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HOMOMORPHISMS ON A CLASS OF COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. We derive representations for homomorphisms and isomorphisms between Banach algebras of Lipschitz functions with values in a sequence space, including ℓ_∞. We show that such homomorphisms are automatically continuous and preserve the $*$ operation. We also give necessary conditions for the compactness of homomorphisms in these settings and give characterizations for the isometric isomorphisms.

1. Introduction. Dunford and Schwartz in [**8**] gave a characterization of homomorphisms between the Banach algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, where X and Y are compact Hausdorff spaces. This theorem was attributed to Gelfand and Kolmogorov, see [**11**].

Theorem 1.1 (cf. [**8**, Theorem 26, page 278]). *If $T : \mathcal{C}(X) \to \mathcal{C}(Y)$ is a linear and multiplicative function, then there exists a continuous function $\tau : Y \to X$ such that, for every $f \in \mathcal{C}(X)$ and $y \in Y$,*

$$T(f)(y) = f(\tau(y)).$$

If T is an isomorphism, then there exists a homeomorphism $\tau : Y \to X$ such that, for every $f \in \mathcal{C}(X)$ and $y \in Y$,

$$T(f)(y) = f(\tau(y)).$$

Sherbert, in [**24**], has shown that algebra homomorphisms on Banach algebras of scalar valued Lipschitz functions are also represented as composition operators.

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Theorem 1.2 (cf. [**24**]). *Let X and Y be compact metric spaces. If $\psi : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ is an algebra homomorphism, then there exists a unique Lipschitz function $\varphi : Y \to X$ such that*

$$\psi(f)(y) = f(\varphi(y)) \quad \textit{for all } f \in \mathrm{Lip}_*(X) \textit{ and } y \in Y.$$

If $\psi : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ is an algebra isomorphism, then φ is a lipeomorphism.

In a previous work, the authors have shown that, under some mild continuity hypothesis, $*$-homomorphisms between operator valued Lipschitz spaces are given as a combination of operators which are unitarily equivalent to composition operators, see [**3**].

In this paper, we study homomorphisms between algebras of Lipschitz functions from a compact metric space into a complex sequence space, $\mathcal{B}$. The space $\mathcal{B}$ is either the space of all convergent sequences, $\mathbf{c}$, or the space of all bounded sequences, ℓ_∞. Our techniques also apply to the finite dimensional case, $\mathbf{C}^n$. All these spaces are C^* algebras with identity under the obvious multiplication and standard $\|\cdot\|_\infty$ norm.

If (X, d) represents a compact metric space, we consider the algebra of Lipschitz functions defined on X and with values in $\mathcal{B}$:

$$\mathrm{Lip}_*(X, \mathcal{B}) = \left\{ f : X \to \mathcal{B} \mid \sup_{x \neq y} \frac{\|f(x) - f(y)\|_\infty}{d(x, y)} < \infty \right\}, \tag{1.2}$$

with norm $\|f\|_* = \|f\|_\infty + \sup_{x\neq y} \|f(x) - f(y)\|_\infty / d(x, y)$.

The quantity $\sup_{x\neq y} \|f(x) - f(y)\|_\infty / d(x, y)$ is said to be the Lipschitz constant of f and is denoted by $L(f)$.

We set $f^*(x) = [f(x)]^*$ for all $x \in X$. The $*$ operation on a sequence $\{x_n\}_{n\in\mathbf{N}}$ yields $\{\overline{x}_n\}_{n\in\mathbf{N}}$, where "overline" stands for complex conjugacy. We observe that $\|f^*\|_* = \|f\|_*$ and $\|f\,g\|_* \le \|f\|_* \, \|g\|_*$; hence, $\mathrm{Lip}_*(X, \mathcal{B})$ is a commutative Banach $*$-algebra with unit 1_X. The function 1_X is given by $1_X(x) = 1$ for every $x \in X$, with 1 representing the constant sequence in $\mathcal{B}$ equal to 1.

The main results in this paper give characterizations for algebra homomorphisms from $\mathrm{Lip}_*(X, \mathcal{B})$ into $\mathrm{Lip}_*(Y, \mathcal{B})$, with X and Y compact metric spaces. We use the fact that $\mathbf{c}$ has a Schauder basis to show that algebra homomorphisms between spaces of $\mathbf{c}$ valued Lipschitz functions are decomposed into algebra homomorphisms between scalar valued Lipschitz functions. Similar results also hold for algebras of Lipschitz functions with values in ℓ_∞; however, the justification follows a completely different strategy since ℓ_∞ does not have a Schauder basis.

As a consequence of our results we obtain characterizations of the isometric isomorphisms as well as the compact homomorphisms. These characterizations rely on the fact that such homomorphisms preserve the subalgebra of all constant functions, $\mathrm{Const}_*(X, \mathcal{B})$. An element in $\mathrm{Const}_*(X, \mathcal{B})$ is a constant function with range consisting of a single element in $\mathcal{B}$. A sequence $\{a_n\}_n$ in $\mathcal{B}$ is simply denoted by $\mathbf{a}$, and the constant function in $\mathrm{Lip}_*(X, \mathcal{B})$ with range equal to $\mathbf{a}$ is denoted by $\mathbf{a}_X$. A homomorphism $\psi : \mathrm{Lip}_*(X, \mathcal{B}) \to \mathrm{Lip}_*(Y, \mathcal{B})$ "preserves constant functions" if

$$\psi(\mathrm{Const}_*(X, \mathcal{B})) \subseteq \mathrm{Const}_*(Y, \mathcal{B}).$$

Furthermore, we also say that ψ "fixes constant functions" if, for every $\mathbf{a}_X \in \mathrm{Const}_*(X, \mathcal{B})$, $\psi(\mathbf{a}_X)(y) = \mathbf{a}$, for all $y \in Y$, or equivalently $\psi(\mathbf{a}_X) = \mathbf{a}_Y$. We apply Sherbert's characterization of algebra homomorphisms on scalar valued Lipschitz spaces to represent homomorphisms that fix constant functions as componentwise composition operators.

2. Definitions and preliminary results. The algebra of scalar valued Lipschitz functions on X is denoted by $\mathrm{Lip}_*(X)$. Given $f \in \mathrm{Lip}_*(X, \mathcal{B})$, we have $f(x) = \{f_i(x)\}_{i \in \mathbf{N}}$, with $f_i \in \mathrm{Lip}_*(X)$. For $j \in \mathbf{N}$, $P_j : \mathrm{Lip}_*(X, \mathcal{B}) \to \mathrm{Lip}_*(X)$ is the projection onto the j-position, i.e., $P_j(f)(x) = f_j(x)$, for all $x \in X$. A map $E_j : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(X, \mathcal{B})$ is a j-embedding of $\mathrm{Lip}_*(X)$ into $\mathrm{Lip}_*(X, \mathcal{B})$, provided that $E_j(f)(x) = \{b_i\}_{i \in \mathbf{N}} \in \mathcal{B}$ with $b_j = f(x)$, for all $x \in X$.

