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ROBUST STABILITY OF DELAY EQUATIONS IN $L^p([-h,0];X)$ UNDER PARAMETER PERTURBATIONS

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ABSTRACT. In this paper we study how the uniform boundedness of an operator associated with delay equation changes under affine parameter perturbations. Characterizations of the stability radius of the operator with respect to this type of disturbances are established. The obtained results are extensions of the recent work presented in [**10**].

1. Introduction. In the last two decades, a considerable amount of attention has been paid to problems of robust stability of dynamic systems in infinite-dimensional spaces. Interested readers are referred to [**1–4, 7, 10, 13, 20**] and the biography therein for further references. One of the most important problems in the study of robust stability is the calculation of the stability radius of a dynamic system subjected to various classes of parameter perturbations. Although there have been many works dedicated to stability radii problems of linear systems, so far there are a few results for the problem of computing the stability radii of delay differential systems under arbitrary affine perturbations, see [**3, 10**].

In this paper, we study the robustness of certain properties of delay equations in $L^p([-h,0];X)$

$$\dot{u}(t) = A_0 u(t) + \sum_{i=1}^{n} A_i u(t-h_i), \quad t \geq 0,$$

under multi-perturbations and affine perturbations, where A_0 is a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Banach lattice X, $A_1, \ldots, A_n$ on X are given bounded linear operators and $0 \leq h_1 < h_2 < \cdots < h_n =: h$.

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This paper consists of three sections. Section 2 recalls some useful results which will be used later. Section 3 establishes some results on robust stability: formulas for stability radii of delay equations in $L^p([-1,0],X)$ will be established, and it has been shown that, in the case of positive delay equations, the complex, real and positive stability radii coincide and can be calculated by a simple formula. The obtained results can be considered to be an extension to the recent work in [**10**].

2. Preliminaries. In this section, we recall some useful results for later use.

Now assume that X, Y are complex Banach lattices. Let X^+ and Y^+ denote positive cones of X and Y, respectively; and let $\mathcal{L}^{\mathbf{R}}(X,Y)$ and $\mathcal{L}^+(X,Y)$ be the sets of all the real and positive linear operators from X to Y, respectively.

For a closed linear operator A on X, let $\sigma(A)$ denote the spectrum of A, $\rho(A) = \mathbf{C} \setminus \sigma(A)$ the resolvent set of A, and $R(\lambda, A) = (\lambda I - A)^{-1} \in \mathcal{L}(X)$ the resolvent of A defined on $\rho(A)$. The spectral radius $r(A)$ and the spectral bound $s(A)$ of A are defined by

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}, \qquad s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

Throughout the paper, we always assume that all underlying spaces are complex Banach lattices.

Definition 2.1. A closed operator A is said to be a Metzler operator if there exists an $\omega \in \mathbf{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R(t, A)$ is positive for $t \in (\omega, \infty)$.

Metzler operators are also called positive resolvent operators in the literature. For an introduction to these operators we refer to [**5**]. The following results concerning Metzler operators and positive operators are taken in [**9, 16**] which will be used in the remainder of the paper.

Theorem 2.2 [**16**]. *Suppose $T \in \mathcal{L}^+(X)$. Then*

i) $r(T) \in \sigma(T)$;

ii) $R(\lambda, T) \geq 0$ *if and only if* $\lambda \in \mathbf{R}$ *and* $\lambda > r(T)$.

Proposition 2.3 [**9**]. *Let A be a Metzler operator on X. Then*

i) $s(A) \in \sigma(A)$

ii) *the function $R(\cdot, A)$ is positive and decreasing for $t > s(A)$, that* *is,*

$$s(A) < t_1 \leq t_2 \Longrightarrow 0 \leq R(t_2, A) \leq R(t_2, A);$$

iii) *If A generates a positive C_0-semigroup, then $R(t, A)$ is positive if and only if $t > s(A)$.*

Lemma 2.4 [**9**]. *Let A be a Metzler operator on X and $E \in \mathcal{L}^+(X, Y)$. Then*

$$|ER(\lambda, A)x| \leq ER(\Re\lambda, A)|x|, \quad \Re\lambda > s(A),\ x \in X.$$

3. Main results. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup generated by the operator A with domain $D(A)$ on the Banach lattice X. The C_0-semigroup $(S(t))_{t\geq 0}$ is called *eventually (norm) continuous*, if there exists a $t_0 \geq 0$ such that the function $t \to S(t)$ is norm continuous from (t_0, ∞) to $\mathcal{L}(X)$. The C_0-semigroup is called *immediately (norm) continuous* if t_0 can be chosen to be $t_0 = 0$. We define the following quantities.

- The *abscissa of uniform boundedness* $s_0(A)$ of the resolvent of A,

$$s_0(A) := \inf\{\omega \in \mathbf{R} : \{\Re\lambda > \omega\} \subset \rho(A) \quad \text{and} \quad \sup_{\Re\lambda > \omega} ||R(\lambda, A)|| < \infty\}.$$

- The *growth bound* or *type* of the C_0-semigroup

$$\omega_1(A) := \inf\{\omega \in \mathbf{R} : \text{ there exists an } M > 0 \text{ such that } \|S(t)x\| \leq Me^{\omega t}||x||_{D(A)}, \text{ for all } t \geq 0, x \in D(A)\},$$

where $||x||_{D(A)} = ||x|| + ||Ax||$.

- The *uniform growth bound* or *type* of the C_0-semigroup

$$\omega_0(A) := \inf\{\omega \in \mathbf{R} : \text{there exists an } M > 0 \text{ such that } \|S(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0\}.$$

We say that $(S(t))_{t\geq 0}$, or the operator A, is *uniformly exponentially stable* (respectively, exponentially stable) if $\omega_0(A) < 0$ (respectively, $\omega_1(A) < 0$). It is known that

$$s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A) < \infty.$$

The inequality $s(A) \leq \omega_1(A) \leq s_0(A) \leq \omega_0(A)$ might be strict, see [**19**], that is, the uniformly exponentially stability of a C_0-semigroup is, in general, not controlled by the spectrum bound or abscissa of its generator. However, if A is bounded on X or A generates a uniformly continuous or eventually continuous C_0 semigroup, then $s(A) = \omega_0(A)$, see [**19**].

