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# NON-AUTONOMOUS PARABOLIC PROBLEMS WITH SINGULAR INITIAL DATA IN WEIGHTED SPACES

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ABSTRACT. In this paper, we investigate the well-posedness of non-autonomous parabolic equations in weighted space $L^r_{\delta(x)}(\Omega)$, where $\delta(x)$ is the distance to the boundary. We first establish regularity properties of the extension Dirichlet heat semigroup in $L^r_{\delta(x)}(\Omega)$ and then, under some assumptions, we obtain the existence, uniqueness and regularity of the positive solutions of parabolic equations with critical and subcritical nonlinearity term in those spaces.

**1. Introduction.** The purpose of this paper is to investigate the following nonlinear parabolic equation

$$(1.1)\qquad \begin{cases} u_t - \Delta u = a(x)u^q + f(x,u) + g(x,t) & t > \tau,\ x \in \Omega, \\ u = 0 & t > \tau,\ x \in \partial\Omega, \\ u(\tau) = u_\tau & \tau \in \mathbf{R},\ x \in \Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^N$ with smooth $C^2$ boundary $\partial\Omega$, $0 < q \leq 1$, and $a(x)$, $f(x,u)$ satisfy some conditions which will be stated bellow.

There has been a great deal of study on problem (1.1) with autonomous cases, i.e., $g(x,t) = 0$. Since the pioneering work of Weissler [**16, 17**], Ni and Sacks [**9**], Brezis and Cazenave in [**3**] studied autonomous problem

$$(1.2)\qquad \begin{cases} u_t - \Delta u = |u|^{p-1}u & t > 0,\ x \in \Omega, \\ u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0) = u_0 & x \in \Omega, \end{cases}$$

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where $p > 1$ and $\Omega \subset \mathbf{R}^N$ is a bounded domain, and obtained that if $r \geq (N/2)(p-1)$, $u_0 \in L^r(\Omega)$, there exists a unique solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. Further studies have been made by Arrita and Carvalho. More precisely, they considered the abstract parabolic problem $\dot{x} = Ax + f(t,x)$, $t > t_0$ with initial data $x(t_0) = x_0 \in X^1$, and obtained existence of a unique solution $x \in C([t_0,\tau], X^1) \cap C((t_0,\tau], X^{1+\varepsilon})$, which is also called an $\varepsilon$-regular solution, where $X^\alpha$ ($\alpha > 0$) is the fractional power space associated to the linear operator $A$ and $f$ satisfies the locally Lipschitz condition on some suitable spaces, see details in [**1**]. These abstract results were applied to autonomous problem (1.2) and autonomous parabolic problems with nonlinear boundary conditions, see details in [**1, 2**]. Recently, Loayza in [**8**] considered autonomous problem (1.1) with $f(x,u) = b(x)u^p$, $p > 1$. When $0 < q < 1$, the nonlinearity lacks the Lipschitz condition. To overcome this obstacle, under some assumptions, Loayza obtained that, if $u_0 \in L^r(\Omega)$, there exists a unique positive solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. In addition, autonomous nonlinear evolution equations with singular initial conditions have been studied by other authors, see [**6, 12, 14, 15**].

Investigation of elliptic and parabolic problems in weighted spaces $L^p_{\delta(x)}(\Omega)$ (see the definition in Section 2) has drawn much attention. In [**12**], Quittner and Souplet presented a priori estimates for the solution of the following equation

$$\begin{cases} -\Delta u = f(x,u,\nabla u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega$ is a smoothly bounded domain of $\mathbf{R}^N$ and $f : \overline{\Omega} \times \mathbf{R}_+ \times \mathbf{R}^N \to \mathbf{R}_+$ is continuous. Under some assumptions on nonlinearity $f$, they first established the $L^1_{\delta(x)}$-estimate for the very weak solution $u$ of (1.3):

$$\|u\|_{L^1_{\delta(x)}} \leq C. \tag{1.4}$$

Let $1 < k_0 < (n+1)/(n-1)$, $k_i = k_0\rho^i$, $i = 0,1,\ldots$, $\rho > 1$. Using (1.4), the estimates $\|u\|_{L^{k_i}_{\delta(x)}} \leq C$ were obtained, and then, by a finite number of steps and using regularity of the Dirichlet Laplacian in the scale of $L^p_{\delta(x)}$-spaces, they obtained that $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty} \leq C$.

Similarly, this $L^p_{\delta(x)}$ bootstrap method was used in [**11**] to establish the $L^\infty$-estimates and existence of positive solutions for the following systems:

$$\begin{cases} -\Delta u = f(x,u,v) & x \in \Omega, \\ -\Delta v = g(x,u,v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega$ is a smoothly bounded domain of $\mathbf{R}^N$. It is pointed out in [**11**] that growth assumptions for systems (1.5) are optimal, that is, the results in [**4**] on establishing the $L^\infty$-estimates and existence by using the method of Hardy-Sobolev inequalities required stronger conditions than those in [**11**]. In [**5**], Fila, Souplet and Weissler studied universal bounds for global nonnegative solutions of (1.2). For $u_0(x) \in L^q_{\delta(x)}(\Omega)$ with $q \geq q_c = [(N+1)/2](p-1)$, using the optimal $L^q_{\delta(x)}$-$L^r_{\delta(x)}$ estimates for the heat semigroup, they proved that there exists a unique classical solution $u$ of (1.2) satisfying

$$u \in C([0,T];\ L^q_{\delta(x)}(\Omega)), \quad u \in C((0,T];\ L^r_{\delta(x)}(\Omega)),\ q < r \leq \infty, \tag{1.6}$$

$$t^{[(N+1)/2][(1/q)-(1/r)]}\|u(t)\|_{L^r_{\delta(x)}(\Omega)} \leq K, \quad 0 < t \leq T,\ q \leq r \leq \infty. \tag{1.7}$$

By (1.7), they obtained that, if $u_0 \in L^\infty(\Omega)$, $1 < p < (N+1)/(N-1)$ and $\tau > 0$, then all nonnegative global classical solutions $u$ of (1.2) satisfy

$$\sup_\Omega u(t,\cdot) \leq C(\Omega, p, \tau), \quad \text{for } t \geq \tau. \tag{1.8}$$

In contrast to the previous results in [**10**] on boundedness of global solutions of (1.2), (1.8) shows that the a priori bound is universal, that is, independent of $u_0$. After that, Quittner, Souplet and Winkler in [**13**] established a universal upper bound on the initial blow-up rate for all positive classical solutions of the following Dirichlet problem

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & 0 < t < T,\ x \in \Omega, \\ u = 0 & 0 < t < T,\ x \in \partial\Omega. \end{cases} \tag{1.9}$$

Applying (1.7), they proved that, if $1 < p < (N+3)/(N+1)$ and $T \in (0,\infty)$, all nonnegative classical solutions of (1.9) satisfy

$$\|u(t,\cdot)\|_{L^\infty} \le C(p,\Omega,T)t^{-(N+1)/2}, \quad 0 < t \le \frac{T}{2},$$

where the exponent $-(N+1)/2$, in contrast to the previous result, is optimal. At this point, we notice that it is important to study the elliptic and parabolic problems in weighted spaces $L^p_{\delta(x)}(\Omega)$.

