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THE RELATIVE BURNSIDE KERNEL–THE ELEMENTARY ABELIAN CASE

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ABSTRACT. We give a conjectural description for the kernel of the map, assigning to each finite $\mathbf{Z}_p$-free $G \times \mathbf{Z}_p$-set its rational permutation module where G is a finite p-group. We prove that this conjecture is true when G is an elementary abelian p-group or a cyclic p-group.

1. Introduction. The relative Burnside module $A(G, H)$ was introduced in [**6**] to study the stable maps between two classifying spaces BG and BH and specializes to the classical Burnside ring $A(G)$ in the case where H is trivial. This study generalizes Segal's conjecture [**3**] which states that the stable co-homotopy group of BG is isomorphic to a certain completion of $A(G)$.

The representation ring $R_F(G)$ of G over a field F has been used to study K-theory. For example, a classical theorem [**2**] states that the complex K-theory of the classifying space BG is isomorphic to a certain completion of $R_F(G)$ with F the field of complex numbers. If F is a finite field, then [**8**] showed that a certain completion of $R_F(G)$ is isomorphic to the stable maps from BG to the plus construction of $BGL(F)$.

The relationship between $A(G)$ and $R_F(G)$ has been studied in [**9, 10**] where it has been shown that the map $f : A(G) \to R_F(G)$ assigning to each finite G-set S its permutation representation $F[S]$ is surjective if G is a finite p-group and F is the field of rational numbers. Moreover, the kernel of the map f has been described in [**11**] for F a field of characteristic zero. The surjectivity of f was used in [**7**] to study algebraic K-theory of rings of integers, and a relative version was used in [**4**].

In this context, it is worthwhile to investigate the map $f : A(G, H) \to R_F(G, H)$ which sends a G-set S with a free H-action to the permutation representation $F[S]$ of G that is a free FH-module. Here F is

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the field of rational numbers, FH the group ring of H over F, and $R_F(G, H)$ a relative version of the representation ring. Very little is known about the map f except its surjectivity [**1**] for H the cyclic group of order p and G a finite p-group. We conjecture that, in this case, the kernel of f is induced from 'small' sub-quotients of $G \times H$, and we prove this conjecture in the case where G is either elementary abelian or cyclic.

Theorem. *Let p be a prime, F the field of rational numbers, $H = \mathbf{Z}_p$ the cyclic group of order p, and G an elementary abelian p-group. The map $f : A(G, H) \to R_F(G, H)$ is surjective and the kernel is induced from sub-quotients of $G \times H$ of the form $\mathbf{Z}_p \times \mathbf{Z}_p \times H$.*

Theorem. *Let p be a prime, F the field of rational numbers, $H = \mathbf{Z}_p$ the cyclic group of order p, and $G = \mathbf{Z}_{p^k}$ the cyclic p-group of order p^k. The map $f : A(G, H) \to R_F(G, H)$ is an isomorphism.*

We begin in Section 2 by giving a brief overview of the classical case, including the construction of induction maps. In Section 3 we consider the case when G is an elementary abelian p-group. In this case we decompose both the classical and relative Burnside rings into graded modules and compute their ranks. The main theorem is given in subsection 3.3 and describes the generators of the relative kernel $N(G, \mathbf{Z}_p)$. In Section 4 we offer a conjecture for describing the kernel of the relative map for any p-group G and offer proofs of the above-described theorems. Extending these results to the case where G is a p-group, but not necessarily elementary abelian or even abelian, poses additional hurdles and are postponed to future work.

2. Preliminaries.

2.1. Burnside and representation rings. For a finite group G the isomorphism classes of finite G-sets form a semiring S with respect to disjoint union and direct product. The Burnside ring $A(G)$ is defined to be the Grothendieck construction of the semiring S. In fact, $A(G)$ is a free $\mathbf{Z}$-module with a basis given by the set of left coset spaces $[G/L]$ where L runs through conjugacy class representatives of subgroups $L < G$. For each such subgroup, we define an induction map $G\uparrow: A(L) \to A(G)$ by sending an L-set X to the G-set $G \times_L X$ where $gl \times x = g \times lx$ for all (g, l, x) in $G \times L \times X$. This definition extends to induction maps $G\uparrow: A(L/C) \to A(G)$ via the pullback map $A(L/C) \to A(L)$ where L/C is a subquotient of G. The induction maps are $\mathbf{Z}$-linear but do not preserve the product.