Proposition 3.1 [**15**]. *For an eventually continuous semigroup* $(S(t))_{t\geq 0}$ *with generator* $(A, D(A))$ *on a Banach space* X*, we have*

$$s(A) = \omega_0(A).$$

Now suppose that A_0 is a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Banach lattice X. We also fix $p \in [1, \infty)$ and non-negative real numbers $0 \leq h_1 < h_2 < \cdots < h_n =: h$. Given bounded linear operators $A_1, \ldots, A_n$ on X, we consider the delay equation of the form

$$(1) \qquad \begin{cases} \dot{u}(t) = A_0 u(t) + \sum_{i=1}^n A_i u(t - h_i) & t \geq 0, \\ u(0) = x, & \\ u(t) = f(t) & t \in [0, -h). \end{cases}$$

Here, $x \in X$ is the initial value and $f \in L^p([-h, 0]; X)$ is the 'history' function. A mild solution of (1) is the function $u(\cdot) \in L^p_{loc}([-h, \infty]; X)$ satisfying

$$u(t) = \begin{cases} T(t)x + \int_0^t T(t-s) \sum_{i=1}^n A_i u(s - h_i)\, ds & t \geq 0, \\ f(t) & t \in [-h, 0). \end{cases}$$

Equation (1) is called *uniformly exponentially stable* if there exist $M > 0$ and $\omega > 0$ such that the solution of (1) satisfies

$$||u(t)|| \leq M e^{-\omega t}(||x|| + ||f||_{L^p([-h,0];X)}), \quad t \geq 0.$$

In order to study the asymptotic behavior of these solutions by semigroup methods, we introduce the product space

$$\mathcal{X} := X \times L^p([-h, 0]; X)$$

endowed with the norm $||(x,f)|| = ||x|| + ||f||_{L^p([-h,0];X)}$ and the operator $\mathcal{A}$ on $\mathcal{X}$ defined by

$$\mathcal{A}(x,f) = \Big(A_0 x + \sum_{i=1}^{n} A_i f(\cdot - h_i), f'\Big),$$

with the domain

$$D(\mathcal{A}) = \{(x,f) \in \mathcal{X} : f \in W^{1,p}([-h,0];X), f(0) = x \in D(A_0)\}$$

(here $W^{1,p}([-h,0];X)$ denotes the space of absolutely continuous functions f on $[-h,0]$ with values on X that are strongly differential, i.e., $f'(\cdot) \in L^p([-h,0];X)$.

It was proven in [**6**] that $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ which is defined by

$$(\mathcal{T}(t))(x,f) = (u(t), u_t), \quad t \geq 0,$$

where $u(t)$ is a mild solution of (1) and $u_t(s) := u(t+s)$, $s \in [-h,0]$. The resolvent and the spectrum of $\mathcal{A}$ are given by following proposition proved in [**10**].

Proposition 3.2. *We have $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(A_0 + \sum_{j=1}^{n} e^{-\lambda h_j} A_j)$. In this case the resolvent of $\mathcal{A}$ is given by*

$$R(\lambda, \mathcal{A}) = E_\lambda R\Big(\lambda, A_0 + \sum_{j=1}^{n} e^{-\lambda h_j} A_j\Big) H_\lambda F + T_\lambda,$$

where $E_\lambda \in \mathcal{L}(X,\mathcal{X}), H_\lambda \in \mathcal{L}(\mathcal{X},X), F \in \mathcal{L}(\mathcal{X},\mathcal{X})$ and $T_\lambda \in \mathcal{L}(\mathcal{X},\mathcal{X})$ are defined by

$$\begin{aligned} E_\lambda x &:= (x, e^{\lambda\cdot} x); \\ H_\lambda(x,f) &:= x + \int_{-h}^{0} e^{\lambda s} f(s)\, ds; \\ F(x,f) &:= \Big(x, \sum_{j=1}^{n} \chi_{[-h_j,0]}(\cdot) A_j f(-h_j - \cdot)\Big); \\ T_\lambda(x,f) &:= \Big(0, \int_{\cdot}^{0} e^{\lambda(\cdot - s)} f(s)\Big)\, ds. \end{aligned}$$

The following proposition gives a necessary condition on the C_0-semigroup $(S(t))_{t\geq 0}$ generated by $(A, D(A))$ so that the semigroup generated by $\mathcal{A}$ is eventually norm continuous.

Proposition 3.3. *If the C_0-semigroup $(S(t))_{t\geq 0}$ generated by $(A, D(A))$ is immediately norm continuous, then $\mathcal{A}$ generates the eventually norm continuous.*

To study the robust stability of equation (1), we define an operator quasi-polynomial

$$P(\lambda) = A_0 + \sum_{i=1}^{n} e^{-\lambda h_i} A_i.$$

The spectral set, resolvent set and the spectral bound of $P(\cdot)$ are defined by

$$\begin{aligned}\sigma(P(\cdot)) &= \{\lambda : \lambda \in \sigma(P(\lambda))\},\\ \rho(P(\cdot)) &= \mathbf{C} \setminus \sigma(P(\cdot)),\\ s(P(\cdot)) &= \sup\{\Re\lambda : \lambda \in \sigma(P(\cdot))\},\end{aligned}$$

respectively.

Then, it is easy to check that $\sigma(P(\cdot)) = \sigma(\mathcal{A})$.

3.1. Multi perturbations. In this section, we consider the robust stability of equation (1) under the subjection of multi perturbations. More precisely, assume that the operators A_i, $i = 0, 1, \dots, n$ are subjected to perturbations of the form

$$A_i \hookrightarrow A_i + \sum_{j=0}^{k} D\Delta_{ij} E_{ij}, \quad i = 0, 1, \dots, n, \tag{2}$$

where $D \in \mathcal{L}(U, X)$ and $E_{ij} \in \mathcal{L}(X, Y_{ij})$, $i \in \overline{N} := \{0, 1, \dots, n\}$, $j \in \overline{K} := \{0, 1, \dots, k\}$ are given operators determining structure of perturbations and $\Delta_{ij} \in \mathcal{L}(Y_{ij}, U)$, $i = 0, 1, \dots, n$, $j = 0, 1, \dots, k$, are unknown operators ($U, Y_{ij},\ i \in \overline{N}, j \in \overline{K}$, are arbitrary Banach lattices).