In this paper, we consider non-autonomous problem (1.1) in weighted space $L^r_{\delta(x)}(\Omega)$. Since the nonlinearity $a(x)u^q + f(x,u)$ and external force $g(x,t)$ of (1.1) depend upon $x$, and $g(x,t)$ belongs to $L^\nu_{\mathrm{loc}}(\mathbf{R};X)$ (see $(P_3)$ in Section 2), which is equipped with the local $\nu$-power mean convergence topology different from the topology $L^r_{\delta(x)}(\Omega)$ upon which the heat semigroup is acting, the results in [**8**] cannot be extended to (1.1) directly. On the other hand, since $L^r(\Omega) \subset L^r_{\delta(x)}(\Omega)$, in contrast to the results in [**8**], we obtain the existence and uniqueness of the positive solution for non-autonomous system (1.1) in weak topology spaces. We first establish regularity properties of extension Dirichlet heat semigroup from $L^r(\Omega)$ to $L^r_{\delta(x)}(\Omega)$. For $0 < q < 1$, the nonlinearity of (1.1) lacks a Lipschitz condition. Motivated by the idea in [**8**], using regularity properties of the extension semigroup in $L^r_{\delta(x)}(\Omega)$ and the weighted Hardy's inequality, we obtain the existence of a unique positive solution $u \in C([0,T]; L^r_{\delta(x)}(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty(\Omega))$ of (1.1). The regularity of the solution is also studied, that is, the solution satisfies $(t-\tau)^{[(N+1)/2][(1/r)-(1/s)]}\|u(t)\|_{L^s_{\delta(x)}(\Omega)} \le C$ for $r \le s \le \infty$, and $(t-\tau)^{[(N+1)/2r]}\|u(t) - T(t)u_\tau\|_{W^{1,N+1}_{0,\delta(x)}} \le C$ for $N > 1$. For $q = 1$, we show that there exists a unique solution of (1.1), which does not need to be positive. By the regularity of solutions proved in Theorem 3.1 and 3.3, as in [**12, 13**], we can obtain optimal results on universal bounds for all global positive solutions and the initial blow-up rate for all positive solutions of (1.1), which are subjects for future study.

This paper is organized as follows. In the next section, we establish the regularity properties of a Dirichlet heat semigroup in $L^q_{\delta(x)}(\Omega)$ and give some lemmas which will be used later. In Section 3, we prove our main results.

**2. Preliminaries.** We first make some assumptions on nonlinearity $f(x,u)$:

($P_1$) There exists a $C > 0$ such that

$$|f(x,u) - f(x,v)| \leq C|b(x)||u-v|(|u|^{\rho-1} + |v|^{\rho-1} + 1),$$

where $\rho > 1$.

($P_2$) $f(x,u)u \geq 0$, $f(x,0) = 0$ almost everywhere in $\Omega$.

Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$. When $\Omega$ has a smooth $C^2$ boundary, it is well known that there exist $C_1$ and $C_2$ such that

$$C_1\phi_1(x) \leq \delta(x) \leq C_2\phi_1(x), \tag{2.1}$$

where $\phi_1(x) > 0$ is the first eigenfunction associated to the first eigenvalue $\lambda_1$ of $-\Delta$ in $H_0^1(\Omega)$, normalized by $\int_\Omega \phi_1(x) = 1$. The weighted Lebesgue spaces $L^p_{\delta(x)}(\Omega)$ are defined as follows. For $1 \leq p \leq \infty$,

$$L^p_\delta := L^p_{\delta(x)}(\Omega) := L^p(\Omega; \delta(x)\,dx).$$

For $1 \leq p < \infty$, $L^p_{\delta(x)}(\Omega)$ is endowed with the norm

$$\|u\|_{L^p_\delta} = \left(\int_\Omega |u|^p \delta(x)\,dx\right)^{1/p}.$$

Since $\delta(x)$ is a bounded function on $\Omega$, it is clear that $L^p(\Omega) \subset L^p_{\delta(x)}(\Omega)$. Analogously, define weighted Sobolev space $W^{l,p}_{\delta(x)}(\Omega)$, $l$ a positive integer and $p \geq 1$ as the space in which $D^i u \in L^p_{\delta(x)}(\Omega)$ for $|i| \leq l$. Denote by $W^{l,p}_{0,\delta(x)}(\Omega)$, $p \geq 1$, the closure of $C_0^l(\Omega)$ in the weighted Sobolev space $W^{l,p}_{\delta(x)}(\Omega)$, which is equipped with the norm $(\sum_{|i|=l} \|D^i u\|^p_{L^p_\delta})^{1/p}$.

We also make the assumption on $g(x,t)$ of (1.1):

($P_3$) $g(x,t) \in L^\nu_{\mathrm{loc}}(\mathbf{R}; X)$, i.e.,

$$\int_{t_1}^{t_2} \|g(x,s)\|^\nu_X ds < \infty, \quad \text{for all } [t_1, t_2] \subset \mathbf{R},$$

where $X = L^r_{\delta(x)}(\Omega) \cap L^\infty(\Omega)$, $1 \leq \nu$, $r < \infty$, and $1/\nu < \min\{(1/2), 1 - (1/\rho)\}$.

We know from [**5**] that, if $\Omega$ is a $C^2$ smooth bounded domain in $\mathbf{R}^N$, the Dirichlet heat semigroup $\{T(t)\}_{t\geq 0}$ admits a unique, densely defined, extension to $L^1_{\phi_1(x)}(\Omega)$. The extension semigroup that restricts on $L^r_{\delta(x)}(\Omega)$, $1<r<\infty$, yields a $C_0$ semigroup on $L^r_{\delta(x)}(\Omega)$, which will still be denoted by $\{T(t)\}_{t\geq 0}$. From now on, we assume that $\Omega$ is $C^2$ smooth. To investigate (1.1), we will use the following smoothing effect of the semigroup $\{T(t)\}_{t\geq 0}$.

**Lemma 2.1** [**5**]. *Let $1\leq q\leq r\leq\infty$ and $\alpha=[(N+1)/2][(1/q)-(1/r)]$. There exists a $C=C(\Omega)>0$ such that, for all $u_0\in L^q_{\delta(x)}(\Omega)$, the following holds*

$$\|T(t)u_0\|_{L^r_\delta}\leq Ct^{-\alpha}\|u_0\|_{L^q_\delta},\quad t>0. \tag{2.2}$$

*Remark* 2.2. Lemma 2.1 shows that, for all $u_0\in L^q_{\delta(x)}(\Omega)$, it holds that $\|T(t)u_0\|_{L^q_{\delta(x)}}\leq C\|u_0\|_{L^q_{\delta(x)}}$. If $1\leq r\leq q\leq\infty$, then $L^q_{\delta(x)}(\Omega)\subset L^r_{\delta(x)}(\Omega)$, the semigroup can be uniquely extended from $L^q_{\delta(x)}(\Omega)$ to $L^r_{\delta(x)}(\Omega)$, and $\|T(t)u_0\|_{L^r_{\delta(x)}}\leq C\|u_0\|_{L^r_{\delta(x)}}\leq C\|u_0\|_{L^q_{\delta(x)}}$. Therefore, for $1\leq r,q\leq\infty$, $u_0\in L^q_{\delta(x)}(\Omega)$, the following holds

$$\|T(t)u_0\|_{L^r_\delta}\leq Ct^{-(N+1)/2\max\{(1/q)-(1/r),0\}}\|u_0\|_{L^q_\delta},\quad t>0. \tag{2.3}$$ □

By Lemma 2.1, we also have the following regularity properties for the semigroup $\{T(t)\}_{t\geq 0}$ defined on the weighted space $L^q_{\delta(x)}(\Omega)$.