Lemma 2.1. *Let X and Y be compact metric spaces and ψ an algebra homomorphism from $\mathrm{Lip}_*(X, \mathcal{B})$ into $\mathrm{Lip}_*(Y, \mathcal{B})$ that fixes constant functions. The map $\psi_j : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$, given by $\psi_j(g) = P_j \psi(E_j(g))$, is independent of the j-embedding E_j.*

Proof. Let E_j^1 and E_j^2 be two distinct j-embeddings of $\mathrm{Lip}_*(X)$ into $\mathrm{Lip}_*(X, \mathcal{B})$. For $j \in \mathbf{N}$, we consider the sequence $\mathbf{e}^j$ in $\mathcal{B}$ given by

$$\mathbf{e}_k^j = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

This sequence determines a constant function in $\mathrm{Const}_*(X, \mathcal{B})$, denoted by $\mathbf{e}_X^j$. Since ψ is multiplicative, we have $\psi(E_j^1(g) - E_j^2(g)) \cdot \psi(\mathbf{e}_X^j) = \psi((E_j^1(g) - E_j^2(g)) \cdot \mathbf{e}_X^j) = 0$, for every $g \in \mathrm{Lip}_*(X)$. This implies that $P_j\psi(E_j^1(g)) = P_j\psi(E_j^2(g))$. $\square$

Remark 2.2. Lemma (2.1) asserts that the value of $\psi_j(g)$ does not depend upon the j-embedding of $g \in \mathrm{Lip}_*(X)$ into $\mathrm{Lip}_*(X, \mathcal{B})$. We then set E_j to be given by $E_j(f) = \{b_n\}_{n \in \mathbf{N}}$ with $b_n = 0$ for $n \neq j$ and $b_j = f(x)$. This embedding is linear and multiplicative; hence, $\psi_j = P_j \psi E_j$ is an algebra homomorphism.

Our first theorem characterizes algebra homomorphisms that fix constant functions. We note that a homomorphism ψ between $\mathrm{Lip}_*(X, \mathcal{B})$ and $\mathrm{Lip}_*(Y, \mathcal{B})$ is not necessarily of the form $\psi(f)(y) = \{f_i(\varphi_i(y))\}$ with φ_i Lipschitz maps, as demonstrated in the next example.

Example 2.3. Consider the homomorphism $\psi : \mathrm{Lip}_*([0,1], \mathcal{B}) \to \mathrm{Lip}_*([0,1], \mathcal{B})$ given by $\psi(f)(y) = \{f_1(y)\}_{i \in \mathbf{N}}$. This map is clearly an algebra homomorphism, but there is no sequence of Lipschitz functions $\{\varphi_i\}_{i \in \mathbf{N}}$ such that $\psi(f)_i(y) = f_i(\varphi_i(y))$.

We set $\prod_{i \in \mathbf{N}} X$ to be the product of $\mathbf{N}$ copies of X equipped with the standard product metric.

Definition 2.4. We say that a homomorphism $\psi : \mathrm{Lip}_*(X, \mathcal{B}) \to \mathrm{Lip}_*(Y, \mathcal{B})$ is a coordinatewise composition operator (CCO) if and only if there exists a map $\Phi : Y \to \prod_{i \in \mathbf{N}} X$ given by $\Phi(y) = \{\varphi_i(y)\}_{i \in \mathbf{N}}$, with $\varphi_i \in \mathrm{Lip}_*(X)$ for all $i \in \Lambda$. Moreover, the sequence of Lipschitz constants $\{L(\varphi_i)\}_{i \in \mathbf{N}}$ is bounded and

$$\psi(f)(y) = \{f_i(\varphi_i(y))\}_{i \in \mathbf{N}} \quad \text{for all } f \in \mathrm{Lip}_*(X, \mathcal{B}) \text{ and } y \in Y. \tag{2.1}$$

We note that the boundedness assumption on the sequence of Lipschitz constants $\{L(\varphi_i)\}_{i\in\mathbf{N}}$ implies that Φ is a Lipschitz map.

The next theorem is a consequence of Theorem 1.2 and Lemma 2.1. The proof is omitted since it follows the argument used by Sherbert for the proof of Theorem 5.1 (see [**24**, page 1397]).

Theorem 2.5. *Let X and Y be compact metric spaces. If ψ from $\mathrm{Lip}_*(X,\mathcal{B})$ into $\mathrm{Lip}_*(Y,\mathcal{B})$ is an algebra homomorphism, then ψ fixes constant functions if and only if ψ is a coordinatewise composition operator.*

3. Continuity of homomorphisms on algebras of Lipschitz Functions into sequence spaces. In this section we establish the continuity of algebra homomorphisms between spaces of $\mathcal{B}$-valued Lipschitz functions. We show that such spaces are semi-simple commutative Banach algebras. We start by recalling a well-known result on maximal ideals of unital rings.

Theorem 3.1 (cf. [**1**, page 34]). *Let A be a ring with unit 1_A. Then the following sets are identical*:

(i) *the intersection of all maximal left ideals of A.*

(ii) *The intersection of all maximal right ideals of A.*

(iii) *The set of all x such that $1_A - xz$ is invertible in A for all $z \in A$.*

(iv) *The set of all x such that $1_A - zx$ is invertible in A for all $z \in A$.*

Definition 3.2 (cf. [**1, 22**]). Let A be a ring with unit 1_A. The radical of A, $\mathrm{rad}\,(A)$ is the two sided ideal equal to the intersection of all maximal left ideals of A. If $\mathrm{rad}\,(A) = \{0\}$, then A is said to be semi-simple.

It is straightforward to check that, for a Banach algebra A with unit 1_A, and X a compact metric space, $\mathrm{Lip}_*(X, A)$ is a Banach algebra with unit 1_X, where $1_X(x) = 1_A$ for every $x \in X$.

Proposition 3.3. *If A is a Banach algebra with unit 1_A, then $\mathrm{Lip}_*(X, A)$ is semi-simple if and only if A is semi-simple.*

Proof. We first notice that A is isomorphically embedded in $\mathrm{Lip}_*(X, A)$, since A is identified with the subalgebra of all constant functions. Corollary 2.3.7 in [**22**, page 57] states that any subalgebra of a semi-simple commutative algebra is semi-simple. This implies that the subalgebra of all constant functions is semi-simple whenever $\mathrm{Lip}_*(X, A)$ is semi-simple. Therefore, A is also semi-simple. Conversely, let $f \in \mathrm{Lip}_*(X, A)$ for which there exist $x_0 \in X$ and $a \in A$ such that $f(x_0) \neq 0$ and $1_A - f(x_0)a$ is not invertible. Let $g \in \mathrm{Lip}_*(X, A)$ be given by $g(x) = a$. Then $1_X - fg$ is not invertible since $(1_X - fg)(x_0)$ is not invertible in A. Consequently, if A is semi-simple, then $\mathrm{Lip}_*(X, A)$ is semi-simple. □

It is easy to see that each sequence space $\mathbf{C}^n$, ℓ_∞, and $\mathbf{c}$ is a semi-simple Banach algebra. Therefore, we have the following corollary.

Corollary 3.4. *If X and Y are compact metric spaces and $\mathcal{B}$ is either $\mathbf{C}^n$, ℓ_∞, or $\mathbf{c}$, then $\mathrm{Lip}_*(X, \mathcal{B})$ is semi-simple. Moreover, every algebra homomorphism from $\mathrm{Lip}_*(X, \mathcal{B})$ into $\mathrm{Lip}_*(Y, \mathcal{B})$ is continuous.*

This continuity statement in Corollary 3.4 is a consequence of the following theorem.

Theorem 3.5 (cf. [**23**, Theorem 11.10]; see also [**22**, Theorem 7.5.17]). *If ψ is a homomorphism of a commutative Banach algebra A_1 into a semi-simple commutative Banach algebra A_2, then ψ is continuous.*

4. Homomorphisms on algebras of Lipschitz Functions with values in $\mathbf{c}$ or $\mathbf{C}^n$. We start by proving that an algebra homomorphism between two Lipschitz spaces of $\mathbf{c}$ valued functions must preserve the subalgebra of the constant functions, $\mathrm{Const}_*(X, \mathbf{c})$. We note that the results in this section also hold for finite dimensional sequence spaces, $\mathbf{C}^n$.