Then the perturbed equation has a form

$$(3)\qquad \begin{cases} \dot{u}(t) = (A_0 + \sum_{j=0}^{k} D\Delta_{0j}E_{0j})u(t) \\ \qquad\qquad + \sum\limits_{i=1}^{n}(A_i + \sum_{j=0}^{k} D\Delta_{ij}E_{ij})u(t-h_i) & t \geq 0 \\ u(0) = x, \\ u(t) = f(t) & t \in [0,-h). \end{cases}$$

We also set

$$\mathcal{A}_\Delta(x,f) = \Big(\Big(A_0 + \sum_{j=0}^{k} D\Delta_{0j}E_{0j}\Big)x + \sum_{i=1}^{n}\Big(A_i + \sum_{j=0}^{k} D\Delta_{ij}E_{ij}\Big)f(-h_i), f'\Big),$$

and

$$P_\Delta(\lambda) = \Big(A_0 + \sum_{j=0}^{k} D\Delta_{0j}E_{0j}\Big) + \sum_{i=1}^{n} e^{-\lambda h_i}\Big(A_i + \sum_{j=0}^{k} D\Delta_{ij}E_{ij}\Big).$$

For $\lambda \in \rho(P(\cdot))$, we introduce the transfer function associated with the triplet $(P(\lambda), D, E_{ij})$

$$G_{ij}(\lambda) = E_{ij}R(\lambda, P(\lambda))D \in \mathcal{L}(U, Y_{ij}), \quad i \in \overline{N}, j \in \overline{K}.$$

It is clear that the function $G_{ij}(\cdot)$, $i \in \overline{N}$, $j \in \overline{K}$ is analytic on $\rho(P(\cdot))$. Since $\mathcal{A}$ generates a C_0-semigroups,

$$(4)\qquad \sup_{\Re\omega \geq 0} ||G_{ij}(\omega)|| = \sup_{\Re\omega = 0} ||G_{ij}(\omega)||, \quad i \in \overline{N},\ j \in \overline{K}.$$

Now, assuming that $s_0(\mathcal{A}) < 0$ (or $\omega_0(\mathcal{A}) < 0$), we will consider how the spectrum of $\mathcal{A}$ changes under small perturbations of disturbances

Δ_{ij}. To do this, we would like to introduce the following quantities defined by

$$r_{\mathbf{C}}^{\gamma} = \inf\Big\{ \sum_{i=0}^{n}\sum_{j=0}^{k} ||\Delta_{ij}|| : \Delta_{ij} \in \mathcal{L}(Y_{ij}, U), \quad i \in \overline{N}, j \in \overline{K} \text{ and } \gamma(\mathcal{A}_{\Delta}) \geq 0 \Big\}$$

$$r_{\mathbf{R}}^{\gamma} = \inf\Big\{ \sum_{i=0}^{n}\sum_{j=0}^{k} ||\Delta_{ij}|| : \Delta_{ij} \in \mathcal{L}^{\mathbf{R}}(Y_{ij}, U), \quad i \in \overline{N},\ j \in \overline{K} \text{ and } \gamma(\mathcal{A}_{\Delta}) \geq 0 \Big\}$$

$$r_{+}^{\gamma} = \inf\Big\{ \sum_{i=0}^{n}\sum_{j=0}^{k} ||\Delta_{ij}|| : \Delta_{ij} \in \mathcal{L}^{+}(Y_{ij}, U), \quad i \in \overline{N},\ j \in \overline{K} \text{ and } \gamma(\mathcal{A}_{\Delta}) \geq 0 \Big\},$$

where $\gamma \in \{s_0, \omega_0\}$ and we set $\inf \varnothing = \infty$.

The quantities $r_{\mathbf{C}}^{\omega_0}, r_{\mathbf{R}}^{\omega_0}$ and $r_{+}^{\omega_0}$ are called complex, real and positive stability radii of $\mathcal{A}$ with respect to the multi-perturbation of form (2), respectively.

The lower bound of the complex stability radius can be obtained easily by the following proposition.

Proposition 3.4. *Assume that $s_0(\mathcal{A}) < 0$ and $\Delta_{ij} \in \mathcal{L}(Y_{ij}, U)$, for all $i \in \overline{N}$, $j \in \overline{K}$. If*

$$\sum_{i=0}^{n}\sum_{j=0}^{k} ||\Delta_{ij}|| < \frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} \sup_{\Re\omega=0} ||G_{ij}(\omega)||}, \tag{5}$$

then $s_0(\mathcal{A}_{\Delta}) < 0$.

Proof. Suppose (5) holds. Then it is possible to choose $\delta \in (0, 1)$ such that

$$\sum_{i=0}^{n}\sum_{j=0}^{k} ||\Delta_{ij}|| < (1-\delta)\frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} \sup_{\Re\omega=0} ||G_{ij}(\omega)||}. \tag{6}$$

For $\lambda \in \mathbf{C}$, $\Re\lambda > 0$, we define the linear operators $E \in \mathcal{L}(X, \prod_{i\in\overline{N},j\in\overline{K}} Y_{ij})$ and $\Delta(\lambda) \in \mathcal{L}(\prod_{i\in\overline{N},j\in\overline{K}} Y_{ij}, U)$ by setting

$$Ex = (E_{00}x, \dots, E_{0k}x, \dots, E_{00}x, \dots, E_{nk}x), \quad x \in X,$$
$$\Delta(\lambda)y = (\Delta_{00}y_{00}, \dots, \Delta_{0k}y_{0k}, \dots, e^{-h_k\lambda}\Delta_{n0}y_{n0}, \dots, e^{-h_k\lambda}\Delta_{nk}y_{nk}),$$

where $y = (y_{ij})_{i\in\overline{N},j\in\overline{K}} \in \prod_{i\in\overline{N},j\in\overline{K}} Y_{ij}$.

