ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 4, 2012

ALTERNATING SUBSETS MODULO m

TOUFIK MANSOUR AND AUGUSTINE O. MUNAGI

ABSTRACT. We enumerate increasing combinations of $\{1, 2, \dots, n\}$ according to parity statistics defined on pairs of adjacent elements. Generating functions are used to devise a framework that addresses all questions on adjacencies of parities with respect to any modulus $m > 1$. In particular, we give a generalization of a classical result on alternating subsets which was previously known for the modulus 2. We also compute some generating functions for the number of combinations possessing special adjacent parity patterns.

1. Introduction. Enumeration of alternating parity sequences (also known as "alternating subsets") of integers have received considerable attention in the literature. These are finite integer sequences

$$(c_1, c_2, \dots, c_k), \quad c_1 < c_2 < \cdots < c_k,$$

that fulfill the condition

$$c_i \not\equiv c_{i-1} \pmod 2, \quad i > 1. \tag{1}$$

The empty sequence and the 1-term sequence are also alternating sequences by convention.

The number $h(n, k)$ of alternating k-subsets of $\{1, 2, \dots, n\}$ is well known (see for example [**2, 12**]):

$$h(n,k) = \binom{\lfloor (n+k)/2 \rfloor}{k} + \binom{\lfloor (n+k-1)/2 \rfloor}{k}, \tag{2}$$

where $\lfloor N \rfloor$ denotes the greatest integer $\leq N$. Moreover, $\sum_{k>0} h(n,k) = F_{n+3} - 2$, where F_n is the nth Fibonacci number ($F_n = F_{n-1} + F_{n-2}$

2010 AMS *Mathematics subject classification.* Primary 05A05, 11B50, 05A15.
Keywords and phrases. Alternating subset, parity statistic, Fibonacci number, generating function.
Received by the editors on May 29, 2009, and in revised form on January 5, 2010.
DOI:10.1216/RMJ-2012-42-4-1313

with $F_0 = 0$ and $F_1 = 1$). We will adopt the notation $[n] = \{1, 2, \dots, n\}$.

Equation (2) has been extended to the enumerator of (α, β)-alternating subsets, that is, combinations consisting of a sequence of blocks of lengths $\alpha, \beta, \alpha, \beta, \dots$, in which the first α elements have the same parity, the next β elements have opposite parity, and so on [**9, 10**].

A famous specialization of (2) relates to the problem of Terquem (see [**10**, page 17]) which asks for the number of k-combinations of $[n]$ with odd elements in odd positions and even elements in even positions. This was generalized by Skolem (see [**8**, pages 313–314]) as follows: find the number of k-combinations in which the jth element is congruent to j modulo m, where $m > 1$ is a fixed integer. Church and Gold [**3**] obtained a proof of Skolem's generalization by counting lattice paths in a rectangular array.

Abramson and Moser [**1**] obtained a further generalization by finding a formula for the number of combinations $(x_1, \dots, x_k)$ of $[n]$ satisfying

(3)
$$c_1 \equiv 1+m_1 \pmod{m}, \qquad c_j \equiv c_{j-1}+1+m_j \pmod{m}, \quad 1 \le j \le k,$$

where $m_1, m_2, \dots, m_k$, $0 \le m_j \le m-1$, is a fixed sequence of integers. Goulden and Jackson [**5**] also discovered a generating function solution to the problem characterized by (3). Recently, Munagi [**10**] found a simple solution by studying the cardinalities of residue classes of individual elements.

However, there was no known *direct* generalization of equation (1) to a higher modulus; the following natural question appears to have been omitted.