Proposition 4.1. *Let X and Y be compact metric spaces, Y connected. If $\psi : \mathrm{Lip}_*(X, \mathbf{c}) \to \mathrm{Lip}_*(Y, \mathbf{c})$ is an algebra homomorphism, then*

$$\psi(\mathrm{Const}_*(X, \mathbf{c})) \subseteq \mathrm{Const}_*(Y, \mathbf{c}).$$

Proof. Let $i \in \mathbf{N}$ and $\mathbf{e}_X^i$ be the constant function in $\mathrm{Lip}_*(X, \mathcal{B})$, given by $(\mathbf{e}_X^i)_j(x) = 1$ if $i = j$ and zero otherwise. We also set $\mathbf{e}_X^\infty$ to represent the constant function everywhere equal to the sequence $\{1\}_n$. For every $i \in \mathbf{N}$ or $i = \infty$, $\mathbf{e}_X^i = \mathbf{e}_X^i \cdot \mathbf{e}_X^i$, we have that $\psi_j(\mathbf{e}_X^i)^2 = \psi_j(\mathbf{e}_X^i)$, with ψ_j representing the jth-coordinate of ψ. Hence, for every $y \in Y$, $\psi_j(\mathbf{e}_X^i)(y) = 1$ or 0. The continuity of $\psi(\mathbf{e}_X^i)$ and the connectedness of Y imply that $\psi(\mathbf{e}_X^i)$ is constant. Since $\{e^1, e^2, \dots, e^\infty\}$ is a Schauder basis for $\mathbf{c}$, we have that each function in $\mathrm{Const}_*(X, \mathcal{B})$ can be written as a linear combination of $\{\mathbf{e}_X^i\}_{i \in \mathbf{N}}$. Therefore, the continuity of ψ establishes the statement in the proposition. $\square$

We now set some notation. Let $\widehat{\mathbf{N}}$ be the Alexandroff compactification of $\mathbf{N}$, i.e., $\widehat{\mathbf{N}} = \{1, 2, \dots, \infty\}$ with standard topology. We set $\widehat{\mathbf{c}} = \{[a_1, a_2, \dots, a_\infty] : \{a_n\}_n \in \mathbf{c} \text{ and } a_\infty = \lim_n a_n\}$. We also denote by $\mathcal{C}(\widehat{\mathbf{N}})$ the space of all continuous and scalar valued functions defined on $\widehat{\mathbf{N}}$. It is clear that $\mathbf{c}$, $\widehat{\mathbf{c}}$ and $\mathcal{C}(\widehat{\mathbf{N}})$ are isometrically isomorphic as Banach algebras. Elements in these spaces, say $\{a_n\} \in \mathbf{c}$, $[a_1, a_2, \dots, a_\infty] \in \widehat{\mathbf{c}}$ or $a \in \mathcal{C}(\widehat{\mathbf{N}})$ are used indistinguishably, as it best fits in our exposition. If $f \in \mathrm{Lip}_*(X, \mathbf{c})$, we denote by $\widehat{f} : X \to \widehat{\mathbf{c}}$ the function $\widehat{f}(x) = [f_1(x), f_2(x), \dots, f_\infty(x)]$ with $f_\infty(x) = \lim_n f_n(x)$. The following lemma is clear.

Lemma 4.2. *If $f \in \mathrm{Lip}_*(X, \mathbf{c})$, then $\widehat{f} \in \mathrm{Lip}_*(X, \widehat{\mathbf{c}})$. The map $f \to \widehat{f}$ is an isometric isomorphism from* $\mathrm{Lip}_*(X, \mathbf{c})$ *onto* $\mathrm{Lip}_*(X, \widehat{\mathbf{c}})$.

Theorem 1.2 motivated our investigation of homomorphisms between sequence valued Lipschitz algebras, and it is used in an essential way in the proof of the next theorem.

Theorem 4.3. *Let X be a compact metric space and Y a compact and connected metric space. If $\psi : \mathrm{Lip}_*(X, \mathbf{c}) \to \mathrm{Lip}_*(Y, \mathbf{c})$ is a homomorphism such that $\psi(1_X) = 1_Y$, then there exist a continuous map $\tau : \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$ and a sequence of Lipschitz maps $\varphi_{n,\tau(n)} : Y \to X$ such that, for all $n \in \mathbf{N}$ and $y \in Y$,*

$$\psi(f)_n(y) = f_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

If ψ is an isomorphism, then there exist a homeomorphism $\tau : \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$

and a sequence of lipeomorphisms $\varphi_{n,\tau(n)} : Y \to X$ *such that, for all* $n \in \mathbf{N}$ *and* $y \in Y$*,*

$$\psi(f)_n(y) = f_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

Proof. The homomorphism ψ induces a homomorphism

$$\widehat{\psi} : \mathrm{Lip}_*(X, \widehat{\mathbf{c}}) \longrightarrow \mathrm{Lip}_*(Y, \widehat{\mathbf{c}})$$

given by $\widehat{\psi}(\widehat{f}) = \widehat{\psi(f)}$. It is clear that $\widehat{\psi}(\mathrm{Const}_*(X, \widehat{\mathbf{c}})) \subseteq \mathrm{Const}_*(Y, \widehat{\mathbf{c}})$. Therefore, $\widehat{\psi}$ restricted to $\mathrm{Const}_*(X, \widehat{\mathbf{c}})$ is given as a composition operator. More precisely, Theorem 1.1 asserts the existence of a continuous mapping $\tau : Y \to X$ such that $\widehat{\psi}(\widehat{a}) = \widehat{a} \circ \tau$. We consider the Schauder basis for $\widehat{\mathbf{c}}$: $\widehat{e}^1$, $\widehat{e}^2$, $\ldots$, $\widehat{e}^\infty$. For each $j \in \mathbf{N}$, we identify $\widehat{e^j}$ with a function in $\mathcal{C}(\widehat{\mathbf{N}})$ given by $\widehat{e^j}(n) = 0$ for $n \neq j$ and $\widehat{e^j}(j) = 1$. The function $\widehat{e^\infty}$ is given by $\widehat{e^\infty}(n) = 1$, for every $n \in \widehat{\mathbf{N}}$.

Given $(i, j) \in \widehat{\mathbf{N}} \times \widehat{\mathbf{N}}$, we define a homomorphism $\psi_{i,j} : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ as follows:

$$\psi_{i,j}(\lambda) = P_i(\widehat{\psi}(\lambda \cdot e^j_X)), \quad \text{for every } \lambda \in \mathrm{Lip}_*(X).$$

Sherbert's theorem (Theorem 1.2) asserts that either $\psi_{i,j} = 0$ or there exists a unique Lipschitz function $\varphi_{i,j} : Y \to X$ such that $\psi_{i,j}(\lambda) = \lambda \circ \varphi_{i,j}$. We now show that $\psi_{i,j} = 0$ if $\tau(i) \neq j$. In fact,

$$\psi_{i,j}(\lambda) = P_i(\widehat{\psi}(\lambda \widehat{e}^j_X)) = P_i(\widehat{\psi}(\lambda \widehat{e}^j_X \cdot \widehat{e}^j_X)) = P_i(\widehat{\psi}(\lambda \cdot \widehat{e}^j_X))P_i(\widehat{e^j} \circ \tau).$$

We observe that $\widehat{e^j} \circ \tau(n) = 1$ if $\tau(n) = j$ and $\widehat{e^j} \circ \tau(n) = 0$ if $\tau(n) \neq j$. For $j \neq \tau(i)$, we have that $P_i(\widehat{e^j} \circ \tau) = 0$ and $\psi_{i.j}(\lambda) = 0$ for $j \neq \tau(i)$. We also conclude that $\psi_{i,\tau(i)}(\lambda) = \lambda \circ \varphi_{i,\tau(i)}$.