By definition, we have, for each $u \in U$,

$$\Delta(\lambda)ER(\lambda, P(\lambda))Du = \sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} e^{-h_i\lambda}\Delta_{ij}G_{ij}(\lambda)u.$$

Therefore,

$$\begin{aligned} ||\Delta(\lambda)ER(\lambda, P(\lambda))Du|| &= \Big\| \sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} e^{-h_i\lambda}\Delta_{ij}G_{ij}(\lambda)u|| \\ &\le \sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} ||\Delta_{ij}G_{ij}(\lambda)||||u|| \\ &\le \max_{\substack{i\in\overline{N}\\ j\in\overline{K}}} ||G_{ij}(\lambda)|| \sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} ||\Delta_{ij}||||u|| \end{aligned}$$

and hence, by (4) and (6), $||\Delta(\lambda)ER(\lambda, P(\lambda))D|| < 1 - \delta$. It follows that the operator $[I - \Delta(\lambda)ER(\lambda, A)D]$ is invertible and $[I - \Delta(\lambda)ER(\lambda, P(\lambda))D]^{-1} \in \mathcal{L}(U)$. Therefore, $[I - D\Delta(\lambda)ER(\lambda, P(\lambda))]$ is invertible and $[I - D\Delta(\lambda)ER(\lambda, P(\lambda))]^{-1} \in \mathcal{L}(X)$. By a simple computation, we have

$$\begin{aligned} [I - D\Delta(\lambda)ER(\lambda, P(\lambda))](\lambda I - P(\lambda)) &= \lambda I - P(\lambda) - D\Delta(\lambda)E \\ &= \lambda I - P_\Delta(\lambda). \end{aligned}$$

Therefore, we obtain that

$$[\lambda I - P_\Delta(\lambda)]^{-1} = R(\lambda, P(\lambda))[I - D\Delta(\lambda)ER(\lambda, P(\lambda))]^{-1},$$

which implies that $\lambda \in \rho(P_\Delta(\cdot))$, or $\{\Re\lambda > 0\} \subset \rho(\mathcal{A})$ and

$$\begin{aligned} ||R(\lambda, \mathcal{A}_\Delta)|| &= ||R(\lambda, P(\lambda))[I - D\Delta(\lambda)ER(\lambda, P(\lambda))]^{-1}|| \\ &\le ||R(\lambda, P(\lambda))|| || \sum_{i=0}^{\infty} [D\Delta(\lambda)ER(\lambda, P(\lambda))]^i || \\ &\le ||R(\lambda, P(\lambda))|| \sum_{i=0}^{\infty} ||[D\Delta(\lambda)ER(\lambda, P(\lambda))]^i|| \\ &\le ||R(\lambda, P(\lambda))|| \sum_{i=0}^{\infty} (1-\delta)^i \\ &\le ||R(\lambda, P(\lambda))|| \frac{1}{1-\delta}. \end{aligned}$$

Due to $s_0(\mathcal{A}) < 0$, $\sup_{\Re\lambda>0} ||R(\lambda, \mathcal{A})|| < \infty$. By invoking Proposition 3.2, we get that $\sup_{\Re\lambda>0} ||P(\lambda)|| < \infty$. Thus, $\sup_{\Re\lambda>0} ||R(\lambda, \mathcal{A}_\Delta)|| < \infty$ which implies that $s_0(\mathcal{A}_\Delta) < 0$.

Theorem 3.5. *Let $s_0(\mathcal{A}) < 0$. Then,*

$$r_{\mathbf{C}}^{s_0} = \frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} \sup_{\Re\omega=0} ||G_{ij}(\omega)||}.$$

Proof. First, due to Proposition 3.4, we obtain that

$$r_{\mathbf{C}}^{s_0} \ge \frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} \sup_{\Re\omega=0} ||G_{ij}(\omega)||}.$$

It remains to show that

$$r_{\mathbf{C}}^{s_0} \le \frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} \sup_{\Re\omega=0} ||G_{ij}(\omega)||}.$$

Indeed, for arbitrary $\omega \in \mathbf{C}$, $\Re\omega = 0$, $i_0 \in \overline{N}$, $j_0 \in \overline{K}$ and $\epsilon > 0$, there exists a $u \in U$ satisfying $||u|| = 1$ and $||G_{i_0j_0}(\omega)|| \ge ||G_{i_0j_0}(\omega)u|| \ge ||G_{i_0j_0}(\omega)|| - \varepsilon$. Applying the Hahn-Banach theorem, there exists a $y^*_{i_0j_0} \in Y^*_{i_0j_0}$ such that $||y^*_{i_0j_0}|| = 1$ and $||y^*_{i_0j_0}(G_{i_0j_0}(\omega)u) = ||G_{i_0j_0}(\omega)u||$.

We set $\overline{\Delta}_{i_0j_0} : Y_{i_0j_0} \to U$ defined by

$$\overline{\Delta}_{i_0j_0} y_{i_0j_0} = \frac{1}{||G_{i_0j_0}(\omega)u||} y^*_{i_0j_0}(y_{i_0j_0})u, \quad \text{for all } y_{i_0j_0} \in Y_{i_0j_0}.$$

It is clear that $\overline{\Delta}_{i_0j_0} \in \mathcal{L}(Y_{i_0j_0}, U)$ and

$$||\overline{\Delta}_{i_0j_0}|| \leq \frac{1}{||G_{i_0j_0}(\omega)u||} \leq \frac{1}{||G_{i_0j_0}(\omega)|| - \varepsilon}.$$

Now we construct the destabilizing operator $\Delta = (\Delta_{ij})_{i\in\overline{N},j\in\overline{K}}$ as follows

$$\Delta_{ij} = \begin{cases} e^{\omega h_{i_0}}\overline{\Delta}_{i_0j_0} & i = i_0,\, j = j_0 \\ 0 & i \neq i_0 \text{ or } j \neq j_0. \end{cases}$$

It can be verified that $\sum_{i\in\overline{N},j\in\overline{K}} ||\Delta_{ij}|| = ||\overline{\Delta}_{i_0j_0}||$. Setting $\widehat{x} = R(\omega, P(\omega))Du \in \mathcal{D}(A)$, then $\widehat{x} \neq 0$ and $(P_\Delta(\omega))\widehat{x} = \omega\widehat{x}$. This implies $\omega \in \sigma(P_\Delta(\cdot))$, and hence $s_0(\mathcal{A}_\Delta) \geq 0$. The proof is complete. □

Let us give some relevant comments on the complex stability radius $r^{\omega_0}_{\mathbf{C}}$. In particular, the case when $A_1 = \cdots = A_n = 0$, the lower bound for $r^{\omega_0}_{\mathbf{C}}$ was obtained, see for example [**13**]. It is more difficult to study the upper bound of $r^{\omega_0}_{\mathbf{C}}$. The aim of this paper is not pursuing the estimates for $r^{\omega_0}_{\mathbf{C}}$ in general cases. Compared with $r^{s_0}_{\mathbf{C}}$ these two complex stability radii $r^{s_0}_{\mathbf{C}}$ and $r^{\omega_0}_{\mathbf{C}}$ may be different. However, if A_0 generates an immediately continuous C_0-semigroup, then the following result can be obtained.