Likewise, let T be the semiring of isomorphism classes of finitely generated $\mathbf{Q}[G]$-modules with respect to the direct sum and tensor product. The rational representation ring $R(G)$ is defined to be the Grothendieck construction of T, and a basis for this ring is given by G-isomorphism classes of simple $\mathbf{Q}G$-modules. For $L < G$, the induction map $G\uparrow: R(L) \to R(G)$ sends a $\mathbf{Q}[L]$-module M to the $\mathbf{Q}[G]$-module $\mathbf{Q}[G] \otimes_{\mathbf{Q}[L]} M$ and extends to subquotients as in the Burnside ring case.

The Burnside and representation rings are related by a natural ring homomorphism $f : A(G) \to R(G)$ sending a G-set X to the permutation $\mathbf{Q}[G]$-module $\mathbf{Q}[X]$. It is immediate that f commutes with the induction maps.

Definition 2.1. The *Burnside kernel* $N(G)$ is the kernel of map f.

2.2. Relative Burnside and representation modules. If $\widetilde{G} = G \times H$ is a direct product of two finite groups, then a $\widetilde{G}$-set is thought of with G acting on the left and H on the right. Let S' be the monoid of isomorphism classes of finite H-free $\widetilde{G}$-sets with respect to disjoint union. The relative Burnside module $A(G, H)$ is the Grothendieck construction of the monoid S'. Then $A(G, H) \subset A(\widetilde{G})$ is a free $\mathbf{Z}$-submodule with a basis given by twisted products $[G \times_\rho H]$ where ρ runs through conjugacy class representatives of homomorphisms $\rho : K \to H$ with $K < G$ and $gk \times h = g \times \rho(k)h$ for all (g, k, h) in $G \times K \times H$.

Similarly, a $\mathbf{Q}[\widetilde{G}]$-module is thought of with $\mathbf{Q}[G]$ acting on the left and $\mathbf{Q}[H]$ on the right. Let T' be the monoid of isomorphism classes of finitely generated $\mathbf{Q}[H]$-free $\mathbf{Q}[\widetilde{G}]$-modules with respect to the direct sum. The relative rational representation module $R(G, H)$ is the Grothendieck construction of the monoid T'. Then the natural ring homomorphism $f : A(\widetilde{G}) \to R(\widetilde{G})$ will restrict to a module homomorphism $f' : A(G, H) \to R(G, H)$.

Definition 2,2, The *relative Burnside kernel* $N(G,H)$ is the kernel of f'.

2.3. Relative induction. The relative induction maps $\widetilde{G}\uparrow$: $\widetilde{A}(L/C) \to A(G,H)$ are defined by the usual induction $\widetilde{G}\uparrow$ restricted to the submodule $\widetilde{A}(L/C)$ made of those elements of $A(L/C)$ that land in $A(G,H)$ under $\widetilde{G}\uparrow$ where L/C is a subquotient of $\widetilde{G}$. The same observation applies to the relative induction map $\widetilde{G}\uparrow$: $\widetilde{R}(L/C) \to R(G,H)$. It is immediate that the natural map f' from the relative Burnside module to the relative representation module commutes with the relative induction maps. We conclude by proving from scratch the following result.

Proposition 2.3. *In the case G is a finite abelian group, all induction maps are injective.*

Proof. Let M be any of the additive monoids S, T, S', T' defining the Burnside and representation modules and their relative versions. The induction map is defined by a homomorphism $L/C\uparrow$: $M \to N$ of monoids extended to the Grothendieck constructions where L/C is a subquotient of $\widetilde{G}$. The Grothendieck construction of M consists of formal differences $[X]-[Y]$ of elements in M such that $[X]-[Y] = [X']-[Y']$ if and only if

$$[X+Y'+Z] = [X'+Y+Z]$$

for some $[Z]$ in M. In particular, if $L/C\uparrow[X] - L/C\uparrow[Y] = 0$, then

$$[L/C\uparrow X + V] = [L/C\uparrow Y + V],$$

for some $[V]$ in N. By restricting the G-structure to an L-structure we have a restriction map $L\downarrow$ such that $L\downarrow(L/C\uparrow[X]) = [\widetilde{G}:L][X]$. In particular,

$$[\widetilde{G}:L][X] + L\downarrow[V] = [\widetilde{G}:L][Y] + L\downarrow[V].$$