For $\widehat{f} \in \mathrm{Lip}_*(X, \widehat{\mathbf{c}})$ and i, $j \in \widehat{\mathbf{N}}$,

$$\widehat{\psi}(\widehat{f})(y) = \widehat{\psi}\Big[f_\infty \widehat{e}^{\,\infty} + \sum_{i \in \mathbf{N}} (f_i - f_\infty)\, \widehat{e}^{\,i}\Big].$$

Therefore,

$$\widehat{\psi}(f_\infty \widehat{e}^{\,\infty}) = \sum_{\{n : \tau(n) = j\}} (f_\infty \circ \varphi_{n,j}) \widehat{e}^{\,n}$$

and

$$\widehat{\psi}[(f_j - f_\infty)\widehat{e^j}] = \sum_{\{n:\tau(n)=j\}} ((f_j - f_\infty) \circ \varphi_{n,j})\widehat{e^n}.$$

If ψ is an isomorphism, then Theorems 1.1 and 1.2 imply the existence of a sequence of lipeomorphisms $\varphi_{n,\tau(n)} : Y \to X$ and a homeomorphism $\tau : \mathbf{N} \to \mathbf{N}$. This completes the proof. $\square$

In general, a homomorphism ψ as described in the previous theorem is not a CCO. We now give a simple example of such a homomorphism to illustrate the broad spectrum of possibilities.

Example 4.4. Let $X = Y = [0, 1]$ and $\tau : \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$ be given by $\tau(2n+1) = \tau(\infty) = \infty$ and $\tau(2n) = n$. We set $\varphi_{n,\tau(n)} = \mathrm{Id}_{[0,\,1]}$. For $f \in \mathrm{Lip}_*([0,1], \mathbf{c})$

$$f(x) = (f_1(x), f_2(x), \dots , f_n(x), \dots) \in \mathbf{c}$$

we denote by $f_\infty(x)$ the limit of $\{f_n(x)\}_n$.

We now define $\psi : \mathrm{Lip}_*([0,1], \mathbf{c}) \to \mathrm{Lip}_*([0,1], \mathbf{c})$ as follows:

$$\psi(f)(y) = (f_\infty(y), f_1(y), f_\infty(y), f_2(y), \dots).$$

It is easy to show that ψ is not a CCO. If we assume otherwise, then φ, a Lipschitz function on $[0,1]$, exists such that

$$f_1(y) = f_2(\varphi(y)),$$

for all $y \in [0,1]$ and for every f_1 and f_2 in $\mathrm{Lip}_*([0,1])$. This is clearly impossible.

The form of homomorphisms derived in Theorem 4.3 implies that algebra homomorphisms preserve the $*$ operation. It is also easy to derive the form for isometric isomorphisms. If ψ is an isomorphism, as in Theorem 4.3, and in addition ψ is an isometry, then $\psi_{n,\tau(n)}$: $\mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$, defined by

$$\psi_{n,\tau(n)}(\lambda) = P_n\psi(E_{\tau(n)}(\lambda)) = \lambda \circ \varphi_{n,\tau(n)},$$

is also an isometry. In [**26**, Theorem 2.6.7], Weaver characterizes the surjective isometric isomorphisms in this scalar setting. Therefore, we have the following corollary.

Corollary 4.5. *Let X and Y be compact and connected metric spaces. If $\psi : \mathrm{Lip}_*(X, \mathbf{c}) \to \mathrm{Lip}_*(Y, \mathbf{c})$ is an isometric isomorphism, then there exist a homeomorphism $\tau : \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$ and a sequence of surjective isometries $\varphi_{n,\tau(n)} : Y \to X$ such that, for all $n \in \mathbf{N}$ and $y \in Y$,*

$$\psi(f)_n(y) = f_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

5. Homomorphisms on algebras of Lipschitz functions with values in ℓ_∞. In this section we consider the algebra of Lipschitz functions with values in ℓ_∞, and we derive a characterization for the algebra homomorphisms that preserve the subalgebra of the constant functions, $\mathrm{Const}_*(X, \mathcal{B})$. We also show that algebra isomorphisms between two Lipschitz spaces of ℓ_∞ valued functions preserve $\mathrm{Const}_*(X, \mathcal{B})$. The space ℓ_∞, contrary to $\mathbf{C}^n$ or $\mathbf{c}$, does not have a Schauder basis so previous techniques do not apply in this setting.

The space ℓ_∞ is isometric to the set of scalar valued and continuous functions defined on the Stone-Čech compactification of $\mathbf{N}$, denoted by $\beta\mathbf{N}$. We set some additional notation. Given $f \in \mathrm{Lip}_*(X, \ell_\infty)$, the function $f^\beta : X \to \mathcal{C}(\beta\mathbf{N})$ is such that $f^\beta(x)$ is the unique continuous extension of $f(x)$ to $\beta\mathbf{N}$.

We now start by proving the following preliminary lemma.

Lemma 5.1. *Let X be a compact metric space. Then $\Phi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N}))$, given by $\Phi(f)(x) = f^\beta(x)$, is an isometric algebra isomorphism.*

Proof. We first observe that f^β is a Lipschitz function on X. In fact, for $x, y \in X$, we have

$$\|f^\beta(x) - f^\beta(y)\|_\infty = |f^\beta(x)(\xi_0) - f^\beta(y)(\xi_0)|,$$

with $\xi_0 \in \beta\mathbf{N}$. Since $\mathbf{N}$ is dense in $\beta\mathbf{N}$, there exists a net $\{n_\alpha\}$ in $\mathbf{N}$ that converges to ξ_0 and

$$|f^\beta(x)(n_\alpha) - f^\beta(y)(n_\alpha)| \longrightarrow_\alpha |f^\beta(x)(\xi_0) - f^\beta(y)(\xi_0)|.$$

Therefore,

$$|f(x)(n_\alpha) - f(y)(n_\alpha)| \leq \|f(x) - f(y)\|_\infty \leq L(f)d(x,y),$$

and

$$|f^\beta(x)(\xi_0) - f^\beta(y)(\xi_0)| = \|f^\beta(x) - f^\beta(y)\|_\infty \leq L(f)\, d(x,y).$$

This shows that f^β is a function in $\mathrm{Lip}_*(Y, \mathcal{C}(\beta\mathbf{N}))$.

The linearity and multiplicative properties of Φ follow from the uniqueness of f^β. It is also easy to see that Φ is an isometry. We recall that $\|f\|_* = \|f\|_\infty + L(f)$. Since $\|f^\beta\|_\infty = |f^\beta(x_0)(\xi)|$ for some $x_0 \in X$ and $\xi \in \beta\mathbf{N}$, it follows that $\|f^\beta\|_\infty = \|f\|_\infty$. Similarly we show that $L(f) = L(f^\beta)$, which completes the proof. $\square$

A homomorphism $\psi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(Y, \ell_\infty)$ induces

$$\psi^\beta : \mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N})) \longrightarrow \mathrm{Lip}_*(Y, \mathcal{C}(\beta\mathbf{N}))$$

given by

$$\psi^\beta(f^\beta)(y) = \psi(f)^\beta(y), \quad \text{for all } y \in Y \text{ and } f^\beta \in \mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N})).$$

It is clear that ψ^β is an algebra homomorphism.