Given an integer $m > 1$, how many combinations $(c_1, \dots, c_k)$, $k \ge 1$, of $[n]$ fulfill the condition:

$$c_i \not\equiv c_{i-1} \pmod{m}, \quad \text{for all } i > 1? \tag{4}$$

A combination satisfying (4) will also be called mod-m alternating. We will obtain the generating function for the number of mod-m alternating combinations of $[n]$ as a corollary of a more general theorem. We also prove, by elementary combinatorial argument (see Section 4), that mod-m alternating combinations are counted by partial sums of the m-generalized Fibonacci numbers $F_n^{(m)}$ (also known as Fibonacci m-step numbers), defined for any positive integer, m, by,

$$
\begin{aligned}
&F_n^{(m)} = 0, \quad \text{if } n \le 0,\\
&F_1^{(m)} = F_2^{(m)} = 1,\ F_3^{(m)} = 2,\\
&F_n^{(m)} = \sum_{i=1}^{m} f_{n-i}^{(m)}, \quad n \ge 3.
\end{aligned}
\tag{5}
$$

In the sequel we treat a combination as a word over the alphabet $[n]$ in which the letters are strictly increasing from left to right and then define parity statistics on pairs of adjacent letters c_i, c_{i+1}. So the word notation $c_1 c_2 \cdots c_k$ will also be used to represent the combination $(c_1, \ldots, c_k)$. Note that, in classical combinatorial language (see, for example, [**4**]), the pair $c_i c_{i+1}$ is a *rise* since $c_i < c_{i+1}$ for all i.

We say that a rise $c_i c_{i+1}$ is endowed with the statistic

$$R_{\alpha\beta}$$

if and only if

$$c_i \equiv \alpha \pmod{m} \quad \text{and} \quad c_{i+1} \equiv \beta \pmod{m}.$$

Our first objective is to enumerate combinations of $[n]$ according to the statistics $R_{\alpha\beta}$. We denote the number of k-combinations $(c_1, c_2, \ldots, c_k)$ of $[n]$ such that the number of rises $c_i c_{i+1}$ with $c_i \equiv \alpha \pmod{m}$, $c_{i+1} \equiv \beta \pmod{m}$ is equal to $R_{\alpha\beta}$ by

$$C(n, k, \mathbf{R}) = C(n, k, R_{00}, \ldots, R_{0(m-1)}, \ldots, R_{(m-1)0}, \ldots, R_{(m-1)(m-1)}),$$

where $\mathbf{R} = R_{00}, \ldots, R_{0(m-1)}, \ldots, R_{(m-1)0}, \ldots, R_{(m-1)(m-1)}$.

In Section 2 we obtain an expression for the generating function of $C(n, k, \mathbf{R})$. Sections 3 and 4 are devoted to a few interesting applications, in the form of specializations, of the general theorem. In particular, we state enumerative generating functions for combinations avoiding certain adjacent parity conditions.

2. A general theorem. In the following generating function construction, our statistics are marked by indeterminates $q_{\alpha\beta}$, the number n, representing the alphabet $[n]$, is marked by x, and the length of a word by y. Hence, we can write the corresponding generating function for $C(n,k,\mathbf{R})$ as

$$C(x,y,\mathbf{q}) = \sum_{\substack{n\geq 0\\ k\geq 0\\ \sharp(R_{\alpha\beta})\geq 0,\\ \alpha,\beta\in\{0,1,\dots,m-1\}}} x^n y^k \prod_{\alpha=0}^{m-1}\prod_{\beta=0}^{m-1} q_{\alpha\beta}^{\sharp(R_{\alpha\beta})}, \tag{6}$$

where $\mathbf{q} = q_{00},\dots,q_{0(m-1)},\dots,q_{(m-1)0},\dots,q_{(m-1)(m-1)}$.

Let $C(n;y,\mathbf{q} \mid a_1\cdots a_r)$ denote the generating function for the number of nonempty k-combinations $c_1c_2\cdots c_{k-r}a_1\cdots a_r$ of $[n]$ (i.e., each combination ends at the subword $a_1\cdots a_r$). Clearly, $C(n;y,\mathbf{q} \mid a) = C(a;y,\mathbf{q} \mid a)$ for all $1\leq a\leq n$. We define

$$C_\beta = C_\beta(x,y,\mathbf{q}) = \sum_{a\equiv\beta \pmod m,\ a\neq 0} C(a;y,\mathbf{q} \mid a)x^a.$$