Since each element of M has a unique decomposition into a sum of irreducible elements, we conclude that $[X]=[Y]$, proving the injectivity of the induction map. ◻

2.4. The p-group case. For G a finite p-group, it was shown by Tornehave [**11**] that $N(G)$ is generated by the induced kernels $G \uparrow N(L/C)$ where L/C runs through all subquotients of G that are isomorphic to the elementary abelian group $\mathbf{Z}_p \times \mathbf{Z}_p$, the dihedral group, or the nonabelian group of order p^3 and exponent p. Combining this with the Ritter-Segal [**9, 10**] proof for the surjectivity of f we get a well-understood short exact sequence:

$$0 \to N(G) \longrightarrow A(G) \xrightarrow{f} R(G) \longrightarrow 0. \tag{1}$$

In the abelian case $G = \mathbf{Z}_p \times \mathbf{Z}_p$; for instance, it is shown in [**5**] that $N(G)$ is the free cyclic group generated by

$$[G/0] - \sum [G/C] + p[G/G] \tag{2}$$

where C runs through all proper cyclic subgroups of G. In the relative case $\widetilde{G} = G \times H$ with G a finite p-group and $H = \mathbf{Z}_p$ it is known by [**1**] only that we have a short exact sequence

$$0 \longrightarrow N(G,H) \longrightarrow A(G,H) \xrightarrow{f'} R(G,H) \longrightarrow 0, \tag{3}$$

and the purpose of this paper is to study $N(G,H)$.

2.5. A useful trick.

Lemma 2.4. *Consider the chain complex of finitely generated free* $\mathbf{Z}$*-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

with α injective and β surjective. If the cokernel of α is a free module and the rank of the image of α equals the rank of the kernel of β, then the sequence is exact.

Proof. Since $\mathrm{Im}\,(\alpha) \subset \mathrm{Ker}\,(\beta)$ and $\mathrm{Coker}\,(\alpha)$ is free, we have the free $\mathbf{Z}$-submodule $\mathrm{Ker}\,(\beta)/\mathrm{Im}\,(\alpha) \subset B/\mathrm{Im}\,(\alpha)$. But the rank of the image of α equals the rank of the kernel of β so that $\mathrm{Ker}\,(\beta)/\mathrm{Im}\,(\alpha)$ is torsion. Therefore, $\mathrm{Ker}\,(\beta)/\mathrm{Im}\,(\alpha) = 0$. □

3. The relative burnside kernel for elementary abelian groups.

3.1. Notation. In this section, let G be an elementary abelian p-group of rank n for some prime p, and assume $H = \mathbf{Z}_p$. We regard $\widetilde{G}$ as an $(n+1)$-dimensional vector space over the field $\mathbf{F}_p$ of order p and denote the vector $(0, 0, \dots, 1) \in \widetilde{G}$ by e. Denote the classical and relative rings defined above by:

$$\begin{aligned} A &= A(\widetilde{G}), & R &= R(\widetilde{G}), & N &= N(\widetilde{G}),\\ A' &= A(G, \mathbf{Z}_p), & R' &= R(G, \mathbf{Z}_p), & N' &= N(G, \mathbf{Z}_p). \end{aligned}$$

We define $A_k \subset A$ to be the set generated by all $[\widetilde{G}/L]$ with $L \subset \widetilde{G}$ of dimension k. Thus, we gain the following decomposition of the classical Burnside ring:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{n+1}.$$

A similar decomposition holds for the relative Burnside ring with $A'_k = A_k \cap A'$. Lastly, define A''_{n-1} to be the module generated by the elements of the form $[\widetilde{G}/L] - [\widetilde{G}/L']$ with $L, L' < \widetilde{G}$ subspaces of dimension $n-1$ not containing e and subject to the relation

$$L \oplus \mathbf{Z}_p e = L' \oplus \mathbf{Z}_p e.$$

3.2. Rank calculations. Let $G(k, n)$ denote the number of k-dimensional subspaces of an n-dimensional vector space $\mathbf{Z}_p^n$.

Proposition 3.1. *The ranks a_k and a'_k of A_k and A'_k are given by the formulas*

$$a_k = G(k, n+1), \qquad a'_k = p^k G(k, n).$$

Proof. The basis elements $[\widetilde{G}/L]$ for A_k are in one-to-one correspondence with the k-dimensional subspaces $L < \widetilde{G} = \mathbf{Z}_p^{n+1}$. Hence, we get the first formula.