Proposition 5.2. *Let X and Y be compact and connected metric spaces. If $\psi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(Y, \ell_\infty)$ is an algebra isomorphism, then*

$$\psi(\mathrm{Const}_*(X, \ell_\infty)) \subseteq \mathrm{Const}_*(Y, \ell_\infty).$$

Before proving Proposition 5.2, we introduce additional notation and a preliminary lemma. We denote the set of all clopen subsets of $\beta\mathbf{N}$ by $\mathrm{clop}\,(\beta\mathbf{N})$. Given a clopen subset of $\beta\mathbf{N}$, A, we define χ_A, a function in $\mathcal{C}(\beta\mathbf{N})$ given by $\chi_A(\xi) = 1$ if $\xi \in A$ and $\chi_A(\xi) = 0$ if $\xi \notin A$. Then we set $\mathbf{X}^A$ to be the constant function in $\mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N}))$, everywhere equal to χ_A. Since ψ^β is multiplicative we have that

$$\psi^\beta(\mathbf{X}^A) = \psi^\beta(\mathbf{X}^A \cdot \mathbf{X}^A) = \psi^\beta(\mathbf{X}^A)^2.$$

This implies that, for $y \in Y$ and $\xi \in \mathbf{N}$, we have

$$\psi^\beta(\mathbf{X}^A)(y)(\xi) = 0, \quad \text{or } 1.$$

Therefore, if A is clopen in $\beta\mathbf{N}$, $y \in Y$ and $\xi \in \beta\mathbf{N}$,

$$\psi^\beta(\mathbf{X}^A)(y)(\xi) = 0 \quad \text{or } 1.$$

We then set

$$B = \{\xi \in \beta\mathbf{N} : \psi^\beta(\mathbf{X}^A)(y)(\xi) = 1\},$$

and $\mathbf{Y}^B = \psi^\beta(\mathbf{X}^A)(y)$. Since $\psi^\beta(\mathbf{X}^A)(y) \in \mathcal{C}(\beta\mathbf{N})$, it follows that B is clopen. For a fixed $y \in Y$, we define $\tau : \mathrm{clop}(\beta\mathbf{N}) \to \mathrm{clop}(\beta\mathbf{N})$ given by

$$\tau(A) = \{\xi \in \beta\mathbf{N} : \psi^\beta(\mathbf{X}^A)(y)(\xi) = 1\}.$$

Lemma 5.3. *If X is a compact metric space, Y is a compact and connected metric space, $y \in Y$, then τ satisfies the following properties*:

1. *τ is independent of y.*
2. *If $A_1 \subseteq A_2$ then $\tau(A_1) \subseteq \tau(A_2)$.*
3. *If $A_1 \cap A_2 = \varnothing$ then*

$$\tau(A_1) \cap \tau(A_2) = \varnothing \quad \textit{and} \quad \tau(A_1 \cup A_2) = \tau(A_1) \cup \tau(A_2).$$

Proof. We show that $\tau(A)$ is independent of y. Suppose that there exist y_0 and y_1 in Y such that $\psi^\beta(\mathbf{X}^A)(y_0) \neq \psi^\beta(\mathbf{X}^A)(y_1)$. We have

$$\|\psi^\beta(\mathbf{X}^A)(y_0) - \psi^\beta(\mathbf{X}^A)(y_1)\|_\infty = 1.$$

We define $h : Y \to \{0, 1\}$ given by

$$h(y) = \|\psi^\beta(\mathbf{X}^A)(y_0) - \psi^\beta(\mathbf{X}^A)(y)\|_\infty.$$

The continuity of h and the connectedness of Y implies that the range of h must be equal to $\{0\}$. We have shown that $\tau(A)$ is independent of y. This proves statement 1.

Statements in 2 and 3 follow from

$$\psi^\beta(\mathbf{X}^{A_1} \cdot \mathbf{X}^{A_2}) = \mathbf{Y}^{\tau(A_1)} \cdot \mathbf{Y}^{\tau(A_1)}$$

and

$$\psi^\beta(\mathbf{X}^{A_1} + \mathbf{X}^{A_2}) = \mathbf{Y}^{\tau(A_1)} + \mathbf{Y}^{\tau(A_2)} = \mathbf{Y}^{\tau(A_1)\cup\tau(A_2)},$$

whenever $A_1 \cap A_2 = \varnothing$. $\square$

Proof of Proposition 5.2. Previous considerations imply that $\psi^\beta(\mathbf{X}^A)$ $(A \in \operatorname{clop}(\beta\mathbf{N}))$ is a constant function in $\operatorname{Lip}_*(Y, \mathcal{C}(\beta\mathbf{N}))$, equal to $\mathbf{Y}^{\tau(A)}$, with $B = \tau(A)$. We notice that τ is a bijection on $\operatorname{clop}(\beta\mathbf{N})$ since ψ is an isomorphism.

We now consider a constant function f in $\operatorname{Lip}_*(X, \ell_\infty)$. It follows that f^β is also constant. Therefore,

$$\psi^\beta(\mathbf{X}^A \cdot f^\beta) = \psi^\beta(\mathbf{X}^A) \cdot \psi^\beta(f^\beta) = \mathbf{Y}^{\tau(A)} \cdot \psi^\beta(f^\beta).$$

For $y \in Y$ and $\xi \in \tau(A)$, we have $\psi^\beta(\mathbf{X}^A \cdot f^\beta)(y)(\xi) = \psi^\beta(f^\beta)(y)(\xi)$, if $\xi \notin \tau(A)$, then $\psi^\beta(\mathbf{X}^A \cdot f^\beta)(y)(\xi) = 0$. We first assume that A is a finite subset of $\mathbf{N}$, i.e., $A = \{n_1, \dots, n_k\}$, then $\mathbf{X}^A = \sum_{i=1}^k \mathbf{X}^{\{n_i\}}$. For simplicity of notation, we set $f^\beta(x)(n_i) = a_i$; thus,

$$\mathbf{X}^A \cdot f^\beta = \sum_{i=1}^k a_i \, \mathbf{X}^{\{n_i\}}$$

and

$$\psi^\beta(\mathbf{X}^A \cdot f^\beta) = \sum_{i=1}^k a_i \mathbf{Y}^{\tau(\{n_i\})}.$$

This implies that $\psi^\beta(\mathbf{X}^A \cdot f^\beta) \in \operatorname{Const}_*(Y, \mathcal{C}(\beta\mathbf{N}))$.

We now assume $\psi^\beta(f^\beta)$ is not constant. This implies that there exists $y_1 \neq y_2$ such that

$$\psi^\beta(f^\beta)(y_1) \neq \psi^\beta(f^\beta)(y_2).$$

For some $\xi \in \beta\mathbf{N}$, we have

$$\psi^\beta(f^\beta)(y_1)(\xi) \neq \psi^\beta(f^\beta)(y_2)(\xi).$$

The continuity of $\psi^\beta(f^\beta)$ implies the existence of a clopen set W containing ξ such that, for every $\eta \in W$,

$$\psi^\beta(f^\beta)(y_1)(\eta) \neq \psi^\beta(f^\beta)(y_2)(\eta). \tag{5.1}$$

Since ψ is an isomorphism then $(\psi^{-1})^\beta(\mathbf{Y}^W) = \mathbf{X}^U$, where $U \in \operatorname{clop}(\beta\mathbf{N})$. Therefore, U is either finite or its intersection with $\mathbf{N}$ is infinite; in either case, we can select a clopen subset A of $U \cap \mathbf{N}$, see [**25**, Proposition 3.10, page 74] (or [**28**]). We have $\psi^\beta(\mathbf{X}^A \cdot f^\beta)$ is a constant function. It follows from Lemma 5.3 that $\tau(A) \subseteq W$. Therefore, for $\eta \in \tau(A)$,

$$\psi^\beta(\mathbf{X}^A \cdot f^\beta)(y_1)(\eta) = \psi^\beta(f^\beta)(y_1)(\eta)$$

and

$$\psi^\beta(\mathbf{X}_A \cdot f^\beta)(y_2)(\eta) = \psi^\beta(f^\beta)(y_2)(\eta).$$

Inequality (5.1) leads to a contradiction and proves that $\psi^\beta(f^\beta)$ is constant. Hence, $\psi(f)$ is constant. $\square$

Remark 5.4. In general a homomorphism between spaces of Lipschitz functions does not necessarily preserve constant functions. For example, let $\psi : \mathrm{Lip}_*([0,\,1], \mathcal{B}) \to \mathrm{Lip}_*([0,\,1] \cup [2,\,3], \mathcal{B})$ be given by $\psi(f)(y) = f(y)$ if $y \in [0,\,1]$ and $\psi(f)(y) = 0$ if $y \in [2,\,3]$.