Lemma 2.1. *The generating function $C(x,y,\mathbf{q})$ is given by*

$$C(x,y,\mathbf{q}) = \frac{1}{1-x}\left(1+\sum_{\beta=0}^{m-1} C_\beta(x,y,\mathbf{q})\right).$$

Proof. Define $C(0;y,\mathbf{q} \mid a)$ to be 1 if $a=0$ and 0 otherwise. Then from the definitions we have

$$\begin{aligned} C(x,y,\mathbf{q}) &= \sum_{n\geq 0}\sum_{a=0}^{n} C(n;y,\mathbf{q} \mid a)x^n = \sum_{n\geq 0}\sum_{a=0}^{n} C(a;y,\mathbf{q} \mid a)x^n\\ &= \frac{1}{1-x}\sum_{a\geq 0} C(a;y,\mathbf{q} \mid a)x^a\\ &= \frac{1}{1-x}\left(1+\sum_{\beta=0}^{m-1} C_\beta(x,y,\mathbf{q})\right), \end{aligned}$$

as claimed. ∎

Lemma 2.2. *For any $\beta = 0, 1, \dots, m-1$,*

$$C_\beta = \frac{y}{1-x^m}\left[x^{e(\beta)} + \sum_{\alpha=0}^{m-1} q_{\alpha\beta} x^{e(\beta-\alpha \pmod m)} C_\alpha\right],$$

where $e(-i) = m - i$ and $e(i) = i$ for $i > 0$, and $e(0) = m$.

Proof. From the definitions, we have

$$\begin{aligned}
C_\beta &= \sum_{\substack{a \equiv \beta \pmod m \\ a \neq 0}} C(a; y, \mathbf{q} \mid a) x^a \\
&= \frac{x^{e(\beta)} y}{1-x^m} + \sum_{\substack{a \equiv \beta \pmod m \\ a \neq 0}} \sum_{b=1}^{a} C(a; y, \mathbf{q} \mid ba) x^a \\
&= \frac{x^{e(\beta)} y}{1-x^m} + \sum_{\alpha=0}^{m-1} \sum_{\substack{a \equiv \beta \pmod m \\ a \neq 0}} \sum_{\substack{b=1 \\ b \equiv \alpha}} \pmod m)^a C(a; y, \mathbf{q} \mid ba) x^a \\
&= \frac{x^{e(\beta)} y}{1-x^m} + y \sum_{\alpha=0}^{m-1} q_{\alpha\beta} \sum_{\substack{a \equiv \beta \pmod m \\ a \neq 0}} \sum_{\substack{b=1 \\ b \equiv \alpha \pmod m}} C(b; y, \mathbf{q} \mid b) x^a \\
&= \frac{x^{e(\beta)} y}{1-x^m} + y \sum_{\alpha=0}^{m-1} q_{\alpha\beta} \frac{x^{e(\beta-\alpha \pmod m)}}{1-x^m} \sum_{\substack{b \equiv \alpha \pmod m \\ b \neq 0}} C(b; y, \mathbf{q} \mid b) x^b \\
&= \frac{x^{e(\beta)} y}{1-x^m} + y \sum_{\alpha=0}^{m-1} q_{\alpha\beta} \frac{x^{e(\beta-\alpha \pmod m)}}{1-x^m} C_\alpha,
\end{aligned}$$

as required. □

Define the following $m \times m$ matrix

$$\mathbf{B}_m = \frac{1}{1-x^m} \times \begin{pmatrix} q_{00}x^m & q_{01}x^1 & q_{02}x^2 & \cdots & q_{0(m-1)}x^{m-1} \\ q_{10}x^{m-1} & q_{11}x^m & q_{12}x^1 & \cdots & q_{1(m-1)}x^{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ q_{(m-1)0}x^1 & q_{(m-1)1}x^2 & q_{(m-1)2}x^3 & \cdots & q_{(m-1)(m-1)}x^m \end{pmatrix},$$

that is, the (i,j) term of the matrix $\mathbf{B}_m$ is given by $[q_{\alpha\beta}x^{e(m+\beta-\alpha(\mathrm{mod}\, m))}]/(1-x^m)$. Let $\mathbf{A}_m = \mathbf{I}_m - y\mathbf{B}_m$, where I_m is the unit $m \times m$ matrix. Then, Lemmas 2.1 and 2.2 give the following result.

