} &= (p-1)\left(\frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2\mathcal{K}} + \frac{\mathcal{E}' - r^2\mathcal{K}'}{rr'^2\mathcal{K}'}\right) \\ &\quad + \frac{r\mathcal{K}}{\mathcal{E} - r'^2\mathcal{K}} + \frac{r\mathcal{K}'}{\mathcal{E}' - r^2\mathcal{K}'} \\ &= (p-1)\frac{\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}'}{rr'^2\mathcal{K}\mathcal{K}'} + \frac{r(\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}')}{(\mathcal{E} - r'^2\mathcal{K})(\mathcal{E}' - r^2\mathcal{K}')} \\ &= \frac{\pi}{2rr'^2\mathcal{K}\mathcal{K}'}\left[p - 1 + \frac{r^2r'^2\mathcal{K}\mathcal{K}'}{(\mathcal{E} - r'^2\mathcal{K})(\mathcal{E}' - r^2\mathcal{K}')}\right]. \end{aligned} \tag{2.5}$$

It follows from Lemma 2.2 that $r^2{r'}^2\mathcal{K}\mathcal{K}'/[(\mathcal{E}-{r'}^2\mathcal{K})(\mathcal{E}'-r^2\mathcal{K}')]$ is strictly decreasing from $(0,\sqrt{2}/2)$ (or strictly increasing from $(\sqrt{2}/2,1)$, respectively) onto $(4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2,\infty)$. Then (2.4) and (2.5) lead to the conclusion that $g(r)$ is strictly increasing in $(0,1)$ if and only if $p\geq 1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2=-3.789\ldots$, and $g(r)<1$ for $r\in(0,\sqrt{2}/2)$ and $g(r)>1$ for $r\in(\sqrt{2}/2,1)$ if $p\geq 1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$. Moreover, if $p<1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$, then from (2.5) we know that there exists $r_1\in(0,\sqrt{2}/2)$, such that $g'(r_1)=g'({r_1}')=0$, $g'(r)>0$ for $r\in(0,r_1)\cup({r_1}',1)$ and $g'(r)<0$ for $r\in(r_1,{r_1}')$. Hence, $g(r)$ is strictly increasing in $(0,r_1)\cup({r_1}',1)$ and strictly decreasing in $(r_1,{r_1}')$. Therefore, Lemma 2.3 follows from (2.4) and the monotonicity of $g(r)$ together with

$$\begin{aligned}\lim_{r\to 0}g(r)&=\lim_{r\to 0}\mathcal{K}^{p-1}\frac{\mathcal{E}-{r'}^2\mathcal{K}}{r^2}\frac{1}{\mathcal{E}'-r^2\mathcal{K}'}\left[\frac{\mathcal{K}'}{(1/r^2)^{1/(1-p)}}\right]^{1-p}\\&=\lim_{r\to 0}\left(\frac{\pi^p}{2^{1+p}}\right)\left[\frac{\mathcal{K}'}{(1/r^2)^{1/(1-p)}}\right]^{1-p}\\&=\lim_{r\to 0}\left(\frac{\pi^p}{2^{1+p}}\right)r^2\left[\frac{(1-p)(\mathcal{E}'-r^2\mathcal{K}')}{2{r'}^2}\right]^{1-p}=0\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 1}g(r)&=\lim_{r\to 1}{\mathcal{K}'}^{1-p}(\mathcal{E}-{r'}^2\mathcal{K})\frac{{r'}^2}{\mathcal{E}'-r^2\mathcal{K}'}\left[\frac{(1/{r'}^2)^{1/(1-p)}}{\mathcal{K}}\right]^{1-p}\\&=\lim_{r\to 1}\left(\frac{2^{1+p}}{\pi^p}\right)\left[\frac{(1/{r'}^2)^{1/(1-p)}}{\mathcal{K}}\right]^{1-p}\\&=\lim_{r\to 1}\left(\frac{2^{1+p}}{\pi^p}\right)\frac{[2r^2/((1-p)(\mathcal{E}-r'^2\mathcal{K}))]^{1-p}}{r'^2}=+\infty.\end{aligned}$$ □

3. Proof of Theorem 1.1. If $p=0$, then inequality (1.2) reduces to inequality (1.1). Thus, we only need to prove inequality (1.2) for $p\neq 0$. Let

$$F(r)=\frac{1}{s}\log\frac{\mathcal{K}(r)^s+\mathcal{K}(r')^s}{2}\quad(s\neq 0). \tag{3.1}$$

Then simple computation leads to

$$
\begin{aligned}
F'(r) &= \frac{1}{s}\frac{s\mathcal{K}^{s-1}(\mathcal{E}-r'^2\mathcal{K})/(rr'^2)-s\mathcal{K}'^{s-1}(\mathcal{E}'-r^2\mathcal{K}')/(rr'^2)}{\mathcal{K}^s+\mathcal{K}'^s}\\
&= \frac{\mathcal{K}^{s-1}(\mathcal{E}-r'^2\mathcal{K})-\mathcal{K}'^{s-1}(\mathcal{E}'-r^2\mathcal{K}')}{rr'^{2(\mathcal{K}^s+\mathcal{K}'^s)}}\\
&= \frac{\mathcal{K}'^{s-1}(\mathcal{E}'-r^2\mathcal{K}')}{rr'^2(\mathcal{K}^s+\mathcal{K}'^s)}\left[\left(\frac{\mathcal{K}}{\mathcal{K}'}\right)^{s-1}\frac{\mathcal{E}-r'^2\mathcal{K}}{\mathcal{E}'-r^2\mathcal{K}'}-1\right].
\end{aligned}
\tag{3.2}
$$

We divide the proof into two cases.