We now characterize algebra homomorphisms between algebras of Lipschitz functions with values in ℓ_∞ that preserve the subalgebra of constant functions. As before, X and Y are compact metric spaces. We denote by ψ an algebra homomorphism from $\mathrm{Lip}_*(X, \ell_\infty)$ into $\mathrm{Lip}_*(Y, \ell_\infty)$ such that

$$\psi(\mathrm{Const}_*(X,\, \mathcal{C}(\beta\mathbf{N}))) \subseteq \mathrm{Const}_*(Y, \mathcal{C}(\beta\mathbf{N})).$$

Therefore, ψ^β induces a homomorphism from $\mathrm{Const}_*(X,\, \mathcal{C}(\beta\mathbf{N}))$ into $\mathrm{Const}_*(Y,\, \mathcal{C}(\beta\mathbf{N}))$.

We use the isometric isomorphism between the $\mathrm{Const}_*(X, \mathcal{C}(\beta\mathbf{N}))$ and $\mathcal{C}(\beta\mathbf{N})$ to define ψ_c^β to be the natural homomorphism on $\mathcal{C}(\beta\mathbf{N})$ determined by ψ. Theorem 1.1 implies that ψ_c^β is a composition operator. More precisely, there exists a continuous map $\tau : \beta\mathbf{N} \to \beta\mathbf{N}$ such that, for every $\alpha \in \mathcal{C}(\beta\mathbf{N})$, $\psi_c^\beta(\alpha) = \alpha \circ \tau$. Given A, a clopen subset of $\beta\mathbf{N}$, we set $\psi_c^\beta(\mathbf{X}^A) = \mathbf{Y}^B$ with $B = \{\xi \in \beta\mathbf{N} : \tau(\xi) \in A\}$. Since $\psi^\beta(\lambda \cdot \mathbf{X}^A \cdot \mathbf{X}^A) = \psi^\beta(\lambda \cdot \mathbf{X}^A) = \psi^\beta(\lambda \cdot \mathbf{X}^A) \cdot \mathbf{Y}^B$, we conclude that

$$\psi^\beta(\lambda \cdot \mathbf{X}_A^\beta)(y)(\xi) = 0, \quad \text{for every } y \in Y \text{ and } \xi \notin B.$$

We consider the A-embedding of $\mathrm{Lip}_*(X)$ into $\mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N}))$ given by

$$\begin{aligned} E_A : \mathrm{Lip}_*(X) &\longrightarrow \mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N})) \\ \lambda &\longrightarrow \lambda \cdot \mathbf{X}^A \end{aligned}$$

Given $n \in \mathbf{N}$ such that $\tau(n) \in A$, we set $\psi_{(n,A)} : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$, given by

$$\psi_{(n,A)} = P_n \psi^\beta E_A.$$

Theorem 1.2 implies the existence of a Lipschitz map $\varphi_{(n,A)} : Y \to X$ such that, for every $\lambda \in \mathrm{Lip}_*(X)$,

$$\psi_{(n,A)}(\lambda) = \lambda \circ \varphi_{(n,A)}.$$

The following lemma is an easy consequence of [**25**, Proposition 1.17].

Lemma 5.5. *If $n \in \mathbf{N}$, then $\mathcal{A}_n = \{A \in \mathrm{clop}\,(\beta\mathbf{N}) : \tau(n) \in A\}$ is an ultrafilter over the ring of clopen subsets of $\beta\mathbf{N}$.*

In a compact Hausdorff space, every ultrafilter converges to a single point, [**27**, Theorem 7.3.6]. Lemma 5.5 implies that $\tau(n) = \lim_n \mathcal{A}_n$. We now show that, given A_1 and A_2 in $\mathcal{A}_n$, then the associated Lipschitz maps are equal, i.e., $\varphi_{(n,A_1)} = \varphi_{(n,A_2)}$.

Lemma 5.6. *If $n \in \mathbf{N}$, A_1 and A_2 are clopen subsets of $\beta\mathbf{N}$ such that $\tau(n) \in A_1 \cap A_2$. Then $\varphi_{(n,A_1)} = \varphi_{(n,A_2)}$.*

Proof. We first assume that $A_1 \subseteq A_2$ and $\tau(n) \in A_1$. If there exists y_0 such that $\varphi_{(n,A_1)}(y_0) \neq \varphi_{(n,A_2)}(y_0)$, we set $\lambda_0(z) = d(\varphi_{(n,A_2)}(y_0), z)$, with d the distance on X and $z \in X$. Then we have that $\lambda_0 \in \mathrm{Lip}_*(X)$. The multiplicative property of ψ implies that

$$\psi_{(n,A_1)}(\lambda) = \psi_{(n,A_2)}(\lambda) \cdot \psi_{(n,A_1)}(\lambda),\ \lambda \in Lip_*(X).$$

Therefore,

$$\lambda \circ \varphi_{(n,A_1)} = \lambda \circ \varphi_{(n,A_2)} \cdot \lambda \circ \varphi_{(n,A_1)}.$$

This implies that, for every $y \in Y$ for which $\lambda \circ \varphi_{(n,A_1)}(y) \neq 0$, we have $\lambda \circ \varphi_{(n,A_2)}(y) = 1$. However, $\lambda_0 \circ \varphi_{(n,A_1)}(y_0) \neq 0$ and $\lambda_0 \circ \varphi_{(n,A_2)}(y_0) = 0$. This contradiction shows that $\varphi_{(n,A_1)} = \varphi_{(n,A_2)}$ whenever $A_1 \subseteq A_2$. In general, given A_1 and A_2 clopen subsets of $\beta\mathbf{N}$ such that $\tau(n) \in A_1 \cap A_2$, we have that $\varphi_{(n,A_1)} = \varphi_{(n,A_1 \cap A_2)}$ and $\varphi_{(n,A_2)} = \varphi_{(n,A_1 \cap A_2)}$. This completes the proof. ◻

Lemma 5.6 allows us to define the sequence of Lipschitz maps $\{\varphi_{n,\tau(n)}\}_n$

$$\varphi_{n,\tau(n)} = \varphi_{(n,A)},$$

for any clopen set $A \in \mathcal{A}_n$.