Theorem 2.3. *The generating function* $C(x,y,\mathbf{q})$ *is given by*

$$C(x,y,\mathbf{q}) = \frac{1}{1-x}\left(1 + \sum_{\beta=0}^{m-1} C_\beta(x,y,\mathbf{q})\right),$$

where

$$(C_0, C_1, \dots, C_{m-1})^T = (\mathbf{A}_m)^{-1}\left(\frac{x^m y}{1-x^m}, \frac{xy}{1-x^m}, \dots, \frac{x^{m-1}y}{1-x^m}\right)^T.$$

This theorem embodies extensive specializations as we indicate in the next two sections.

3. Combinations avoiding adjacent parity statistics. In this section we apply the general result stated in Section 2 to count combinations of $[n]$ with restricted adjacent parity statistics. For example, when $n=2$, the generating function for combinations $c_1c_2\cdots c_k$ of $[n]$ avoiding the statistic R_{01} (i.e., avoiding rises c_ic_{i+1} of type even-odd) is given by $C(x,y,\mathbf{q}) = C(x,y,1,0,1,1)$. Other variations are similarly interpreted. We give a detailed account of only the cases $m=2$ and $m=3$ in this section.

3.1. The case $m=2$. Theorem 2.3 for $m=2$ gives

$$C_0 = \frac{(1-x^2+y(q_{10}-q_{11}x^2))yx^2}{(1-x^2)^2 - x^2(1-x^2)(q_{00}+q_{11})y + x^2(q_{00}q_{11}x^2 - q_{01}q_{10})y^2}$$

$$C_1 = \frac{(1-x^2-yx^2(q_{00}-q_{01}))xy}{(1-x^2)^2 - x^2(1-x^2)(q_{00}+q_{11})y + x^2(q_{00}q_{11}x^2 - q_{01}q_{10})y^2},$$

which implies that

$$(7)\quad C(x,y,\mathbf{q}) = \frac{1}{1-x}\left(1 + \frac{(1-x^2)(1+x)xy - x^2(q_{11}x^2 - q_{01}x + q_{00} - q_{10})y^2}{(1-x^2)^2 - x^2(1-x^2)(q_{00}+q_{11})y + x^2(q_{00}q_{11}x^2 - q_{01}q_{10})y^2}\right).$$

TABLE 1. Generating functions for the number of combinations of n satisfying $R_{\alpha\beta} = 0$ with $\alpha\beta = 00, 01, 10, 11$.

$\mathbf{q}$	$C(x, 1, \mathbf{q})$
$(0, 1, 1, 1)$	$\frac{1+2x}{1-4x^2+2x^4}$
$(1, 0, 1, 1)$	$\frac{1+2x-2x^3}{(1-2x^2)^2}$
$(1, 1, 0, 1)$	$\frac{(1-x^2)(1+2x)}{(1-2x^2)^2}$
$(1, 1, 1, 0)$	$\frac{(1+x)(1+x-x^2)}{1-4x^2+2x^4}$

Example 3.1. Equation (7) for $q_{00} = q_{11} = 0$ and $y = q_{01} = q_{10} = 1$ gives that

$$C(x, 1, 0, 1, 1, 0) = \frac{1}{(1-x)(1-x-x^2)},$$

which implies that the number of nonempty mod $-$ 2 alternating combinations of $[n]$ is given by $F_{n+3} - 1$ (i.e., combinations avoiding R_{00} and R_{11}). For a reconciliation of this result with the one stated in Section 1, namely $F_{n+3} - 2$, see Remark 4.3 below.

Moreover, equation (7) for different substitutions gives the specific generating functions shown in Table 1.