**Lemma 2.3.** *Let $1\leq p\leq q\leq\infty$, $0\leq r\leq s<\infty$, $r$ and $s$ are nonnegative integers. $T(t):W^{r,p}_{0,\delta(x)}(\Omega)\to W^{s,q}_{0,\delta(x)}(\Omega)$ for $t>0$. Then there exists a constant $C>0$ such that*

$$\|T(t)u_0\|_{W^{s,q}_{0,\delta(x)}}\leq Ct^{-[(N+1)/2][(1/p)-(1/q)]+(r-s)/2}\|u_0\|_{W^{r,p}_{0,\delta(x)}}, \tag{2.4}$$

*for all $t>0$ and $u_0\in W^{r,p}_{0,\delta(x)}(\Omega)$.*

*Proof.* The mapping $T(t):W^{r,p}_{0,\delta(x)}(\Omega)\to W^{s,q}_{0,\delta(x)}(\Omega)$ can be decomposed into

$$T(t):W^{r,p}_{0,\delta(x)}(\Omega)\longrightarrow W^{s,p}_{0,\delta(x)}(\Omega)\longrightarrow W^{s,q}_{0,\delta(x)}(\Omega),\quad\text{for } t>0.$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}
&\|T(t)u_0\|_{W^{s,q}_{0,\delta(x)}} \\
&\quad = \|T(t)(\Delta)^{s/2}u_0\|_{L^q_\delta} \\
&\quad \le Ct^{-[(N+1)/2][(1/p)-(1/q)]}\|T(t)(\Delta)^{s/2}u_0\|_{L^p_\delta} \\
&\quad = Ct^{-[(N+1)/2][(1/p)-(1/q)]}\|(\Delta)^{(s-r)/2}T(t)(\Delta)^{r/2}u_0\|_{L^p_\delta} \\
&\quad \le Ct^{-[(N+1)/2][(1/p)-(1/q)]+(r-s)/2}\|(\Delta)^{r/2}u_0\|_{L^p_\delta} \\
&\quad = Ct^{-[(N+1)/2][(1/p)-(1/q)]+(r-s)/2}\|u_0\|_{W^{r,p}_{\delta(x)}}, \quad \text{for } t > 0. \qquad \square
\end{aligned}$$

In order to obtain a uniform time for the existence of the solution, we will need the following results.

**Lemma 2.4.** *Given a compact* $\mathcal{K} \subset L^q_{\delta(x)}(\Omega)$ *and* $q < r \le \infty$, *there exists a function* $\gamma(t) : (0,1] \to (0,\infty)$ *with* $\lim_{t\to 0}\gamma(t) = 0$, *such that*

$$t^{[(N+1)/2][(1/q)-(1/r)]}\|T(t)u_0\|_{L^r_\delta} \le \gamma(t),$$

*for all* $t \in (0,1)$ *and all* $u_0 \in \mathcal{K}$.

*Proof.* By using the smoothing effect of Lemma 2.1, the proof is similar to the one of Lemma 8 in [**3**]. $\square$

We will also use the following generalized Gronwall's inequality.

**Lemma 2.5** [3]. *Let* $T > 0$, $A \ge 0$, $\alpha \ge 0$, $0 \le \beta, \nu < 1$. *Consider* $\varphi \in L^\infty(0,T)$ *to be a nonnegative function such that*

$$\varphi(t) \le A + t^\alpha \int_0^t (t-\tau)^{-\beta}\tau^{-\nu}\varphi(\tau)\,d\tau \quad \textit{almost everywhere in } (0,T).$$

*If* $1+\alpha > \beta+\nu$, *then there exists a positive constant* $C = C(T,\alpha,\beta,\nu) > 0$ *such that*

$$\varphi(t) \le CA \quad \textit{almost everywhere in } (0,T).$$

When $0 < q < 1$, the nonlinearity of (1.1) does not satisfy the Lipschitz condition. In order to obtain the existence, uniqueness and regularity of the solution of (1.1), we will use the weighted Hardy inequality, that is, for bounded Lipschitz domains $\Omega$ in $\mathbf{R}^N$, if $1 < p < \infty$, then for all $\beta \in [0, p-1)$, there exists a $C > 0$ such that, for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |u(x)|^p \delta(x)^{-p+\beta} dx \le C \int_\Omega |\nabla u(x)|^p \delta(x)^\beta dx, \tag{2.5}$$

where $\delta(x) := \text{dist}\,(x, \partial\Omega)$. See details in [**7**].

We will need the following technical results.

**Lemma 2.6.** *Let* $0 < q \le 1 \le \rho < \infty$, $\alpha, \beta, r \ge 1$ *with* $\alpha > (N+1)/(q+1)$ *and* $(1/\alpha) + (q/r) < q + [(1-q)/(N+1)] + [2q/(N+1)\rho]$. *Let* $(1/\beta) + [(\rho-1)/r] < [2/(N+1)]$ *or* $(1/\beta) + [(\rho-1)/r] = [2/(N+1)]$ *with* $r > 1$. *Then there exists an* $\eta > r$ *such that*

(i) $(1/\alpha) + (q/\eta) < q + [(1-q)/(N+1)]$,

(ii) $(1/\beta) + (\rho/\eta) < 1$,

(iii) $(1/\beta) + [(\rho-1)/\eta] < [2/(N+1)]$,

(iv) $\rho[(N+1)/2][(1/r) - (1/\eta)] < 1$.

*Proof.* From the assumptions, we have $1/r - [2/(N+1)\rho] < 1 + [(1-q)/(N+1)q] - (1/\alpha q)$ and $(1/\beta) + (\rho/r) \le (1/r) + (2/N+1) < 1 + (2/N+1)$ which implies that $(1/r) - [2/(N+1)\rho] < (1/\rho)[1-(1/\beta)]$. This allows us to choose $\eta > r$ such that

$$\begin{aligned} \frac{1}{r} - \frac{2}{(N+1)\rho} &< \frac{1}{\eta} \\ &< \min\left\{\frac{1}{\rho}\left(1 - \frac{1}{\beta}\right),\ 1 + \frac{1-q}{(N+1)q} - \frac{1}{\alpha q}, \frac{1}{\rho-1}\left(\frac{2}{N+1} - \frac{1}{\beta}\right)\right\}. \end{aligned}$$

The proof is completed. ◻

*Remark* 2.7. From the proof of Lemma 2.6, we find that, if $N \ge 2$, there exists an $\eta$ such that $\eta > \max\{r, [2(N+1)/(N-1)]\}$, and all results of Lemma 2.6 hold; if $N = 1$, we can choose $\eta_0 > 0$ small enough such that

$$\frac{1}{\alpha} + \frac{q}{\eta} < q + \frac{1-q}{2} - \eta_0 q = \frac{1+q}{2} - \eta_0 q, \qquad \frac{\rho}{\eta} + \frac{1}{\beta} < 1 - \eta_0. \qquad \square$$

Define $m(\eta)$ by

$$\frac{1}{m(\eta)} = \begin{cases} (1/\eta) + [1/(N+1)] & \eta > \max\{r, [2(N+1)/(N-1)]\},\ N \geq 2, \\ (1/2) - \eta_0 & \eta_0 \text{ is small enough, } N = 1. \end{cases}$$

The definition of $m(\eta)$ implies that $m(\eta) > 2$. Moreover, by Remark 2.7 and direct computations, we have the following results.