Case 1. $s\geq 1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$. Then from (3.2) and Lemma 2.3 we know that $F'(r)<0$ for $r\in(0,\sqrt{2}/2)$ and $F'(r)>0$ for $r\in(\sqrt{2}/2,1)$. Hence, $F(r)$ is strictly decreasing in $(0,\sqrt{2}/2)$ and strictly increasing in $(\sqrt{2}/2,1)$. Then (3.1) leads to the conclusion that

$$\frac{1}{s}\log\frac{\mathcal{K}(r)^s+\mathcal{K}(r')^s}{2}\geq\log\mathcal{K}\big(\sqrt{2}/2\big) \tag{3.3}$$

for all $r\in(0,1)$.

Therefore, inequality (1.2) follows from (3.3).

Case 2. $s<1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$. Then, from (3.2) and Lemma 2.3, we clearly see that $F'(r)<0$ for $r\in(0,r_0)\cup(\sqrt{2}/2,r_0{}')$ and $F'(r)>0$ for $r\in(r_0,\sqrt{2}/2)\cup(r_0{}',1)$. Hence, $F(r)$ is strictly decreasing in $(0,r_0)\cup(\sqrt{2}/2,r_0{}')$, strictly increasing in $(r_0,\sqrt{2}/2)\cup(r_0{}',1)$, and

$$
\begin{aligned}
\sup_{r\in(0,1)}F(r) &= \max\Big\{\lim_{r\to0}F(r),F(\sqrt{2}/2),\lim_{r\to1}F(r)\Big\}\\
&= \max\Big\{\log(\pi/2)-\frac{1}{s}\log 2,\log\mathcal{K}\Big(\sqrt{2}/2\Big)\Big\}.
\end{aligned}
\tag{3.4}
$$

Further, if $(\log 2)/[\log(\pi/2)-\log\mathcal{K}(\sqrt{2}/2)]<s<1-4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$, then from (3.4) we have $\sup_{r\in(0,1)}F(r)=\log(\pi/2)-(\log 2)/s$ and

$$\frac{1}{s}\log\frac{\mathcal{K}(r)^s+\mathcal{K}(r')^s}{2}<\log(\pi/2)-\frac{1}{s}\log 2 \tag{3.5}$$

for all $r\in(0,1)$; if $s\leq(\log 2)/[\log(\pi/2)-\log\mathcal{K}(\sqrt{2}/2)]$, then from (3.4) we get $\sup_{r\in(0,1)}F(r)=\log\mathcal{K}(\sqrt{2}/2)$ and

$$\frac{1}{s}\log\frac{\mathcal{K}(r)^s+\mathcal{K}(r')^s}{2}\leq\log\mathcal{K}\big(\sqrt{2}/2\big)\quad\text{for all } r\in(0,1). \tag{3.6}$$

Therefore, inequality (1.3) follows from (3.6).

Next, we prove that the parameters $p = 1 - 4[\mathcal{K}(\sqrt{2}/2)^4/\pi]^2$ and $q = (\log 2)/[\log(\pi/2) - \log \mathcal{K}(\sqrt{2}/2)]$ are the best possible such that inequalities (1.2) and (1.3) hold for all $r \in (0,1)$, respectively.

If $q < s < p$, then, from the monotonicity of $F(r)$, we know that there exists an $r \in (\sqrt{2}/2, r_0')$, such that $F(r) < F(\sqrt{2}/2)$ and $M_s(\mathcal{K}(r), \mathcal{K}(r')) < \mathcal{K}(\sqrt{2}/2)$. Moreover, equation (3.4) and inequality (3.5) imply that there exists a $\delta = \delta(s) \in (0,1)$, such that $F(r) > \log \mathcal{K}(\sqrt{2}/2)$ and $M_s(\mathcal{K}(r), \mathcal{K}(r')) > \mathcal{K}(\sqrt{2}/2)$ for $r \in (0, \delta)$. ◻

Remark 3.1. For all $r \in (0,1)$, we have

$$M_s(\mathcal{K}(r), \mathcal{K}(r')) < \pi/2^{1+1/s} \tag{3.7}$$

if $s \in ((\log 2)/[\log(\pi/2) - \log \mathcal{K}(\sqrt{2}/2)], 0)$.

Proof. We divide the proof into two cases.

Case A. $(\log 2)/[\log(\pi/2) - \log \mathcal{K}(\sqrt{2}/2)] < s < 1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2$. Then inequality (3.7) follows from (3.5).

Case B. $1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2 \le s < 0$. Then inequality (3.7) follows from the monotonicity of $F(r)$ and the limiting values $\lim_{r\to 0} F(r) = \lim_{r \to 1} F(r) = \log(\pi/2) - (\log 2)/s$. ◻

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