Corollary 3.6. *Let $\omega_0(\mathcal{A}) < 0$. If A_0 generates a C_0-semigroup which is immediately continuous, then*

$$r^{\omega_0}_{\mathbf{C}} = r^{s_0}_{\mathbf{C}} = \frac{1}{\max_{i\in\overline{N},j\in\overline{K}} \sup_{\omega\in\mathbf{R}} ||G_{ij}(i\omega)||}.$$

Proof. Since A_0 generates an immediately continuous C_0-semigroup, the operator $\mathcal{A}$ and $A_0 + \sum_{j=0}^{k} D\Delta_{0j}E_{0j}$ also generate an immediately continuous C_0-semigroup, see [**19**]. Thus, thanks to Proposition 3.3,

$\mathcal{A}_\Delta$ generates an eventually continuous C_0-semigroup. Then, using Propositions 3.1 and 3.5, the required result is obtained.

In general, these three radii $r^{s_0}_{\mathbf{C}}, r^{s_0}_{\mathbf{R}}$ and $r^{s_0}_{+}$ may be different. It is therefore natural to investigate for which kind of systems these three radii coincide. Motivated by the recent works, see for example [**2, 3, 4, 9, 10, 12**], the positive answer will be addressed for the class of positive equations. We recall that equation (1) is called positive if A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$ for all $i = 1, \dots, n$. It is well known that, if equation (1) is positive, then $\mathcal{A}$ generates a positive C_0-semigroup; moreover, $s(\mathcal{A}) = s_0(\mathcal{A})$, see [**10, 19**]. We are now in a position to introduce some properties on the operator polynomial $P(\cdot)$, see [**3**].

Lemma 3.7. *Suppose that A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$, for all $i = 1, \dots, n$. Then the resolvent $R(\cdot, P(\cdot))$ is positive and decreasing for $t > s(P(\cdot)) = s(\mathcal{A})$, that is,*

$$s(\mathcal{A}) = s(P(\cdot)) < t_1 \le t_2 \Longleftrightarrow 0 \le R(t_2, P(t_2)) \le R(t_1, P(t_1)).$$

Proposition 3.8. *Suppose that A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$ for all $i = 1, \dots, n$. For $E \in \mathcal{L}^+(X, Y), x \in X$, we have*

$$|ER(\lambda, P(\lambda))x| \le ER(\Re\lambda, P(\Re\lambda))|x|, \quad \Re\lambda > s(A) = s(P(\cdot)).$$

We now establish a necessary and sufficient condition for which $s_0(\mathcal{A}) < 0$.

Theorem 3.9. *Let equation* (1) *be positive. Then the following statements are equivalent*:

i) $s(\mathcal{A}) = s_0(\mathcal{A}) < 0$,

ii) $s(A_0 + A_1 + \cdots + A_n) < 0$.

Proof. i) $\Rightarrow$ ii). Assume that $s(\mathcal{A}) = s_0(\mathcal{A}) < 0$. Since $\mathcal{A}$ generates a positive C_0-semigroup, $R(0, \mathcal{A})$ is a positive operator. Using

Proposition 3.2, we get that $R(0,P(0)) = -(A_0+A_1+\cdots+A_n)^{-1} \geq 0$. On the other hand, since A_0 generates a positive C_0-semigroup and $A_i \in \mathcal{L}^+(X)$ for all $i = 1,\ldots,n$, $(A_0 + A_1 + \cdots + A_n)$ also generates a positive C_0-semigroup. Thus, as a consequence of Proposition 2.3, we conclude that $s(A_0 + A_1 + \cdots + A_n) < 0$.

ii) $\Rightarrow$ i). Suppose that $s(A_0+A_1+\cdots+A_n) < 0$. Then $R(0,P(0)) = -(A_0+A_1+\cdots+A_n)^{-1} \geq 0$. By Lemma 3.7, we get that $0 < s(P(\cdot)) = s(\mathcal{A}) = s_0(\mathcal{A})$. The proof is complete. ◻

Theorem 3.10. *Let equation* (1) *be positive and* $s_0(\mathcal{A}) < 0$. *If all operators* D, E_{ij}, $i \in \overline{N}$, $j \in \overline{K}$ *are positive, then*

$$r_{\mathbf{C}}^{s_0} = r_{\mathbf{R}}^{s_0} = r_{+}^{s_0} = \frac{1}{\max_{i\in\overline{N},j\in\overline{K}} ||G_{ij}(0)||}.$$

Proof. Due to Lemma 3.7 and Proposition 3.8, we get that

$$\sup_{\Re\omega=0} ||G_{ij}(\omega)|| = ||G_{ij}(0)||, \quad \text{for all } i \in \overline{N},\ j \in \overline{K}.$$

Therefore, by Theorem 3.5,

$$r_{\mathbf{C}}^{s_0} = \frac{1}{\max_{i\in\overline{N},j\in\overline{K}} ||G_{ij}(0)||}.$$

To show that $r_{\mathbf{C}}^{s_0} \leq r_{+}^{s_0}$, let us fix $i_0 \in \overline{N}, j_0 \in \overline{K}$ and an arbitrary $\epsilon > 0$. As in the proof of Theorem 3.5, using the Hahn-Banach theorem for positive operators, one can construct a one-rank positive destabilizing perturbation $\Delta = (\Delta_{ij})_{i\in\overline{N},j\in\overline{K}}$ with $\Delta_{ij}, i \in \overline{N}, j \in \overline{K}$ are positive operators such that $\sum_{i\in\overline{N},j\in\overline{K}} ||\Delta_{ij}|| = ||\Delta_{i_0j_0}|| < ||G_{i_0j_0}(0)||^{-1} + \epsilon$. This implies that

$$r_{+}^{s_0} < \frac{1}{||G_{i_0j_0}(0)||} + \epsilon = r_{\mathbf{C}}^{s_0} + \epsilon,$$

concluding the proof. ◻

By a simple implication, from Theorem 3.10, we deduce the following result.