**Lemma 2.8.** *Let the assumptions of Lemma* 2.6 *hold, and let* $\eta$ *be given by Lemma* 2.6. *Then*

(i) $(1/\alpha) + (q/\eta) + [(1-q)/m(\eta)] < 1$,

(ii) $(1/\alpha) + (q/\eta) - [q/m(\eta)] < [1/(N+1)]$,

(iii) $[1/m(\eta)] < (1/\eta) + [1/(1-q)][(2/N+1) - (1/\alpha)]$,

(iv) $(\rho/\eta) + (1/\beta) - [1/(N+1)] < [1/m(\eta)]$.

We will use the following lemma.

**Lemma 2.9.** *Let* $\Omega$ *be a* $C^2$ *bounded domain in* $\mathbf{R}^N$ *and* $f \in L^1((0,T), L_{\delta(x)}(\Omega))$. *For* $t \in (0,T)$, *let* $\phi(t) = \int_0^T T(t-s)f(s)\,ds$. *If, for some* $1 < r < \infty$, $\phi(t) \in L^r_{\delta(x)}(\Omega)$ *and* $\nabla T(t-\cdot)f(\cdot) \in L^1((0,t), L^r_{\delta(x)}(\Omega))$, *then* $\phi(t) \in W^{1,r}_{0,\delta(x)}(\Omega)$ *for every* $t \in (0,T)$.

*Proof.* Using Lemma 2.3, the proof is similar to the one of Lemma 2.3 in [**8**]. $\square$

**3. Main results.** By a solution $u \in C([\tau,T], L^r_{\delta(x)}(\Omega)) \cap L^\infty_{\text{loc}}((\tau,T), L^\infty(\Omega))$ of (1.1), we mean that

(3.1)
$$\begin{cases} u(t) = T(t-\tau')u(\tau') + \int_{\tau'}^t T(t-s)[au^q(s) + f(x,u(s)) + g(x,s)]\,ds, \\ u(\tau') \longrightarrow u_\tau \quad \text{in } L^r_{\delta(x)}(\Omega) \text{ as } \tau' \to \tau^+. \end{cases}$$

Now, for $0<q<1$, we give our main results.

**Theorem 3.1.** *Suppose that* $(\mathrm{P}_1)$–$(\mathrm{P}_3)$ *hold. Let* $a(x)\in L^{\alpha}_{\delta(x)}(\Omega)$, $b(x)\in L^{\beta}_{\delta(x)}(\Omega)$ *with* $1<\alpha,\beta\le+\infty$ *and* $a(x)\ge 0$ *almost everywhere in* $\Omega$. *Let* $1\le r<+\infty$, $0<q<1$, $\alpha>(N+1)/(q+1)$, $(1/\alpha)+(q/r)\le q+[(1-q)/(N+1)]+[2q/(N+1)\rho]$ *and* $(1/\beta)+[(\rho-1)/r]<[2/(N+1)]$ (*respectively,* $(1/\beta)+[(\rho-1)/r]=[2/(N+1)]$, $r>1$). *Assume that* $u_\tau\in L^r_{\delta(x)}(\Omega)$, $g(x,t)\ge 0$ *almost everywhere in* $\Omega$, *and there exists a* $\kappa_0>0$ *such that* $u_\tau\ge\kappa_0\delta(x)$ *almost everywhere in* $\Omega$. *Then there exist* $T=T(u_\tau)>\tau$ *and* $\kappa>0$ *such that there is a unique positive solution* $u\in C([\tau,T],L^r_{\delta(x)}(\Omega))$ *of* (1.1) *satisfying* $u(t)\ge\kappa\delta(x)$ *for all* $t\in(\tau,T]$.

*Moreover, there exists a* $C>0$ *such that*

(3.2)
$$u(t)\in C((\tau,T),L^s_{\delta(x)}(\Omega)),\quad r\le s\le+\infty,$$

(3.3)
$$(t-\tau)^{[(N+1)/2][(1/r)-(1/s)]}\|u(t)\|_{L^s_\delta}\le C,\quad \tau<t<T,\ r\le s\le+\infty,$$

*and, if* $N\ge 2$, *then*

$$(t-\tau)^{(N+1)/(2r)}\|u(t)-T(t)u_0\|_{W^{1,N+1}_{0,\delta(x)}}\le C.$$

*This solution is unique in the class*

$$C([\tau,T],L^r_{\delta(x)}(\Omega))\cap L^\infty_{\mathrm{loc}}((\tau,T),L^\infty(\Omega)).$$

*Furthermore, for any bounded set* (*respectively, compact set*) $\mathcal{K}$ *in* $L^r_{\delta(x)}(\Omega)$), *there is a* (*uniform*) *time* $T=T(\mathcal{K})$ *such that, for any* $u_\tau\in\mathcal{K}$, *the positive solution of* (1.1) *exists on* $[\tau,T]$.

*Proof.* To study problem (1.1) with singular initial data, we formally define

$$\Phi(u)(t)=T(t)u_\tau+\int_\tau^t T(t-s)[au(s)^q+f(x,u(s))+g(x,s)]\,ds.$$

The proof of the theorem can be divided into three parts.

**Part one: Existence of solution.** We will apply the contraction mapping principle to $\Phi$ in some complete metric space. Now consider two situations.

**Case a: Subcritical nonlinearity term** $(1/\beta) + [(\rho - 1)/r] < [2/(N+1)]$**.** Let $\eta$ be given by Lemma 2.6, and let $m(\eta)$ be defined by (2.6). Now, fix $\|u_\tau\|_{L^r_{\delta(x)}} \le M$ and define

$$E = C((\tau, T), L^\eta_{\delta(x)}(\Omega)) \bigcap C((\tau, T), W^{1,m(\eta)}_{0,\delta(x)}(\Omega)),$$
$$X = \{u \in E : u(t) \ge \kappa\delta(x),\ (t-\tau)^{\tilde{\alpha}} \|u(t)\|_{L^\eta_\delta} \le M + 1,$$
$$(t-\tau)^{\tilde{\beta}} \|\nabla(u(t) - T(t)u_\tau)\|_{L^{m(\eta)}_\delta} \le 1,\ \text{for } t \in (\tau, T)\},$$

where $0 < T - \tau < 1$, $\widetilde{\alpha} = [(N+1)/2][(1/r) - (1/\eta)]$, $\widetilde{\beta} = -[(N+1)/2m(\eta)] + (1/2) + [(N+1)/2r]$, and $\kappa$ is a positive constant which will be given later. Equip $X$ with the distance

$$d(u, v) = \max\left\{ \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t) - v(t)\|_{L^\eta_\delta},\ \sup_{\tau<t<T} (t-\tau)^{\tilde{\beta}} \|\nabla(u(t) - v(t))\|_{L^{m(\eta)}_\delta} \right\}; \tag{3.4}$$

then $(X, d)$ is a nonempty complete metric space.