Theorem 5.7. *Let X and Y be compact and connected metric spaces. If $\psi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(Y, \ell_\infty)$ is a homomorphism that preserves constant functions, then there exist a continuous map $\tau : \beta\mathbf{N} \to \beta\mathbf{N}$ and a sequence of Lipschitz maps $\varphi_{n,\tau(n)} : Y \to X$ such that, for every $n \in \mathbf{N}$ and $y \in Y$,*

$$\psi(f)_n(y) = f^{\beta}_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

If ψ is an isomorphism, then there exist a homeomorphism $\tau : \beta\mathbf{N} \to \beta\mathbf{N}$ and a sequence of lipeomorphisms $\varphi_{n,\tau(n)} : Y \to X$ such that, for every $n \in \mathbf{N}$ and $y \in Y$,

$$\psi(f)_n(y) = f^{\beta}_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

Proof. Since ψ preserves constants, ψ^{β} induces a linear and multiplicative map on $\mathcal{C}(\beta\mathbf{N})$. Theorem 1.1 asserts the existence of

$\tau : \beta\mathbf{N} \to \beta\mathbf{N}$. Lemmas 5.5 and 5.6 assert the existence of Lipschitz maps $\varphi_{n,\tau(n)} : Y \to X$ such that

$$\psi^\beta(\lambda \mathbf{X}^A)_n(y) = \begin{cases} 0 & \text{if } \tau(n) \notin A \\ \lambda \circ \varphi_{n,\tau(n)}(y) & \text{if } \tau(n) \in A. \end{cases}$$

Given $f \in \mathrm{Lip}_*(X, \ell_\infty)$, we denote by f^β the corresponding function in $\mathrm{Lip}_*(X, \mathcal{C}(\beta\mathbf{N}))$, f^β is clearly continuous relative to $\|\cdot\|_\infty$. It is shown in [**5**, page 224], also [**19**, Theorem 1.13], that the tensor product space $\mathcal{C}(X) \otimes \mathcal{C}(\beta\mathbf{N})$, with the least crossnorm, is dense in $\mathcal{C}(X, \mathcal{C}(\beta\mathbf{N}))$, the space of all continuous $\mathcal{C}(\beta\mathbf{N})-$valued functions defined on X and equipped with $\|\cdot\|_\infty$. Therefore, there exists a sequence $\{F_n\}$ in $\mathcal{C}(X) \otimes \mathcal{C}(\beta\mathbf{N})$ that converges uniformly to f^β. We identify the space $\mathcal{C}(X) \otimes \mathcal{C}(\beta\mathbf{N})$ with all the functions of the form $\sum_{i=1}^m \lambda_i\, \alpha_i$ with $\lambda_i \in \mathcal{C}(X)$ and $\alpha_i \in \mathcal{C}(\beta\mathbf{N})$. Each function F_n is represented as follows:

$$F_n(x) = \sum_{i=1}^{k_n} \lambda_i^n(x) \alpha_i^n,$$

with $\lambda_i^n \in \mathcal{C}(X)$ and $\alpha_i^n \in \mathcal{C}(\beta\mathbf{N})$. Without loss of generality, we may assume that λ_i^n are Lipschitz functions, (see [**13**, Theorem 6.8]). On the other hand, the following class of characteristic functions in $\mathcal{C}(\beta\mathbf{N})$

$$\mathcal{A} = \{\chi_A \in \mathcal{C}(\beta\mathbf{N}) : A \text{ is a clopen subset of } \beta\mathbf{N}\}$$

(we recall that $\chi_A(\xi) = 1$ if $\xi \in A$ and zero otherwise) separates points. An application of the Stone-Weierstrass theorem (cf. [**7**]) asserts that the algebra generated by $\mathcal{A}$ is dense in $\mathcal{C}(\beta\mathbf{N})$. Consequently, we consider a sequence $\{F_n^\beta\}_n$ converging uniformly to f^β with F_n^β given by

$$F_n^\beta(x) = \sum_{i=1}^{k_n} \lambda_i^n(x) \mathbf{X}^{A_{i,n}}(x),$$

with $\lambda_i^n \in \mathrm{Lip}_*(X)$ and $A_{i,n}$ clopen subsets of $\beta\mathbf{N}$.

The continuity of ψ^β allows us to conclude that

$$\begin{aligned} \psi^\beta(f^\beta)_n(y) &= \psi(f)_n(y) \\ &= \lim_{j\to\infty} \sum_{i=1}^{k_j} \lambda_i^j(\varphi_{n,\tau(n)}(y)) \cdot \mathbf{X}^{A_{i,j}}(\varphi_{n,\tau(n)}(y)) \end{aligned}$$

for every $y \in Y$ and $f \in \mathrm{Lip}_*(X, \ell_\infty)$. Therefore, we have

$$\psi(f)_n(y) = \lim_{j\to\infty} \sum_{i=1}^{k_j} \left(\lambda_i^j \cdot \mathbf{X}^{A_{i,j}}\right) (\varphi_{n,\tau(n)}(y)).$$

If ψ is an isomorphism, then Theorems 1.1 and 1.2 imply the existence of a sequence of lipeomorphisms $\varphi_{n,\tau(n)} : Y \to X$ and a homeomorphism τ. Proposition 5.2 asserts that ψ preserves constant functions. The result now follows from the first statement. This completes the proof. □

Theorem 5.7 gives a procedure to construct examples of homomorphisms between spaces of ℓ_∞-valued Lipschitz functions. We now give an illustrative example.

Example 5.8. We set $X = Y = [0, 1]$ and select a sequence of elements in the growth of $\mathbf{N}$ (i.e., in $\beta\mathbf{N} \setminus \mathbf{N}$). We represent this sequence by $\{\xi_n\}_n$. We now set $\tau(n) = \xi_n$, for $n \in \mathbf{N}$, and extend τ continuously to $\beta\mathbf{N}$. We also assign $\varphi_{(n,\tau(n))} = \mathrm{Id}_{[0,\,1]}$. Given $f \in \mathrm{Lip}_*(X, \ell_\infty)$ we have

$$f^\beta(x) = \{f_\xi^\beta(x)\}_{\xi\in\beta\mathbf{N}}.$$

We define ψ as follows:

$$\psi(f)(y) = (f_{\xi_1}^\beta(y),\, f_{\xi_2}^\beta(y), \dots, f_{\xi_n}^\beta(y), \dots\}.$$

The map ψ is a homomorphism on $\mathrm{Lip}_*([0,\,1], \ell_\infty)$.

As for the case of $\mathbf{C}$ valued Lipschitz functions, [**26**, Theorem 2.6.7] implies the following characterization of isometric isomorphisms.

Corollary 5.9. *Let X and Y be compact and connected metric spaces. If $\psi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(Y, \ell_\infty)$ is an isometric isomorphism, then there exist a homeomorphism map $\tau : \beta\mathbf{N} \to \beta\mathbf{N}$ and a sequence of surjective isometries $\varphi_{n,\tau(n)} : Y \to X$ such that, for every $n \in \mathbf{N}$ and $y \in Y$,*

$$\psi(f)_n(y) = f_{\tau(n)}^\beta \circ \varphi_{n,\tau(n)}(y).$$

6. Final remarks. Kamowitz and Scheinberg, in [**18**], gave a characterization of compact composition operators between spaces of scalar valued Lipschitz functions. We consider compact metric spaces (X, d) and (Y, D) and a Lipschitz function $\varphi : Y \to X$, the composition operator determined by φ, $\psi : \mathrm{Lip}_*(X, d) \to \mathrm{Lip}_*(Y, D)$ given by $\psi(f) = f \circ \varphi$ is compact if and only φ is super-contractive, i.e.,

$$\lim_{d(x,y)\to 0} \frac{d(\varphi(x), \varphi(y))}{d(x,\, y)} = 0.$$

We use this result to derive the form of compact homomorphisms between algebras of $\mathcal{B}$ valued Lipschitz functions.