3.2. The case $m = 3$**.** If $m = 3$, we deduce from Theorem 2.3 that

$$C_0 = \frac{x^3 y \Delta_0}{\Delta}, \qquad C_1 = \frac{xy\Delta_1}{\Delta}, \qquad C_2 = \frac{x^2 y \Delta_2}{\Delta},$$

where

$$\begin{aligned}
\Delta_0 &= (1-x^3)^2 - (1-x^3)(q_{11}x^3 + q_{22}x^3 - q_{20} - q_{10})y \\
&\quad + (q_{11}q_{22}x^6 - q_{10}q_{22}x^3 - q_{21}q_{12}x^3 + q_{10}q_{21}x^3 - q_{20}q_{11}x^3 + q_{20}q_{12})y^2, \\
\Delta_1 &= (1-x^3)^2 - x^3(1-x^3)(q_{00} - q_{01} + q_{22} - q_{21})y \\
&\quad - x^3(q_{01}q_{22}x^3 + q_{00}q_{21}x^3 - q_{21}q_{02}x^3 - q_{22}q_{00}x^3 - q_{20}q_{01} + q_{02}q_{20})y^2, \\
\Delta_2 &= (1-x^3)^2 - (1-x^3)(q_{11}x^3 - q_{02}x^3 + q_{00}x^3 - q_{12})y \\
&\quad + x^3(q_{00}q_{11}x^3 - q_{02}q_{11}x^3 - q_{12}q_{00} + q_{01}q_{12} + q_{02}q_{10} - q_{10}q_{01})y^2,
\end{aligned}$$

and

$$\begin{aligned}\Delta = &(1-x^3)^3 - x^3(1-x^3)^2(q_{00}+q_{11}+q_{22})y\\ &+x^3(1-x^3)(x^3q_{11}q_{00}+x^3q_{22}q_{00}+x^3q_{22}q_{11}-q_{01}q_{10}-q_{12}q_{21}-q_{02}q_{20})y^2\\ &-x^3(x^6q_{00}q_{11}q_{22}+x^3q_{02}q_{10}q_{21}-x^3q_{00}q_{21}q_{12}\\ &-x^3q_{01}q_{10}q_{22}-x^3q_{02}q_{20}q_{11}+q_{01}q_{20}q_{12})y^3,\end{aligned}$$

which implies that

$$C(x,y,\mathbf{q}) = \frac{1}{1-x}\left(1+\frac{x^3y\Delta_0+xy\Delta_1+x^2y\Delta_2}{\Delta}\right). \tag{8}$$

Examples of generating functions for the numbers of combinations avoiding several parity statistics, modulo 3, are shown in Table 2.

TABLE 2. Generating functions for the number of combinations of $[n]$ satisfying $R_{\alpha\beta}=0$, for certain $\alpha\beta \in \{00,01,02,10,11,12,20,21,22\}$.

$\mathbf{q}$	$C(x,1,\mathbf{q})$
$(0,1,1,1,1,1,1,1,1)$	$\frac{1+2x+4x^2}{1-8x^3+4x^6}$
$(1,0,1,1,1,1,1,1,1)$	$\frac{1+2x+4x^2+x^3-2x^4-4x^5}{1-7x^3+8x^6}$
$(1,1,0,1,1,1,1,1,1)$	$\frac{1+2x+4x^2-4x^5}{1-8x^3+8x^6}$
$(1,1,1,0,1,1,1,1,1)$	$\frac{(1-x^3)(1+2x+4x^2)}{1-8x^3+8x^6}$
$(1,1,1,1,0,1,1,1,1)$	$\frac{1+2x+4x^2-x^4-2x^5}{1-8x^3+4x^6}$
$(1,1,1,1,1,0,1,1,1)$	$\frac{1+2x+3x^2-x^3-2x^4-4x^5}{1-7x^3+8x^6}$
$(1,1,1,1,1,1,0,1,1)$	$\frac{(1-x^3)(1+2x+4x^2)}{1-7x^3+8x^6}$
$(1,1,1,1,1,1,1,0,1)$	$\frac{1+2x+4x^2-2x^4-4x^5}{1-8x^3+8x^6}$
$(1,1,1,1,1,1,1,1,0)$	$\frac{1+2x+4x^2-2x^5}{1-8x^3+4x^6}$
$(0,1,1,1,0,1,1,1,0)$	$\frac{1}{(1-x)(1-x-x^2-x^3)}$
$(0,1,1,0,1,1,0,1,1)$	$\frac{1+2x+4x^2}{(1-x^3)(1-4x^3)}$
$(1,0,1,1,0,1,1,0,1)$	$\frac{1+2x+4x^2+3x^3-x^4-2x^5}{(1-x^3)(1-4x^3)}$
$(1,1,0,1,1,0,1,1,0)$	$\frac{1+2x+3x^2+x^3+2x^4-2x^5}{(1-x^3)(1-4x^3)}$