From now on, we denote by $C$ a variable positive constant whose value may vary from line to line or even in the same line.

From Lemma 2.6, we get that $(1/\alpha) + [(q-1)/\eta] < (1/\alpha) < [2/(N+1)]$, $(1/\beta) + [(\rho - 1)/\eta] < [2/(N+1)]$, $(1/\alpha) + (q/\eta) < 1$, $(1/\beta) + (\rho/\eta) < 1$ and $\tilde{\alpha}q < 1$. Let $(1/G_1) = (1/\alpha) + (q/\eta)$, $(1/G_2) = (1/\beta) + (\rho/\eta)$. Thus, by assumptions $(P_1)$–$(P_3)$, Lemma 2.1 and Remark 2.2,

(3.5)
$$(t-\tau)^{\tilde{\alpha}} \|\Phi(u)(t)\|_{L^\eta_\delta} \le (t-\tau)^{\tilde{\alpha}} \|T(t)u_\tau\|_{L^\eta_\delta}$$
$$+ (t-\tau)^{\tilde{\alpha}} \int_\tau^t \|T(t-s)[au^q(s) + f(x, u(s)) + g(x, s)]\|_{L^\eta_\delta} ds$$
$$\le M + (t-\tau)^{\tilde{\alpha}} \int_\tau^t \|T(t-s)au^q(s)\|_{L^\eta_\delta} ds$$

$$
\begin{aligned}
&+ C(t-\tau)^{\tilde{\alpha}} \int_\tau^t \|T(t-s)b(|u(s)|^\rho + 1)\|_{L^\eta_\delta} ds \\
&+ (t-\tau)^{\tilde{\alpha}} \int_\tau^t \|T(t-s)g(x,s)\|_{L^\eta_\delta} ds \\
\le{}& M + C(t-\tau)^{\tilde{\alpha}} \\
&\times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}} \|au^q(s)\|_{L^{G_1}_\delta} ds \\
&+ C(t-\tau)^{\tilde{\alpha}} \int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+(\rho-1/\eta)]} \|b(|u(s)|^\rho+1)\|_{L^{G_2}_\delta} ds \\
&+ (t-\tau)^{\tilde{\alpha}} \int_\tau^t (t-s)^{-[(N+1)/2][(1/r)-(1/\eta)]} \|g(x,s)\|_{L^r_\delta} ds \\
\le{}& M + C\|a\|_{L^\alpha_\delta} (t-\tau)^{\tilde{\alpha}} \\
&\times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}} \|u(s)\|^q_{L^\eta_\delta} ds \\
&+ C\|b\|_{L^\beta_\delta}(t-\tau)^{\tilde{\alpha}} \int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]} \left(\|u(s)\|^\rho_{L^\eta_\delta}+1\right) ds \\
&+ (t-\tau)^{\tilde{\alpha}} \left( \int_\tau^t (t-s)^{-[(N+1)/2][(1/r)-(1/\eta)]\nu'} \right)^{1/\nu'} \\
&\times \left( \int_\tau^t \|g(x,s)\|^\nu_{L^r_\delta} ds \right)^{1/\nu} \\
\le{}& M + C\|a\|_{L^\alpha_\delta} \left( \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t)\|_{L^\eta_\delta} \right)^q (t-\tau)^{\tilde{\alpha}} \\
&\times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}} (s-\tau)^{-\tilde{\alpha}q} ds \\
&+ C\|b\|_{L^\beta_\delta}(t-\tau)^{\tilde{\alpha}} \Bigg[ \left( \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t)\|_{L^\eta_\delta} \right)^\rho \\
&\times \int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]} (s-\tau)^{-\tilde{\alpha}\rho} ds \\
&\qquad\qquad + \int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]} ds \Bigg] \\
&+ C(t-\tau)^{1/\nu'} \|g(x,t)\|_{L^\nu_{\mathrm{Loc}}([\tau,T];L^r_\delta(\Omega))} \\
\le{}& M + C(M+1)^q (t-\tau)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}+1+\tilde{\alpha}(1-q)}
\end{aligned}
$$

$$+ C(M+1)^{\rho}(t-\tau)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/r]+1]}$$
$$+ C(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]+[(N+1)/2][(1/r)-(1/\eta)]}$$
$$+ C(t-\tau)^{1/\nu'}\|g(x,t)\|_{L^{\nu}_{\mathrm{loc}}([\tau,T];L^r_\delta(\Omega))},$$

where $(1/\nu') + (1/\nu) = 1$. Similarly, by Lemma 2.8, we get that $(1/2) + [(N+1)/2][(1/\alpha) + (q/\eta) - (1/m(\eta))] < 1$ and $(1/2) + [(N+1)/2][(1/\beta)+(\rho/\eta)-(1/m(\eta))] < 1$. Furthermore, we can assume that $[1/m(\eta)] \leq (\rho/\eta) + (1/\beta)$. Using Lemma 2.3, we have

(3.6)
$$(t-\tau)^{\tilde\beta}\|\nabla(\Phi(u)(t) - T(t)u_\tau)\|_{L^{m(\eta)}_\delta}$$
$$\leq (t-\tau)^{\tilde\beta}\int_\tau^t \|\nabla[T(t-s)(au^q(s) + f(x,u(s)) + g(x,s))]\|_{L^{m(\eta)}_\delta} ds$$
$$\leq (t-\tau)^{\tilde\beta}\int_\tau^t \|\nabla(T(t-s)au^q(s))\|_{L^{m(\eta)}_\delta} ds$$
$$+ C(t-\tau)^{\tilde\beta}\int_\tau^t \|\nabla[T(t-s)b(|u(s)|^\rho+1)]\|_{L^{m(\eta)}_\delta} ds$$
$$+ (t-\tau)^{\tilde\beta}\int_\tau^t \|\nabla T(t-s)g(x,s)\|_{L^{m(\eta)}_\delta} ds$$
$$\leq C\|a\|_{L^\alpha_\delta}(t-\tau)^{\tilde\beta}$$
$$\times\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}}\|u(s)\|^q_{L^\eta_\delta} ds$$
$$+ C\|b\|_{L^\beta_\delta}(t-\tau)^{\tilde\beta}$$
$$\times\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}(\|u(s)\|^\rho_{L^\eta_\delta}+1)\, ds$$
$$+ (t-\tau)^{\tilde\beta}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/r)-(1/m),0\}}\|g(x,s)\|_{L^r_\delta} ds$$
$$\leq C(M+1)^q(t-\tau)^{(1/2)+\tilde\beta-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}-\tilde\alpha q}$$
$$+ C(M+1)^\rho(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]}$$
$$+ C(t-\tau)^{1-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/r)]}$$
$$+ C(t-\tau)^{(1/\nu')-(1/2)-[(N+1)/2]\max\{(1/r)-(1/m),0\}+\tilde\beta}$$
$$\times\|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))}.$$