The basis elements $[G \times_\rho H]$ for A'_k are in one-to-one correspondence with pairs (K, ρ) with $K < G$ a k-dimensional subspace and $\rho : K \to H$ a homomorphism. Given a subgroup K, the group of homomorphisms from K to H is isomorphic to the dual vector space of K and hence has cardinality $|K| = p^k$. Thus, we gain the formula for a'_k. □

Let ζ denote a primitive p-root of unity and $F = \mathbf{Q}[\zeta]$ be the associated cyclotomic field. For each $s \in \mathbf{Z}_p^{n+1}$, let F_s be the $\mathbf{Q}[\widetilde{G}]$-module F obtained by letting the ith canonical generator of $\widetilde{G}$ act on F via the automorphism sending ζ to ζ^{s_i} where s_i is the ith coordinate of s.

Proposition 3.2. *The ranks r and r' of R and R' are given by the formulas*

$$r = G(1, n+1) + 1, \qquad r' = G(1, n+1).$$

Proof. Since $\widetilde{G}$ is finite, the group R is a free group with rank equal to the number of conjugacy classes of cyclic subgroups of $\widetilde{G}$. Immediately we gain the formula for r.

For the second formula, we claim that a basis for R' is given by the elements

$$[F_{s' \times 1}] + [\mathbf{Q}], \qquad [F_{t \times 0}] - (p-1)[\mathbf{Q}]$$

indexed by $s' \in \mathbf{Z}_p^n$ and $[t]$ an element of $(n-1)$-dimensional projective space over $\mathbf{F}_p$. Let $\mathcal{B}$ denote the set of these elements and $\mathcal{M}$ the $\mathbf{Z}$-module generated by $\mathcal{B}$. Since $F_{0\times 1} + \mathbf{Q} = \mathbf{Q}[H]$, it follows that by forgetting the G-action, the elements:

$$\begin{aligned} &[F_{s' \times 1}] + [\mathbf{Q}], \\ &[F_{t \times 0}] + (p-1)[F_{0 \times 1}], \end{aligned}$$

and

$$(p-1)[F_{0\times 1}] + (p-1)[\mathbf{Q}]$$

are all represented by the $\mathbf{Q}[H]$-free modules $\mathbf{Q}[H]$ or $(p-1)\mathbf{Q}[H]$. Thus, $\mathcal{M} \subset R'$, and it is immediate that $\mathcal{B}$ is a linearly independent set. Now, by inspection, $R/\mathcal{M}$ is the free module generated by $[\mathbf{Q}]$ and $m[\mathbf{Q}] \in R'$ implies $m = 0$. Thus, the rank of R' equals the rank of $\mathcal{M}$. In particular, Lemma 2.4 applies to the sequence:

$$0 \longrightarrow \mathcal{M} \longrightarrow R \longrightarrow R/R' \longrightarrow 0,$$

implying that $R' = \mathcal{M}$. □

From Propositions 3.1 and 3.2 and the short exact sequences (1) and (3), we deduce the following result.

Corollary 3.3. *The ranks b and b' of N and N' are given by the formulas*

$$b = \sum_{k=0}^{n-1} G(k, n+1), \qquad b' = \sum_{k=0}^{n} p^k G(k, n) - G(1, n+1).$$

3.3. The main theorem.

Lemma 3.4. *The submodule $A'_k \subset A'$ is the free abelian group on the set of projective subspaces $(L) \subset P^n$ with $L < \widetilde{G}$ of dimension k not containing e.*

Proof. It is easy to see that the basis element $[G \times_\rho H]$ of A' associated with a pair (K, ρ) is of the form $['/L]$ where $K < G$, $\rho : K \to H$ is a homomorphism, and $L = \{(k, \rho(k)) | k \in K\}$ is a linear subspace of $\widetilde{G}$ not containing e.