4. Mod-m alternating combinations. We first obtain an efficient corollary from Theorem 2.3. Denote by $CP(x,y,q_0,\dots,q_{m-1})$ the generating function for the number of k-combinations of $[n]$ according to the statistics $p_s=\sum_{|\alpha-\beta|=s}R_{\alpha\beta}$, $s=0,1,\dots,m-1$, that is, combinations $c_1c_2\cdots c_k$ according to the number of rises c_ic_{i+1} such that $c_{i+1}-c_i\equiv s \pmod m$. Theorem 2.3 for $q_{i,j}=q_{i-j \pmod m}$ gives that

$$CP(x,y,q_0,\dots,q_{m-1})=\frac{1}{1-x}\left(1+\sum_{j=0}^{m-1}CP_j\right),$$

where $CP_j=C_j(x,y,q_0,\dots,q_{m-1})$ and

$$\mathbf{A}_m(CP_0,CP_1,\dots,CP_{m-1})^T=\left(\frac{x^my}{1-x^m},\frac{xy}{1-x^m},\dots,\frac{x^{m-1}y}{1-x^m}\right)^T.$$

Therefore, by summing all the equations of this system of equations, we obtain

$$\left(1-\frac{q_0x^my+q_1x^1y+\cdots+q_{m-1}x^{m-1}y}{1-x^m}\right)\sum_{j=0}^{m-1}CP_j=\frac{(x+x^2+\cdots+x^m)y}{1-x^m},$$

which implies that

$$\sum_{j=0}^{m-1}CP_j=\frac{(x+x^2+\cdots+x^m)y}{(1-x^m)\,(1-(q_0x^my+q_1x^1y+\cdots+q_{m-1}x^{m-1}y)/1-x^m)}.$$

Hence, we can state the following result.

Corollary 4.1. *The generating function $CP(x,y,q_0,\dots,q_{m-1})$ for the number of k-combinations of $[n]$ according to the statistics p_s is given by*

$$CP(x,y,q_0,\dots,q_{m-1})$$
$$=\frac{1}{1-x}\left(1+\frac{(x+x^2+\cdots+x^m)y}{1-x^m-q_0x^my-q_1xy-q_2x^2y-\cdots-q_{m-1}x^{m-1}y}\right).$$

For instance, Corollary 4.1 for $m=3$ gives that the generating function $CP(x,y,q_0,q_1,q_2)$ is given by

$$CP(x,y,q_0,q_1,q_2)=\frac{1}{1-x}\left(1+\frac{x(1+x+x^2)y}{1-x^3-q_0x^3y-q_1xy-q_2x^2y}\right).$$

In particular, $CP(x,1,1,1,1) = 1/(1-2x)$, which agrees with the fact that the number of all combinations of $[n]$ is 2^n.