Corollary 3.11. *Let $\omega_0(\mathcal{A}) < 0$. If A_0 generates an immediately continuous C_0-semigroup, equation* (1) *is positive and all operators D, E_{ij}, $i \in \overline{N}$, $j \in \overline{K}$, are positive. Then we have*

$$r_{\mathbf{C}}^{\omega_0} = r_{\mathbf{R}}^{\omega_0} = r_{+}^{\omega_0} = \frac{1}{\max_{i\in\overline{N}, j\in\overline{K}} ||G_{ij}(0)||}.$$

3.2. Affine perturbation. Now suppose that the operators A_i, $i \in \overline{N}$, are subjected to perturbations of the form

$$A_i \hookrightarrow A_i + \sum_{j=0}^{k} \delta_{ij} A_{ij},$$

where A_{ij}, $i \in \overline{N}$, $j \in \overline{K}$, are given operators defining the structure of the perturbations and δ_{ij}, $i \in \overline{N}$, $j \in \overline{K}$, are scalars presenting parameter uncertainties. Then, we can write the perturbed equation of the form

$$\begin{cases} \dot{u}(t) = (A_0 + \sum_{j=0}^{k} \delta_{0j} A_{0j}) u(t) & \\ \qquad\qquad + \sum_{i=1}^{n} (A_i + \sum_{j=0}^{k} \delta_{ij} A_{ij}) u(t - h_i) & t \geq 0 \\ u(0) = x, & \\ u(t) = f(t) & t \in [0, -h). \end{cases}$$

We also set

$$\mathcal{A}_\delta(x, f) := \Big(\Big(A_0 + \sum_{j=0}^{k} \delta_{0j} A_{0j} \Big) x + \sum_{i=1}^{n} \Big(A_i + \sum_{j=0}^{k} \delta_{ij} A_{ij} \Big) f(-h_i), f' \Big)$$

and

$$P_\delta(\lambda) := \Big(A_0 + \sum_{j=0}^{k} \delta_{0j} A_{0j} \Big) + \sum_{i=1}^{n} e^{-\lambda h_i} \Big(A_i + \sum_{j=0}^{k} \delta_{ij} A_{ij} \Big).$$

And we also define

$$r_{\delta,\mathbf{C}}^{\gamma} = \inf \{ ||\delta||_\infty : \delta = (\delta_{ij})_{\substack{i\in\overline{N}\\ j\in\overline{K}}} \in \mathbf{C}^{nk}, \gamma(\mathcal{A}) \geq 0 \}$$

$$r_{\delta,\mathbf{R}}^{\gamma} = \inf \{ ||\delta||_\infty : \delta = (\delta_{ij})_{\substack{i\in\overline{N}\\ j\in\overline{K}}} \in \mathbf{R}^{nk}, \gamma(\mathcal{A}) \geq 0 \}$$

$$r_{\delta,+}^{\gamma} = \inf \{ ||\delta||_\infty : \delta = (\delta_{ij})_{\substack{i\in\overline{N}\\ j\in\overline{K}}} \in \mathbf{R}_{+}^{nk}, \gamma(\mathcal{A}) \geq 0 \}$$

where we set $||\delta||_\infty = \max\{|\delta_{ij}| : i \in \overline{N}, j \in \overline{K}\}$, $\gamma \in \{s_0, \omega_0\}$ and we set $\inf \varnothing = \infty$. First, the estimate for $r^{s_0}_{\delta,\mathbf{C}}$ will be established in the following proposition.

Proposition 3.12. *Let $s_0(\mathcal{A}) < 0$ and $A_0 \in \mathcal{L}(X)$. Then,*

$$r^{s_0}_{\delta,\mathbf{C}} = \frac{1}{\sup_{\substack{\Re s \geq 0 \\ |z_{ij}| \leq 1, i \in \overline{N}, j \in \overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N}, j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij})]},$$

where we set $h_0 = 0$.

Proof. Assume that $\delta = (\delta_{ij})_{i\in\overline{N}, j\in\overline{K}} \in \mathbf{C}^{nk}$ satisfying

$$||\delta||_\infty < \frac{1}{\sup_{\substack{\Re s \geq 0 \\ |z_{ij}| \leq 1, i \in \overline{N}, j \in \overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N}, j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij})]}.$$

Then it is possible to choose $\epsilon \in (0,1)$ such that

$$||\delta||_\infty < (1-\epsilon) \frac{1}{\sup_{\substack{\Re s \geq 0 \\ |z_{ij}| \leq 1, i \in \overline{N}, j \in \overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N}, j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij})]}.$$

This implies that

$$r\left[R(s,P(s))\left(\sum_{\substack{i\in\overline{N} \\ j\in\overline{K}}} ||\delta||_\infty z_{ij} e^{-sh_i} A_{ij}\right)\right] < 1-\epsilon,$$

whenever $\Re s \geq 0$ and $|z_{ij}| \leq 1$. Choosing $z_{ij} = \delta_{ij}/||\delta||_\infty$, we obtain that

$$r\left[R(s,P(s))\left(\sum_{i\in\overline{N}, j\in\overline{K}} \delta_{ij} e^{-sh_i} A_{ij}\right)\right] < 1-\epsilon.$$

It follows that the operator $I - R(s,P(s))(\sum_{i\in\overline{N}, j\in\overline{K}} \delta_{ij} e^{-sh_i} A_{ij})$ is invertible. By a straightforward computation, we have

$$R(s,P_\delta(s)) = \left[I - R(s,P(s))\left(\sum_{\substack{i\in\overline{N} \\ j\in\overline{K}}} \delta_{ij} e^{-sh_i} A_{ij}\right)\right]^{-1} R(s,P(s)),$$

$$\text{for all } s, \Re s \geq 0,$$

which implies $\{\Re\lambda > 0\} \subset \rho(P_\delta(\cdot))$. Furthermore, due to $\sup_{\Re s>0} ||R(s, P(s))|| < \infty$ and $r[R(s, P(s))(\sum_{i\in\overline{N}, j\in\overline{K}} \delta_{ij}e^{-sh_i}A_{ij})] < 1-\epsilon$,

$$\sup_{\Re s>0}\left\|\left[I - R(s,P(s))\left(\sum_{i\in\overline{N},j\in\overline{K}} \delta_{ij}e^{-sh_i}A_{ij}\right)\right]^{-1}\right\| < \infty.$$

Therefore,

$$\sup_{\Re s>0} ||R(s, P_\delta(s))|| < \infty,$$

which leads to $s_0(\mathcal{A}_\delta) < 0$. Thus,

$$r^{s_0}_{\delta,\mathbf{C}} \geq \frac{1}{\sup_{\substack{\Re s\geq 0\\ |z_i|\leq 1, i\in\overline{N}, j\in\overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij})]}.$$