From Lemmas 2.6 and 2.8, by direct computations, we find that $-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}+1+\tilde{\alpha}(1-q)>0$, $(1/2)+\tilde{\beta}-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}-\tilde{\alpha}q>0$. Thus, from (3.5) and (3.6), we can choose $0<T-\tau<1$ such that

$$
\begin{aligned}
(3.7)\quad & (t-\tau)^{\tilde{\alpha}}\|\Phi(u)(t)\|_{L^{\eta}_{\delta}} \le M+1,\\
& (t-\tau)^{\tilde{\beta}}\|\nabla(\Phi(u)(t)-T(t)u_\tau)\|_{L^{m(\eta)}_{\delta}} \le 1, \quad \text{for all } t\in(\tau,T).
\end{aligned}
$$

Since $a\ge 0$, $f(x,u)u\ge 0$ and $g(x,t)\ge 0$, we get that, if $u\in X$, then $\Phi(u)(t)\ge T(t)u_\tau\ge \kappa\delta(x)$, where $\kappa=\kappa_0 C_1 C_2^{-1}e^{-\lambda_1}$.

Next, we show that $\Phi(X)\subset X$. Proceeding as in the derivation of (3.5),

(3.8)

$$
\begin{aligned}
&(t-\tau)^{\tilde{\beta}}\Big\|\int_\tau^t T(t-s)[au^q(s)+f(x,u(s))+g(x,s)]\,ds\Big\|_{L^{m(\eta)}_{\delta}}\\
&\le (t-\tau)^{\tilde{\beta}}\int_\tau^t \|T(t-s)au^q(s)\|_{L^{m(\eta)}_{\delta}}\,ds\\
&\quad + C(t-\tau)^{\tilde{\beta}}\int_\tau^t \|T(t-s)b(|u(s)|^\rho+1)\|_{L^{m(\eta)}_{\delta}}\,ds\\
&\quad + (t-\tau)^{\tilde{\beta}}\int_\tau^t \|T(t-s)g(x,s)\|_{L^{m(\eta)}_{\delta}}\\
&\le C\|a\|_{L^{\alpha}_{\delta}}(t-\tau)^{\tilde{\beta}}\\
&\quad\times\int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/(m(\eta))],0\}}\|u(s)\|^q_{L^{\eta}_{\delta}}\,ds\\
&\quad + C\|b\|_{L^{\beta}_{\delta}}(t-\tau)^{\tilde{\beta}}\\
&\quad\times\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}(\|u(s)\|^\rho_{L^{\eta}_{\delta}}+1)\,ds\\
&\quad + (t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/r)-(1/m),0\}}\|g(x,s)\|_{L^{r}_{\delta}}\,ds\\
&\le C(M+1)^q(t-\tau)^{1+\tilde{\beta}-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}-\tilde{\alpha}q}\\
&\quad + C(M+1)^\rho(t-\tau)^{(3/2)-[(N+1)/2][(1/\beta)+[\rho-1]/r]}\\
&\quad + C(t-\tau)^{(3/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/r)]}
\end{aligned}
$$

$$+ C(t-\tau)^{\tilde{\beta}-[(N+1)/2]\max\{(1/r)-(1/m),0\}+1/\nu'} \times \|g(x,t)\|_{L^{\nu}_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))}.$$

With (3.7) and (3.8), by Lemma 2.9 we get that $\Phi(u)(t) - T(t)u_\tau \in W^{1,m}_{0,\delta(x)}$. Proceeding as in the derivation of (3.5) and (3.6), for $\tau < t_1 < t < T$,

(3.9)
$$\begin{aligned}
&\left\| \int_{t_1}^{t} T(t-s)[au^q(s) + f(x,u(s)) + g(x,s)]\,ds \right\|_{L^\eta_\delta} \\
&\le C\|a\|_{L^\alpha_\delta}(M+1)^q \\
&\quad \times \int_{t_1}^{t} (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}}(s-\tau)^{\tilde{\alpha}q}ds \\
&\quad + C\|b\|_{L^\beta_\delta}(M+1)^\rho \\
&\quad \times \int_{t_1}^{t} (t-\tau)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}(s-\tau)^{\tilde{\alpha}\rho}ds \\
&\quad + C\|b\|_{L^\beta_\delta}\int_{t_1}^{t} (t-\tau)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}ds \\
&\quad + C(t-t_1)^{(1/\nu')-[(N+1)/2][(1/r)-(1/\eta)]} \\
&\quad \times \|g(x,t)\|_{L^{\nu}_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))} \longrightarrow 0, \quad \text{as } t \to t_1^+,
\end{aligned}$$

(3.10)
$$\begin{aligned}
&\left\| \int_{t_1}^{t} T(t-s)[au^q(s) + f(x,u(s)) + g(x,s)]\,ds \right\|_{W^{1,m(\eta)}_{0,\delta(x)}} \\
&\le C\|a\|_{L^\alpha_\delta}(M+1)^q \\
&\quad \times \int_{t_1}^{t} (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}}(s-\tau)^{\tilde{\alpha}q}ds \\
&\quad + C\|b\|_{L^\beta_\delta}(M+1)^\rho \\
&\quad \times \int_{t_1}^{t} (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}(s-\tau)^{\tilde{\alpha}\rho}ds \\
&\quad + C\|b\|_{L^\beta_\delta}\int_{t_1}^{t} (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}ds
\end{aligned}$$

$$+ C(t-t_1)^{1/\nu' - (1/2) - [(N+1)/2]\max\{(1/r)-(1/m),0\}} \| \times g(x,t)\|_{L^{\nu}_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))} \longrightarrow 0, \quad \text{as } t \to t_1^+,$$

which imply that $\Phi(u)(t) \in C((0,T), L^{\eta}_{\delta(x)}(\Omega)) \cap C((0,T), W^{1,m(\eta)}_{0,\delta(x)}(\Omega))$. Therefore, $\Phi(X) \subset X$.

In the following, we show that $\Phi(u)(t)$ is a strict contraction mapping. Let $u, v \in X$,

$$\Phi(u)(t) - \Phi(v)(t) = \int_{\tau}^{t} T(t-s)a[u^q(s) - v^q(s)]\, ds + \int_{s}^{t} T(t-s)[f(x,u(s)) - f(x,v(s))]\, ds. \tag{3.11}$$

Let

$$I_1(t) = \int_{\tau}^{t} T(t-s)a[u^q(s) - v^q(s)]\, ds,$$
$$I_2(t) = \int_{\tau}^{t} T(t-s)[f(x,u(s)) - f(x,v(s))]\, ds.$$