Conversely, let $(L) \subset P^n$ with $L < \widetilde{G}$ of dimension k not containing e and define K to be the image of the canonical projection $\widetilde{G} \to G$. If (g, h) is an element in L which maps to 0 under the projection, then $g = 0$. This would imply $he \in L$ so $h = 0$. Thus, the projection induces an isomorphism $L \cong K$. Let $\alpha : K \to L$ be the inverse, and define $\rho : K \to H$ by composing α with the canonical projection $\widetilde{G} \to H$. We can then check that $[\widetilde{G}/L] = [G \times_\rho H]$. □

In particular, we see that A''_{n-1} is a submodule of A'_{n-1} and, for strictly notational purposes, we define $A_{\leq k}$ to be the submodule of A

generated by $[\widetilde{G}/L]$ with the dimension of L being less than or equal to k. In the relative version we see:

$$A'_{\leq n-1} = (A' \cap A_{\leq n-2}) \oplus A''_{n-1}.$$

For each subspace $L < \widetilde{G}$ of codimension at least 2, let $L^* < \widetilde{G}$ be a distinguished subspace such that the following two conditions are both satisfied. We observe that L^* always exists subject to the two conditions and, in particular, if L has codimension exactly 2, then $L^* = \widetilde{G}$ is the only choice without violating condition 2.

(1) L^* contains L and L^*/L has rank 2.

(2) If L does not contain e and has codimension at least 3, then L^* does not contain e.

Definition 3.5. For each such L, define

$$t(L) = (L) - \sum (C) + p(L^*)$$

where the sum is over all proper subspaces $L < C < L^*$.

Lemma 3.6. *There is a short exact sequence*:

$$0 \longrightarrow A_{\leq n-1} \xrightarrow{t} A \xrightarrow{f} R \longrightarrow 0.$$

Proof. By the definition, the matrix representation of t with respect to a suitable basis is upper triangular, implying that t is injective and the Ritter-Segal result implies f is surjective. Citing Tornehave's result, the kernel of f is $N = \sum_Q \widetilde{G} \uparrow N(Q)$ where $Q = P/L$ is a subquotient of $\widetilde{G}$ isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. By Laitinen, each kernel $N(Q)$ is generated by an element of the form:

$$x = [Q/0] - \sum [Q/C] + p[Q/Q]$$

where the sum is taken over all cyclic subgroups $C < Q$. Applying the induction map $\widetilde{G} \uparrow$ to a generator x, we gain $t[\widetilde{G}/L]$. ◻

Theorem 3.7. *There is a short exact sequence*

$$0 \longrightarrow A'_{\leq n-1} \xrightarrow{t'} A' \xrightarrow{f'} R' \longrightarrow 0.$$

Proof. As a restriction of t, the map t' is injective and, by Anton, f' is surjective. The result will follow immediately after proving the rank condition below:

$$\operatorname{rank}(A') = \operatorname{rank}(R') + \operatorname{rank}(A'_{\leq n-1}).$$

Recognizing that

$$\operatorname{rank}(A'_{\leq n-1}) = \sum_{k=0}^{n-2} \operatorname{rank}(A'_k) + \operatorname{rank}(A''_{n-1})$$
$$\operatorname{rank}(A') = \sum_{k=0}^{n} \operatorname{rank}(A'_k)$$

we are left to show an equivalent condition:

$$\operatorname{rank}(A'_{n-1}) + \operatorname{rank}(A'_n) = \operatorname{rank}(R') + \operatorname{rank}(A''_{n-1}).$$

To determine the rank of A''_{n-1}, note that there are $p^{n-1}G(n-1,n)$ subspaces $L < \widetilde{G}$ of dimension $n-1$ not containing e. For each such L, the number of subspaces $L' < L + \mathbf{Z}_p e$ of dimension $n-1$ not containing e is $p^{n-1}G(n-1,n-1)$. Thus, the desired rank is the number of differences $[\widetilde{G}/L] - [\widetilde{G}/L']$ with L and L' as described is:

$$(p^{n-1}-1)\frac{p^{n-1}G(n-1,n)}{p^{n-1}G(n-1,n-1)} = G(1,n)(p^{n-1}-1).$$

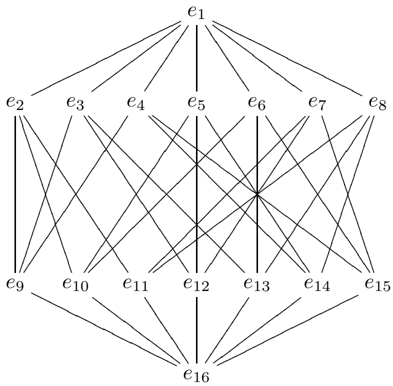
The other three ranks were calculated in subsection 3.2, and substituting into the original equation, we want to verify:

$$p^{n-1}G(n-1,n) + p^n = G(1,n+1) + G(1,n)(p^{n-1}-1).$$

Canceling like terms and substituting, in fact we see:

$$p^n = \frac{p^{n+1}-1}{p-1} - \frac{p^n-1}{p-1}. \qquad \square$$

3.4. An illustration for $n = 2$ and $p = 2$. Order $\mathbf{Z}_2$ such that $0 < 1$ and order $\mathbf{Z}_2^3$ lexicographically. We gain a labeling of the basis of $A(\widetilde{G})$ $\{e_i\}$ such that $e_1 < e_2 < \cdots < e_{16}$. With this labeling of the basis of $A(\widetilde{G})$, the subgroup lattice of $\widetilde{G}$ can be represented by the graph E below and offers a visual description of the relationship between basis elements e_i and e_j.



Theorem 3.7 implies that we have the commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_{\leq 1} & \xrightarrow{t'} & A' & \xrightarrow{f'} & R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A_{\leq 1} & \xrightarrow{t} & A & \xrightarrow{f} & R & \longrightarrow & 0 \end{array}$$

Using our basis we see that

$$A_0 = A'_0 = \mathbf{Z}e_1, \qquad A_1 = \sum_{i=2}^{8} \mathbf{Z}e_i$$

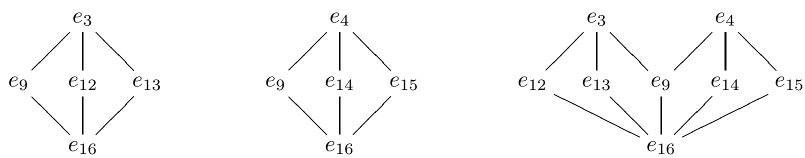
and

$$A'_1 = \mathbf{Z}(e_3 - e_4) + \mathbf{Z}(e_5 - e_6) + \mathbf{Z}(e_7 - e_8).$$

Hence, t is well defined on A_1 while we define:

$$t(e_1) = e_1 - e_3 - e_5 - e_7 + 2e_{12}.$$

Define the subgraph E_i to be the full subgraph of E where the vertices are the terms occurring in $t(e_i)$. Then the image $t'(e_i - e_j)$ is associated with the subgraph $E_i - E_j$ whose vertices are those in E_i and E_j. For example, if $i = 3$ and $j = 4$ these subgraphs are:



Conversely, given a subgraph E_i, the image $t(e_i)$ is uniquely determined by taking a weighted sum of the vertices of E_i. Moreover, given a subgraph $E_i - E_j$, the image $t(e_i - e_j)$ is also uniquely determined by the vertices of $E_i - E_j$. It follows that the kernel of f is generated by all of the subgraphs E_i for $i = 1, 2, \ldots, 8$ and the kernel of f' is generated by all the non-singular subgraphs E_1, $E_3 - E_4$, $E_5 - E_6$, $E_7 - E_8$.

4. Final remarks. We would like to develop a description for the kernel $N(G, H)$ with $H \cong \mathbf{Z}_p$ similar to that given by Tornehave in [**11**] for $N(G)$ with G an arbitrary finite p-groups G. Define $\widetilde{N}(L/C)$ for L/C a subquotient of $\widetilde{G} = G \times H$ to be the intersection of $N(L/C)$ with the submodule $\widetilde{A}(L/C)$ of $A(L/C)$ that lands inside $A(G, H)$ under the induction $\widetilde{G}\uparrow$ of subsection 2.3.

Conjecture 4.1. *Let p be a prime, G any finite p-group, and $H \cong \mathbf{Z}_p$. Then*

$$N(G, H) = \sum \widetilde{G} \uparrow \widetilde{N}(L/C)$$

where the sum is taken over subquotients L/C of $\widetilde{G}$ isomorphic to $T \times H$ where T is the elementary abelian group $\mathbf{Z}_p \times \mathbf{Z}_p$, a dihedral group of order 2^n where $n \geq 3$, or the nonabelian group of order p^3 and exponent p.

For G elementary abelian or cyclic this conjecture can readily be checked using Theorem 3.7 and rank arguments.