Our task is now to show that

$$r^{s_0}_{\delta,\mathbf{C}} \leq \frac{1}{\sup_{\substack{\Re s\geq 0\\ |z_i|\leq 1, i\in\overline{N}, j\in\overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij})]}.$$

Indeed, for every $s \in \mathbf{C}$, $\Re s \geq 0$ and $|z_{ij}| \leq 1$, if we set

$$z_0 = \frac{1}{r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij})]},$$

then $r[R(s, P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_0z_{ij}e^{-sh_i}A_{ij})] = 1$. Thus, there exists $z \in \mathbf{C}$ such that $|z| = 1$ and $z \in \sigma(R(s, P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_0z_{ij}e^{-sh_i}A_{ij}))$. Considering the following equation

$$\begin{aligned} z&\left[sI - \left(P(s) + \sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} z^{-1}z_0z_{ij}e^{-sh_i}A_{ij}\right)\right] \\ &= (sI - P(s))\left[zI - R(s,P(s))\left(\sum_{\substack{i\in\overline{N}\\ j\in\overline{K}}} z_0z_{ij}e^{-sh_i}A_{ij}\right)\right], \end{aligned}$$

we see that since $z \in \sigma(R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_0 z_{ij} e^{-sh_i} A_{ij}))$, $s \in \sigma(P_\delta(\cdot))$, where $\delta = (z_0 z_{ij})_{i\in\overline{N},j\in\overline{K}}$ which implies $s_0(\mathcal{A}_\delta) \geq 0$. Thus,

$$r^{s_0}_{\delta,\mathbf{C}} \leq \frac{1}{\sup\limits_{\substack{\Re s\geq 0 \\ |z_{ij}|\leq 1, i\in\overline{N},j\in\overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij})]}.$$

The proof is complete. □

From this result and by an argument similar to those used in the proof of Corollary 3.6, the following corollary may be deduced.

Corollary 3.13. *Let $\omega_0(\mathcal{A}) < 0$ and $A_0 \in \mathcal{L}(X)$. Then*

$$r^{\omega_0}_C = r^{s_0}_C = \frac{1}{\sup\limits_{\substack{\Re s\geq 0 \\ |z_{ij}|\leq 1, i\in\overline{N},j\in\overline{K}}} r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij})]}.$$

Remark 3.14. Theorem 3.12 give us the formulas to compute $r^{s_0}_{\delta,\mathbf{C}}$. In fact, this is very complicated because we must solve the optimization problem with many variations. So, in some sense, formulas to compute the complex radii are not "very interesting" but they are used to determine the positive radii which are established in next section.

Theorem 3.15. *Let equation (1) be positive, $s_0(\mathcal{A}) < 0$ and $A_0 \in \mathcal{L}(X)$. If all operators A_{ij}, $i \in \overline{N}$, $j \in \overline{K}$ are positive, then*

$$r^{s_0}_{\delta,\mathbf{C}} = r^{s_0}_{\delta,\mathbf{R}} = r^{s_0}_{\delta,+} = \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

Proof. Using Lemma 3.7 and Proposition 3.8, we have

$$\left|R(s,P(s))\Big(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij} e^{-sh_i} A_{ij}\Big)x\right| \leq \left|R(0,P(0))\Big(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij}\Big)x\right|$$

whenever $\Re s \geq 0$ and $|z_{ij}| \leq 1$, $i \in \overline{N}$, $j \in \overline{K}$. By reduction, we get that

$$\left|\left[R(s,P(s))\Big(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij}\Big)\right]^m x\right| \leq \left|\left[R(0,P(0))\Big(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij}\Big)\right]^m x\right|.$$

Therefore, applying the lattice norm property, we conclude

$$\left\|\left[R(s,P(s))\Big(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij}\Big)\right]^m\right\| \leq \left\|\left[R(0,P(0))\Big(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij}\Big)\right]^m\right\|,$$

for all $\Re s \geq 0$, $|z_{ij}| \leq 1$, $i \in \overline{N}$, $j \in \overline{K}$ which implies that $r[R(s,P(s))(\sum_{i\in\overline{N},j\in\overline{K}} z_{ij}e^{-sh_i}A_{ij})]^m|| \leq r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]$, for all $\Re s \geq 0$, $|z_{ij}| \leq 1$, $i \in \overline{N}$, $j \in \overline{K}$. Thus,

$$r_{\delta,\mathbf{C}}^{s_0} = \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

It remains to show that

$$r_{\delta,+}^{s_0} \leq \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

To do this, we set

$$z_0 = \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

Then, $r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} z_0A_{ij})] = 1$. Since $R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} z_0A_{ij})$ is positive, by Theorem 2.2, $1 \in \sigma(R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} z_0A_{ij}))$. The following equation

$$\left[-\Big(P(0)+\sum_{i\in\overline{N},j\in\overline{K}} z_0A_{ij}\Big)\right] = (-P(0))\left[I-R(0,P(0))\Big(\sum_{i\in\overline{N},j\in\overline{K}} z_0A_{ij}\Big)\right],$$

gives that, since $1 \in \sigma(R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} s_0 A_{ij}))$, $0 \in \sigma(P_\delta(\cdot))$, where $\delta = (s_{ij})_{i\in\overline{N},j\in\overline{K}}$, $s_{ij} = z_0$, $i \in \overline{N}$, $j \in \overline{K}$ which implies $s_0(\mathcal{A}_{\delta_0}) \geq 0$. Thus,

$$r^{s_0}_{\delta,+} \leq \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

The proof is complete. □

From Theorem 3.15, the following corollary can be deduced.

Corollary 3.16. *Let $\omega_0(\mathcal{A}) < 0$ and $A_0 \in \mathcal{L}(X)$. If equation* (1) *is positive and all operators A_{ij}, $i \in \overline{N}$, $j \in \overline{K}$ are positive, then*

$$r^{\omega_0}_{\delta,\mathbf{C}} = r^{\omega_0}_{\delta,\mathbf{R}} = r^{\omega_0}_{\delta,+} = \frac{1}{r[R(0,P(0))(\sum_{i\in\overline{N},j\in\overline{K}} A_{ij})]}.$$

Now we will consider the following example to illustrate the above results.