Since $u, v \geq \kappa\delta(x)$ for $t \in (\tau, T)$, we get that $|u^q - v^q| \leq C|u-v|^q[|u-v|/\delta(x)]^{1-q}$. By Lemma 2.8 (i) we have $(1/\alpha) + (q/\eta) + [(1-q)/m] < 1$. Let $(1/G_1) = (1/\alpha) + (q/\eta) + [(1-q)/m(\eta)]$,

$$\begin{aligned}
\|I_1(t)\|_{L^{\eta}_{\delta}} &\leq \int_{\tau}^{t} \|T(t-s)a[u^q(s) - v^q(s)]\|_{L^{\eta}_{\delta}} ds \\
&\leq C\int_{\tau}^{t} (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \times \|a(u^q(s) - v^q(s))\|_{L^{G_1}_{\delta}} ds \\
&\leq C\|a\|_{L^{\alpha}_{\delta}} \times \int_{\tau}^{t} (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \\
&\qquad \times \|u(s) - v(s))\|^q_{L^{\eta}_{\delta}} \times \left(\int_{\Omega} \left|\frac{|u(s)-v(s)|}{\delta}\right|^{m(\eta)} \delta(x)\, dx\right)^{[(1-q)/m(\eta)]} ds.
\end{aligned} \tag{3.12}$$

The definition of $m(\eta)$ implies that $m(\eta) > 2$. Since $C_0^\infty(\Omega)$ is dense in $W_{0,\delta(x)}^{1,m(\eta)}$, we get that (2.5) holds for $u \in W_{0,\delta(x)}^{1,m(\eta)}$. From (3.12), we have

(3.13)

$$\begin{aligned}\|I_1(t)\|_{L_\delta^\eta} \le C\|a\|_{L_\delta^\alpha} & \\ \times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} & \\ \times \|u(s)-v(s))\|_{L_\delta^\eta}^q & \\ \times \|\nabla(u(s)-v(s))\|_{L_\delta^{m(\eta)}}^{1-q} ds. &\end{aligned}$$

Lemma 2.8 (iii) shows that $(1/\alpha) + [(q-1)/\eta] + [(1-q)/m(\eta)] < [2/(N+1)]$, together with the fact $\widetilde{\alpha}q + (1-q)\widetilde{\beta} < 1$, from (3.13) we have

(3.14)

$$\begin{aligned}(t-\tau)^{\tilde{\alpha}}\|I_1(t)\|_{L_\delta^\eta} &\le C\|a\|_{L_\delta^\alpha}(\sup_{\tau<t<T}(t-\tau)^{\tilde{\alpha}}\|u(t)-v(t)\|_{L_\delta^\eta})^q \\ &\quad \times \left(\sup_{\tau<t<T} t^{\tilde{\beta}}\,\|\nabla(u(t)-v(t))\|_{L_\delta^{m(\eta)}}\right)^{1-q}(t-\tau)^{\tilde{\alpha}} \\ &\quad \times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \\ &\qquad (s-\tau)^{-\tilde{\alpha}q-(1-q)\tilde{\beta}} ds \\ &\le C\|a\|_{L_\delta^\alpha} d(u,v) \\ &\quad \times (t-\tau)^{1-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})}.\end{aligned}$$

Similarly, by Lemma 2.8 (ii), $(1/\alpha) + (q/\eta) - [q/m(\eta)] < [1/(N+1)]$, we get that

(3.15)

$$\begin{aligned}(t-\tau)^{\tilde{\beta}}\|I_1(t)\|_{W_{0,\delta(x)}^{1,m(\eta)}} &\le (t-\tau)^{\tilde{\beta}} \\ &\quad \times \int_\tau^t \|T(t-s)a[u^q(s)-v^q(s)]\|_{W_{0,\delta(x)}^{1,m(\eta)}} ds \\ &\le C\|a\|_{L_\delta^\alpha}(t-\tau)^{\tilde{\beta}}\end{aligned}$$

$$\times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/m(\eta)],0\}}$$
$$\times \|u(s)-v(s)\|_{L_\delta^\eta}^q \|u(s)-v(s)\|_{W_{0,\delta(x)}^{1,m(\eta)}}^{1-q} ds$$
$$\leq C\|a\|_{L_{\delta(x)}^\alpha} d(u,v)$$
$$\times (t-\tau)^{(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/m(\eta)],0\}+q(\tilde{\beta}-\tilde{\alpha})}.$$

By Lemma 2.6 (iii) and Lemma 2.8 (iv), we get that $(1/\beta)+[(\rho-1)/\eta] < [2/(N+1)]$, and $(1/\beta)+(\rho/\eta)-[1/m(\eta)] < [1/(N+1)]$. Thus, proceeding as in the above derivation,

$$\begin{aligned}(3.16)\quad &(t-\tau)^{\tilde{\alpha}}\|I_2(t)\|_{L_\delta^\eta}\\
&\leq C\|b\|_{L_\delta^\beta}(t-\tau)^{\tilde{\alpha}}\\
&\quad\times \int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}\|u(s)-v(s)\|_{L_\delta^\eta}\\
&\qquad\times (\|u(s)\|_{L_\delta^\eta}^{\rho-1}+\|v(s)\|_{L_\delta^\eta}^{\rho-1}+1)ds\\
&\leq C\|b\|_{L_\delta^\beta}\left(\sup_{\tau<t<T}(t-\tau)^{\tilde{\alpha}}\|u(t)-v(t)\|_{L_\delta^\eta}\right)(t-\tau)^{\tilde{\alpha}}\\
&\quad\times\Big[(M+1)^{\rho-1}\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+(\rho-1/\eta))}(s-\tau)^{-\tilde{\alpha}\rho}ds\\
&\qquad+\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}(s-\tau)^{-\tilde{\alpha}}ds\Big]\\
&\leq C\|b\|_{L_\delta^\beta}d(u,v)[(M+1)^{\rho-1}(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]}\\
&\qquad+(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}],\end{aligned}$$

$$\begin{aligned}(3.17)\quad &(t-\tau)^{\tilde{\beta}}\|I_2(t)\|_{W_{0,\delta}^{1,m}} \leq C\|b\|_{L_\delta^\beta}(t-\tau)^{\tilde{\beta}}\\
&\times\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}\\
&\times\|u(s)-v(s)\|_{L_\delta^\eta}(\|u(s)\|_{L_\delta^\eta}^{\rho-1}+\|v(s)\|_{L_\delta^\eta}^{\rho-1}+1)\,ds\\
&\leq C\|b\|_{L_\delta^\beta}d(u,v)\\
&\times[(M+1)^{\rho-1}(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]}\\
&\quad+(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}].\end{aligned}$$

Using Lemma 2.8, by direct computations, we get that $1-[(N+1)/2] \times \max\{(1/\alpha) + [(q-1)/\eta] + [(1-q)/m(\eta)], 0\} + (1-q)(\widetilde{\alpha} - \widetilde{\beta}) > 0$, $(1/2) - [(N+1)/2]\max\{(1/\alpha) + (q/\eta) - [q/m(\eta)], 0\} + q(\tilde{\beta} - \tilde{\alpha}) > 0$. Since $(1/\beta) + [(\rho-1)/r] < 1$, by (3.14)–(3.17) we can choose $0 < T - \tau < 1$ such that

$$
\begin{aligned}
(3.18)\quad & (t-\tau)^{\tilde{\alpha}}\|\Phi(u)(t) - \Phi(v)(t)\|_{L^{\eta}_{\delta}} \\
&\le C\Big\{\|a\|_{L^{\alpha}_{\delta}}(t-\tau)^{1-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})} \\
&\qquad + C\|b\|_{L^{\beta}_{\delta}}[(M+1)^{\rho-1}(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]} \\
&\qquad\qquad + (t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}]\Big\}\, d(u,v) \\
&\le \frac{1}{2} d(u,v),
\end{aligned}
$$

$$
\begin{aligned}
(3.19)\quad & (t-\tau)^{\tilde{\beta}}\|\Phi(u)(t) - \Phi(v)(t)\|_{W^{1,m}_{0,\delta(x)}} \\
&\le C\Big\{\|a\|_{L^{\alpha}_{\delta(x)}}(t-\tau)^{(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-(q/m),0\}-q(\tilde{\alpha}-\tilde{\beta})} \\
&\qquad + \|b\|_{L^{\beta}_{\delta}}[(M+1)^{\rho-1}(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]} \\
&\qquad\qquad + (t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}]\Big\}\, d(u,v) \\
&\le \frac{1}{2}\, d(u,v).
\end{aligned}
$$