Proposition 4.2. *Let p be any prime, G an elementary abelian p-group, and $H \cong \mathbf{Z}_p$. Then*

$$N(G,H) = \sum L/C \uparrow \widetilde{N}(L/C)$$

with the sum taken over all subquotients $L/C \cong \mathbf{Z}_p^3$.

Proof. From Theorem 3.7, we know that the image of t generates the kernel $N(G,H)$. If $(L) \in A_i'$ with $0 \leq i \leq n-2$, then there exist subgroups $L < L^* < B < \widetilde{G}$ such that $B/L \cong \mathbf{Z}_p^3$ where L^* is the distinguished element used to define t in Definition 3.5. In addition, regardless of our choice of B,

$$t((L)) \in \widetilde{G} \uparrow \widetilde{N}(B/L).$$

If $(L) - (L') \in \widetilde{A}_{n-1}$, let

$$C = L + \mathbf{Z}_p e = L' + \mathbf{Z}_p e, \qquad D = L \cap L'.$$

We see immediately that $\widetilde{G}/D \cong \mathbf{Z}_p^3$ and also that $t((L) - (L'))$ is an element of $\widetilde{G}/D \uparrow \widetilde{N}(\widetilde{G}/D)$. Hence we conclude that

$$N(G,H) \subset \sum \widetilde{G} \uparrow \widetilde{N}(L/C).$$

The converse is immediate. □

Proposition 4.3. *Lt p be any prime, G the cyclic p-group with order p^k, and $H \cong \mathbf{Z}_p$. Then f' is an isomorphism between $A(G,H)$ and $R(G,H)$.*

Proof. Let $\widetilde{G} = G \times H$ and ξ be the primitive p^k-root of unity. Since G is cyclic, the rank of $A(G,H)$ is easily equal to $kp+1$ as G has $k+1$ subgroups and, for a nontrivial subgroup $K < G$, there are p homomorphisms $\rho : K \to H$. Let $F_{\nu,\phi} = \mathbf{Q}(\xi^{p^{k-\nu}}, \xi^{\phi p^{k-1}})$ be the $\mathbf{Q}[\widetilde{G}]$-module with the generators of G and H acting by multiplication by $\xi^{p^{k-\nu}}$ and $\xi^{\phi p^{k-1}}$, respectively, where $\nu = 0, 1, \ldots, k$ and $\phi =$

$0, 1, \ldots, p-1$. Then the irreducible $\mathbf{Q}[\widetilde{G}]$-modules as seen from the decomposition of the group ring $\mathbf{Q}[\widetilde{G}]$ are:

$$\begin{aligned} &F_{0,0} = \mathbf{Q} \\ &F_{0,1} = \mathbf{Q}(\xi^{\phi p^{k-1}}) \\ &F_{\nu,\phi} \quad \text{with } \nu = 1, \ldots, k \text{ and } \phi = 0, 1, \ldots, p-1. \end{aligned}$$

This implies the rank of $R(\widetilde{G})$ is $kp+2$.

For $[M], [M'] \in R(\widetilde{G})$, define $[M] \equiv [M']$ if we have $[M] - [M'] \in R(G,H)$. Using this relation we immediately gain the following equivalences from [**1**]:

$$\begin{aligned} &[F_{\nu,\phi}] \equiv -p^{\nu-1}[\mathbf{Q}] && \text{for } \nu = 1, \ldots, k \text{ and } \phi = 1, \ldots, p-1, \\ &[F_{0,1}] \equiv -[\mathbf{Q}], && \\ &[F_{\nu,0}] \equiv p^{\nu-1}(p-1)[\mathbf{Q}] && \text{for } \nu = 1, \ldots, k. \end{aligned}$$

The equivalences imply that the rank of $R(\widetilde{G})/R(G,H)$ is less than or equal to 1. In addition, since f' is surjective, the rank of $A(G,H)$ is $kp+1$, and the rank of $R(\widetilde{G})$ is $kp+2$, we see that the rank of $R(\tilde{G})/R(G,H)$ is at least 1. Thus the rank of $R(\tilde{G})/R(G,H)$ is exactly 1, which implies the rank of $R(G,H) = kp+1$. As $A(G,H)$ is a free module, the rank of $A(G,H)$ is equal to the rank of $R(G,H)$, and f' is a surjection, we see that f' is an isomorphism. □

As a corollary, Conjecture 4.1 is true for G a cyclic p-group.

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