As another example, Corollary 4.1 for $q_0 = 0$ and $q_1 = \cdots = q_{m-1} = 1$ gives that the generating function for the number of mod-m alternating combinations of $[n]$, with respect to any length, is given by

$$\begin{aligned} AC(x,y) &= CP(x,y,0,1,1,\ldots,1) \\ &= \frac{1-x^m+x^m y}{(1-x)(1-x^m-xy-x^2y-\cdots-x^{m-1}y)} \\ &= \frac{1+[x^m/(1-x^m)]y}{(1-x)\,(1-(x+x^2+\cdots+x^{m-1})/1-x^m y)}. \end{aligned}$$

Moreover, by finding the coefficient of y^k in the generating function $AC(x,y)$, we obtain that the generating function for the number of mod-m alternating k-combinations of $[n]$ is given by

$$\frac{(x+x^2+\cdots+x^{m-1})^{k-1}(x+x^2+\cdots+x^m)}{(1-x)(1-x^m)^k} = \frac{x^k(1-x^{m-1})^{k-1}}{(1-x)^{k+1}(1-x^m)^{k-1}}.$$

Let $h_m(n,k)$ be the number of mod-m alternating k-combinations of $[n]$, and let $h_m(n) = \sum_k h_m(n,k)$. Then

$$\sum_{n\geq 0} h_m(n)x^n = AC(x,1) = \frac{1}{(1-x)(1-x-\cdots-x^m)}.$$

This result can be explained combinatorially by the following theorem.

Theorem 4.2. *For all $n \geq 0$,*

$$h_m(n) = \sum_{j=1}^{n+1} F_j^{(m)}. \tag{9}$$

Proof. Let $H_m(j)^*$ denote the set of mod-m alternating combinations of $[j]$ with last term $j > 0$. Then, each member of $H_m(j)^*$ may be constructed by appending j to a member of $H_m(v)^*$, where $j > v$ and $v \not\equiv j \pmod m$, and then including the singleton $\{j\}$. In other words, the sets contributing to $H_m(j)^* \setminus \{j\}$ form blocks of $m-1$ contiguous objects which are separated by the sets $H_m(u)^*$, $u \equiv j \pmod{m}$. This results in the following equation of cardinalities of sets:

$$(10)\qquad |H_m(j)^* \setminus \{j\}| = \left| \bigcup_{t=1}^{q} \bigcup_{i=(t-1)m+1}^{tm-1} H_m(j-i)^* \bigcup A(j,m) \right|,$$

where $j = mq + r$, $0 \le r \le m-1$, and $A(j,m)$ is the only incomplete block,

$$A(j,m) = \bigcup_{s=1}^{r-1} H_m(r-s)^*.$$

Note that $A(j,m)$ is empty for $r = 0, 1$. Since the sets on the right-hand side of (10) are disjoint, we have

$$(11)\qquad |H_m(j)^* \setminus \{j\}| = \sum_{t=1}^{q} \sum_{i=(t-1)m+1}^{tm-1} |H_m(j-i)^*| + |A(j,m)|.$$

Now we consider the difference $|H_m(j)^* \setminus \{j\}| - |H_m(j-m)^* \setminus \{j-m\}|$. This can be evaluated using equation (11). Since $j \equiv j-m \pmod{m}$, it follows that the sets of contributing blocks to $H_m(j)^* \setminus \{j\}$ and $H_m(j-m)^* \setminus \{j-m\}$ coincide except for the (extra) first block of $H_m(j)^* \setminus \{j\}$, namely,

$$\bigcup_{s=1}^{m-1} H_m(j-s)^*.$$

Hence, we obtain

$$|H_m(j)^* \setminus \{j\}| - |H_m(j-m)^* \setminus \{j\}| = \sum_{s=1}^{m-1} |H_m(j-s)^*|.$$

That is,

$$(12)\qquad |H_m(j)^*| = \sum_{s=1}^{m} |H_m(j-s)^*|.$$

The initial conditions are $|H_m(1)^*| = 1$, $|H_m(2)^*| = 2$, since $H_m(1)^* = \{\{1\}\}$ and $H_m(2)^* = \{\{2\},\{1,2\}\}$.