Theorem 6.1. *Let X be a compact metric space and Y a compact and connected metric space. If $\psi : \mathrm{Lip}_*(X, \mathbf{c}) \to \mathrm{Lip}_*(Y, \mathbf{c})$ is a compact homomorphism such that $\psi(1_X) = 1_Y$, then there exist a continuous map $\tau : \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$ and super-contractive Lipschitz maps $\varphi_{n,\tau(n)} : Y \to X$ ($n \in \widehat{\mathbf{N}}$) such that, for all $n \in \mathbf{N}$ and $y \in Y$*

$$\psi(f)_n(y) = f_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

Proof. We assume that ψ is compact. For a fixed $n \in \widehat{\mathbf{N}}$, $\psi_{n,\tau(n)} : \mathrm{Lip}_*(X) \to \mathrm{Lip}_*(Y)$ is given by

$$\psi_{n,\tau(n)}(\lambda) = P_n \psi E_{\tau(n)}(\lambda) = \lambda \circ \varphi_{n,\tau(n)},$$

following Theorem 4.3. We also recall that P_n is a projection into the n position and $E_{\tau(n)}$ is the embedding into the $\tau(n)$ position, as described in Remark 2.2. Therefore $\psi_{n,\tau(n)}$ is compact and Theorem 1 in [**18**] implies that $\varphi_{n,\tau(n)}$ is super-contractive. $\square$

Similarly, the following theorem is a consequence of Theorem 5.7 and Theorem 1 in [**18**].

Theorem 6.2. *Let X be a compact metric space and Y a compact and connected metric space. If $\psi : \mathrm{Lip}_*(X, \ell_\infty) \to \mathrm{Lip}_*(Y, \ell_\infty)$ is a* *compact homomorphism that preserves constant functions, then there exist a continuous map $\tau : \beta\mathbf{N} \to \beta\mathbf{N}$ and super-contractive maps $\varphi_{\xi,\tau(\xi)} : Y \to X$ ($\xi \in \beta\mathbf{N}$) such that, for every $n \in \mathbf{N}$ and $y \in Y$,*

$$\psi(f)_n(y) = f^{\beta}_{\tau(n)} \circ \varphi_{n,\tau(n)}(y).$$

6.1. Lipschitz algebras without unit. The spaces $\mathbf{c}_0$ and ℓ_p with $1 \leq p < \infty$ are Banach algebras without unit. Following a construction in [**22**], we can isometrically embed each space into an algebra with unit, $\mathbf{c}_0 \oplus \mathbf{C}$ and $\ell_p \oplus \mathbf{C}$, respectively. The multiplication is then defined by

$$(x, \lambda_1) \cdot (y, \lambda_2) = (\lambda_2 x + \lambda_1 y + xy, \lambda_1\lambda_2)$$

and norm

$$\|(x, \lambda)\| = \|x\|_\infty + |\lambda|.$$

It is easy to see that $(\{0\}_n, 1)$ is the unit and the spaces $\mathbf{c}_0$, ℓ_p are isometrically isomorphic to $\mathbf{c}_0 \oplus \{0\}$ and $\ell_p \oplus \{0\}$, respectively. A similar argument to the one presented for the proof of Corollary 3.4 shows that $\mathbf{c}_0 \oplus \mathbf{C}$ and $\ell_p \oplus \mathbf{C}$ are semi-simple algebras. Corollary 2.3.7 in [**22**] says that every subalgebra of a semi-simple algebra is semi-simple. This implies that $\mathbf{c}_0$ and ℓ_p are semi-simple and therefore Proposition 3.3 implies that $\mathrm{Lip}_*(X, \mathbf{c}_0)$ and $\mathrm{Lip}_*(X, \ell_p)$ are also semi-simple.

In addition, a homomorphism from $\mathrm{Lip}_*(X, \mathbf{c}_0)$ into $\mathrm{Lip}_*(X, \mathbf{c}_0)$, with X a compact and connected metric space, is lifted in a natural way to a homomorphism $\widetilde{\psi}$ from $\mathrm{Lip}_*(X, \mathbf{c}_0 \oplus \mathbf{C})$ into $\mathrm{Lip}_*(X, \mathbf{c}_0 \oplus \mathbf{C})$:

$$\widetilde{\psi}(\widetilde{f})(x) = (\psi(\widetilde{f}_1)(x), \widetilde{f}_2(x)), \quad \text{for all } x \in X.$$

It is straightforward to check that $\widetilde{\psi}$ is a homomorphism that extends ψ. Most of the results of the previous sections hold for a homomorphism $\widetilde{\psi}$ on $\mathrm{Lip}_*(X, \mathbf{c}_0 \oplus \mathbf{C})$ and therefore for ψ on $\mathrm{Lip}_*(X, \mathbf{c}_0)$. Similar considerations also apply to ℓ_p with $1 \leq p < \infty$.

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approach which uses an isomorphism between $\mathrm{Lip}_*(X, \ell_\infty)$ and a Lipschitz space of scalar valued functions defined on a noncompact metric space.

REFERENCES

1. B. Aupetit, *A primer on spectral theory*, Springer Verlag New York, 1991.

2. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer Verlag, Berlin, 1970.

3. F. Botelho and J. Jamison, *Homomorphisms on algebras of Lipschitz functions*, Stud. Math. **199** (2010), 95–106.

4. M. Day, *Normed linear spaces*, Third edition, Ergeb. Math. Grenzg. **21** (1973), Springer Verlag, New York.

5. J. Diestel and J. Uhl, *Vector measures*, Math. Surv. **15**, Mathematical Surveys, American Mathematical Society, Providence, RI, 1977.

6. R. Douglas, *Banach algebras techniques in operator theory*, Academic Press, New York, 1972.

7. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1970.

8. N. Dunford and J. Schwartz, *Linear operators. Part* I: *General topology*, Interscience Publishers, Inc., New York, 1957.

9. R. Fleming and J. Jamison, *Hermitian operators on $C(X, E)$ and the Banach-Stone theorem*, Math. Z. **170** (1980), 77–84.

10. A. Friedman, *Foundation of modern analysis*, Dover Publications, Inc., New York, 1982.

11. I. Gelfand and A. Kolmogoroff, *On rings of continuous functions on a topological space*, Dokl. Akad. Nauk **22** (1939), 11–15.

12. J.B. Gonzàlez and J.R. Ramírez, *Homomorphisms on Lipschitz spaces*, Monats. Math. **129** (2000), 25–30.

13. J. Heinonen, *Lectures on analysis on metric spaces*, Springer Verlag, New York, 2001.

14. B. Johnson, *The uniqueness of the* (*complete*) *norm topology*, AMS Bull. **73** (1967), 537–539.

15. R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras–Elementary theory*, Vol. I, Academic Press, Inc., London, 1983.

16. H. Kamowitz and S. Scheinberg, *Some properties of endomorphisms of Lipschitz algebras*, Stud. Math. **96** (1990), 255–261.

17. I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, Reg. Conf. Ser. Math., American Mathematical Society, Providence, RI, 1970.

18. I. Kaplansky, *Fields and rings*, Chicago Lect. Math., Chicago University Press, Chicago, 1969.

19. W. Light and E. Cheney, *Approximation theory in tensor product spaces*, Lect. Notes Math. **1169**, Springer Verlag, New York, 1980.

20. L. Molnár, *A reflexivity problem concerning the C^*-algebra $C(X)\bigotimes\mathcal{B}(\mathcal{H})$*, Proc. Amer. Math. Soc. **129** (2002), 531–537.

21. ———, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Lect. Notes Math. **1895**, Springer Verlag, Berlin, 2007.

22. C. Rickart, *General theory of Banach algebras*, D. Van Nostrand Company, Inc., New York, 1960.

23. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

24. D. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math. **13** (1963), 1387–1399.

25. R. Walker, *The Stone-Čech compactification*, Springer Verlag, Berlin, 1974.

26. N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co., Singapore, 1999.

27. A. Wilansky, *Functional analysis*, Blaisdell Publishing Company, New York, 1964.

28. ———, *Topology for analysis*, Robert E. Krieger Publishing Company, Malabar, Florida, 1970.

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