Example. Let $X = l^1(\mathbf{C})$ be a space of all sequences $(x_i)_{i\in\mathbf{N}} \subset \mathbf{C}$ satisfying $\sum_{i=1}^{\infty} |x_i| < +\infty$. Then X is a complex Banach lattice space if it is endowed with the norm

$$||x|| = \sum_{i=1}^{\infty} |x_i|, x = (x_i)_{i\in\mathbf{N}} \in X$$

and the module

$$|x| = (|x_i|)_{i\in\mathbf{N}}.$$

Consider the following system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$$

where A_0 and A_1 are operators on X defined by

$$A_0 x = \Big(-\frac{11}{12}x_1, \frac{1}{3}x_1 - \frac{11}{12}x_2, \cdots, \frac{1}{3}x_n - \frac{11}{12}x_{n+1}, \cdots \Big),$$
$$x = (x_i)_{i\in\mathbf{N}} \in X.$$
$$A_1 x = \Big(\frac{1}{4}x_1, \frac{1}{6}x_1 + \frac{1}{4}x_2, \cdots, \frac{1}{6}x_n + \frac{1}{4}x_{n+1}, \cdots \Big),$$
$$x = (x_i)_{i\in\mathbf{N}} \in X.$$

It is easy to see that $(I + A_0) \in \mathcal{L}^+(X)$, and thus A_0 is a generator of a positive semigroup. On the other hand, one can check that

$$(-A_0 - A_1)^{-1}x = (y_1, y_2, \dots, y_n, \dots), x = (x_i)_{i\in\mathbf{N}} \in X,$$

where

$$y_1 = \frac{3}{2}x_1, y_2 = \frac{3}{2}x_2 + \frac{3^2}{2^3}x_1, \dots ,$$
$$y_n = \frac{3}{2}x_n + \frac{3^2}{2^3}x_{n-1} + \frac{3^3}{2^5}x_{n-2} + \dots + \frac{3^{n-1}}{2^{2n-3}}x_2 + \frac{3^n}{2^{2n-1}}x_1, \quad n \geq 3.$$

This means that $(-A_0 - A_1)^{-1} \in L^+(X)$. Due to Theorem 2.3, $s(A_0 + A_1) < 0$. By invoking Theorem 3.9, $s_0(A_0 + A_1) < 0$. Now we assume that A_0 and A_1 are subjected to perturbations of the form

$$A_0 \hookrightarrow A_0 + D_{01}\Delta_{01}E_{01} + D_{02}\Delta_{02}E_{02}$$
$$A_1 \hookrightarrow A_1 + D_{11}\Delta_{11}E_{11} + D_{12}\Delta_{12}E_{12},$$

where $D_{ij} = I_X$, $i = 0, 1$, $j = 1, 2$ and $E_{ij} \in \mathcal{L}^+(X)$, $i = 0, 1$, $j = 1, 2$ are defined by

$$E_{01}x = (0, x_2, x_3, \dots , x_n, \dots), \quad x = (x_i)_{i\in\mathbf{N}} \in X.$$
$$E_{02}x = (x_1, 0, x_3, \dots , x_n, \dots), \quad x = (x_i)_{i\in\mathbf{N}} \in X.$$
$$E_{11}x = (x_1, 0, 0, \dots , 0, \dots), \quad x = (x_i)_{i\in\mathbf{N}} \in X.$$
$$E_{12}x = (x_1, x_2, 0, 0, \dots , 0, \dots), \quad x = (x_i)_{i\in\mathbf{N}} \in X.$$

Thus, using Theorem 3.9 for stable radii for the positive systems, we have

$$r^{s_0}_{\mathbf{C}} = r^{s_0}_{\mathbf{R}} = r^{s_0}_{+} = \frac{1}{\max_{i\in\{0,1\},j\in\{1,2\}} ||G_{ij,ij}(1)||}.$$

By a simple calculation, we can compute that

$$||G_{00,00}(1)|| = ||E_{00}(I - A)^{-1}|| = \sum_{i=1}^{\infty} \frac{3^n}{2^{2n-1}} = 6,$$
$$||G_{01,01}(1)|| = ||E_{01}(I - A)^{-1}|| = \frac{3}{2} + \sum_{i=3}^{\infty} \frac{3^n}{2^{2n-1}} = \frac{39}{8},$$

$$||G_{10,10}(1)|| = ||E_{10}(I-A)^{-1}|| = \frac{3}{2},$$
$$||G_{11,11}(1)|| = ||E_{11}(I-A)^{-1}|| = \frac{3}{2} + \frac{3^2}{2^3} = \frac{21}{8},$$

Therefore, we obtain

$$r^{s_0}_{\mathbf{C}} = r^{s_0}_{\mathbf{R}} = r^{s_0}_{+} = \frac{1}{6}.$$

Next we assume that operators A_i, $i = 0, 1$, are subjected to affine perturbations of the form

$$A_0 \hookrightarrow A_0 + \delta_{01}A_{01} + \delta_{02}A_{02},$$
$$A_1 \hookrightarrow A_1 + \delta_{11}A_{11} + \delta_{12}A_{12},$$

where $A_{ij} \in \mathcal{L}^+(X)$, i, j are defined by

$$A_{01}x = \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \dots, 0, \dots\right),$$
$$A_{02}x = \left(0, \frac{1}{2}x_2, 0, \dots, 0, \dots\right),$$
$$A_{11}x = \left(\frac{1}{3}x_1, \frac{1}{2}x_2, \dots, 0, \dots\right),$$
$$A_{12}x = \left(\frac{7}{6}x_1, \frac{2}{3}x_2, 0, \dots, 0, \dots\right).$$

By invoking Theorem 3.9, we have

$$r^{s_0}_{\delta,\mathbf{C}} = r^{s_0}_{\delta,\mathbf{R}} = r^{s_0}_{\delta,+} = \frac{1}{r[(-A_0 - A_1)^{-1}(A_{01} + A_{02} + A_{11} + A_{12})]}.$$

By a simple computation, one has

$$r[(-A_0 - A_1)^{-1}(A_{01} + A_{02} + A_{11} + A_{12})] = \lim_{n\to\infty} \sqrt[n]{||(I - A_0 - A_1)^{-1}(A_{01} + A_{02} + A_{11} + A_{12})||} = 3.$$

Thus, we obtain

$$r^{s_0}_{\delta,\mathbf{C}} = r^{s_0}_{\delta,\mathbf{R}} = r^{s_0}_{\delta,+} = \frac{1}{3}.$$

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