Thus, $|H_m(j)^*|$ satisfies the same recurrence relation as the number $F_{j+1}^{(m)}$ (see (5)) and fulfills the same initial conditions. Hence,

$$|H_m(j)^*| = F_{j+1}^{(m)}. \tag{13}$$

Lastly, since $H_m(j)^*$ is nonempty, we have $h_m(n) - 1 = \sum_{j=1}^{n} F_{j+1}^{(m)}$, which proves the theorem. □

Remark 4.3. • It can be shown that $h_2(n) = \sum_{j=1}^{n+1} F_j^{(2)} = F_{m+3}^{(2)}$. This formula includes the empty combination $\{\varnothing\}$ enumerated as a single object. Thus the number of nonempty mod-2 combinations of $[n]$ is $F_{m+3}^{(2)} - 1$. However, there is a convention in the literature to assign a count of 2 to $\{\varnothing\}$ since this is consistent with the recurrence relation for $h(n,k)$ (see for example [**6**]). This clarifies the formula for $h_2(n) = \sum_{k>0} h(n,k)$ stated in Section 1.

• Equation (11) corresponds to the following Fibonacci identity

$$F_n^{(m)} = 1 + \sum_{t=1}^{q} \sum_{i=1}^{m-1} F_{n-i-(t-1)m}^{(m)} + \sum_{i=1}^{r-1} F_{r-i}^{(m)}, \tag{14}$$

where $n = mq + r$, $0 \le r \le m-1$.

Lastly, we observe that

$$\begin{aligned} h_m(n) = \sum_{j=1}^{n+1} F_j^{(m)} &= F_1^{(m)} + \sum_{j=2}^{n} \left(F_{j-1}^{(m)} + \cdots + F_{j-m}^{(m)} \right) \\ &= 1 + \sum_{i=1}^{m} \sum_{j=i+1}^{n+1} F_{j-i}^{(m)} \\ &= 1 + \sum_{i=1}^{m} \sum_{j=1}^{n+1-i} F_j^{(m)} \\ &= 1 + \sum_{i=1}^{m} h_m(n-i). \end{aligned}$$

Thus, $h_m(n)$ satisfies the following equivalent recurrence relations, where $h_m(n) = 0$ if $n < 0$, and $h_m(0) = 1$.

$$h_m(n) = 1 + \sum_{i=1}^{m} h_m(n-i), \tag{15}$$

$$h_m(n) = 2h_m(n-1) - h_m(n-m-1). \tag{16}$$

Acknowledgments. The authors wish to thank an anonymous referee for several valuable comments leading to a substantial improvement in the exposition of the paper.

REFERENCES

1. M. Abramson and W. Moser, *Generalizations of Terquem's problem*, J. Combin. Theory **7** (1969), 171–180.

2. L. Carlitz, *Alternating sequences*, Discrete Math. **17** (1977), 133–138.

3. C.A. Church and H.W. Gould, *Lattice point solution of the generalized problem of Terquem and an extension of Fibonacci numbers*, Fibonacci Quart. **5** (1967), 59–68.

4. I.P. Goulden and D.M. Jackson, *Combinatorial enumeration*, Dover, New York, 2004.

5. ———, *The enumeration of generalized alternating subsets with congruences*, Discrete Math. **22** (1978), 99–104.

6. A.O. Munagi, *Alternating subsets and permutations*, Rocky Mountain J. Math. **40** (2010), 1965–1977.

7. ———, *Alternating subsets and successions*, ARS Combinatoria, to appear.

8. E. Netto, *Lehrbuch der Combinatorik*, 2nd ed., Leipzig, 1927, reprinted by Chelsea, New York, 1958.

9. J.W. Reilly, *Generalized alternating subsets by generating function methods*, ARS Combin. **3** (1977), 151–163.

10. J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.

11. N.J.A. Sloane, *The on-line encyclopedia of integer sequences*, published electronically at www.research.att.com/ njas/sequences/.

12. S.M. Tanny, *An application of a generalization of Terquem's problem*, Canad. Math. Bull. **18** (1975), 769–770.

Department of Mathematics, University of Haifa, 31905 Haifa, Israel
Email address: toufik@math.haifa.ac.il

School of Mathematics, University of the Witwatersrand, 2050 Johannesburg, South Africa
Email address: augustine.munagi@wits.ac.za