Equations (3.18)–(3.19) show that $d(\Phi(u), \Phi(v)) \le (1/2)d(u,v)$, that is, $\Phi$ is a strict contraction mapping on $X$; thus, $\Phi$ has a unique fixed point in $X$.

We now prove that $u \in C([\tau,T], L^r_{\delta(x)}(\Omega))$. First note that $u \in X$, $\eta > r$. This implies that $u \in C((\tau,T], L^{\eta}_{\delta(x)}(\Omega)) \subset C((\tau,T], L^r_{\delta(x)}(\Omega))$. It remains to show that $u$ is continuous at $t = \tau$ in the norm $L^r_{\delta(x)}$. Note that $\alpha > [(N+1)/(q+1)]$ implies $(1/\alpha) + (q/\eta) - (1/r) < [2/(N+1)]$, and $(1/\beta) + [(\rho-1)/\eta] < [2/(N+1)]$ implies $(1/\beta) + (\rho/\eta) - (1/r) < [2/(N+1)]$. By direct computations, we get that $1 - [(N+1)/2]\max\{(1/\alpha) + (q/\eta) - (1/r), 0\} - \widetilde{\alpha} q > 0$, $1 - [(N+1)/2]\max\{(1/\beta) + (\rho/\eta) - (1/r), 0\} - \widetilde{\alpha}\rho > 0$. Thus, by

Lemma 2.1 and Remark 2.2,

$$\begin{aligned}
&\|u(t)-T(t)u_\tau\|_{L^r_\delta} \le C\|a\|_{L^\alpha_\delta} \\
&\quad \times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-(1/r),0\}} \|u(s)\|^q_{L^\eta_\delta}\, ds \\
&\quad + C\|b\|_{L^\beta_\delta} \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\beta)+(\rho/\eta)-(1/r),0\}} (\|u(s)\|^\rho_{L^\eta_\delta}+1)\, ds \\
&\quad + \int_\tau^t \|g(x,s)\|_{L^r_\delta}\, ds \\
&\le C(t-\tau)^{1-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-(1/r),0\}-\tilde{\alpha}q} \\
&\quad + C(t-\tau)^{1-[(N+1)/2]\max\{(1/\beta)+(\rho/\eta)-(1/r),0\}-\tilde{\alpha}\rho} \\
&\quad + C(t-\tau)^{1-[(N+1)/2]\max\{(1/\beta)+(\rho/\eta)-(1/r),0\}} \\
&\quad + \int_\tau^t \|g(x,s)\|_{L^r_\delta}\, ds \longrightarrow 0, \quad \text{as } t \to \tau^+.
\end{aligned}$$

**Case b: Critical nonlinearity term** $(1/\beta) + [(\rho-1)/r] = [2/(N+1)]$, $r > 1$. The proof of this case is essentially the same as the previous case. Let $\eta$, $\widetilde{\alpha}$, $\widetilde{\beta}$ be the same as previously, and define

$$\begin{aligned}
&E = \{C((\tau,T), L^\eta_{\delta(x)}(\Omega)) : \lim_{t\to\tau}(t-\tau)^{\tilde{\alpha}}u(t)=0\} \bigcap C((\tau,T), W^{1,m(\eta)}_{0,\delta(x)}(\Omega)), \\
&X = \{u \in E : u(t) \ge \kappa\delta(x),\ (t-\tau)^{\tilde{\alpha}}\|u(t)\|_{L^\eta_\delta} \le \sigma, \\
&\qquad (t-\tau)^{\tilde{\beta}}\|\nabla(u(t)-T(t)u_\tau)\|_{L^{m(\eta)}_{\delta(x)}} \le 1,\ \text{for all } t \in (\tau,T)\},
\end{aligned}$$

where $0 < \sigma < 1$. Equip $X$ with the distance

$$\begin{aligned}
d(u,v) = \max\Big\{ &\sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}}\|u(t)-v(t)\|_{L^\eta_\delta}, \\
&\sup_{\tau<t<T} (t-\tau)^{\tilde{\beta}}\|\nabla(u(t)-v(t))\|_{L^{m(\eta)}_\delta}\Big\};
\end{aligned}$$

then $(X,d)$ is a nonempty complete metric space.

Here, we only show that $\Phi(X) \subset X$, and $\Phi$ is a strict contraction mapping on $X$, the regularity and continuous parts following as in the previous case. As the arguments of the previous case show,

$\Phi(u)(t) \geq T(t)u_0 \geq \kappa\delta(x)$, $\Phi(u)(t) - T(t)u_0 \in W^{1,m(\eta)}_{0,\delta(x)}(\Omega)$. Proceeding as in the derivation of (3.5) and (3.6),

(3.20)
$$\begin{aligned}
(t-\tau)^{\tilde{\alpha}}&\|\Phi(u)(t)\|_{L^\eta_\delta} \\
&\leq (t-\tau)^{\tilde{\alpha}}\|T(t)u_\tau\|_{L^\eta_\delta} \\
&\quad + (t-\tau)^{\tilde{\alpha}}\int_\tau^t \|T(t-s)[au^q(s) + f(x,u(s)) + g(x,s)]\|_{L^\eta_\delta}\,ds \\
&\leq (t-\tau)^{\tilde{\alpha}}\|T(t)u_\tau\|_{L^\eta_\delta} + C\|a\|_{L^\alpha_\delta}(t-\tau)^{\tilde{\alpha}} \\
&\quad \times \int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}}\|u(s)\|^q_{L^\eta_\delta}ds + C\|b\|_{L^\beta_\delta} \\
&\quad \times (t-\tau)^{\tilde{\alpha}}\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}(\|u(s)\|^\rho_{L^\eta_\delta}+1)\,ds \\
&\quad + C(t-\tau)^{1/\nu'}\|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))} \\
&\leq (t-\tau)^{\tilde{\alpha}}\|T(t)u_\tau\|_{L^\eta_\delta} \\
&\quad + C\|a\|_{L^\alpha_\delta}\sigma^q(t-\tau)^{1+\tilde{\alpha}(1-q)-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta],0\}} \\
&\quad + C\|b\|_{L^\beta_\delta}\sigma^\rho + C(t-\tau)^{1-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/r)]} \\
&\quad + C(t-\tau)^{1/\nu'}\|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))},
\end{aligned}$$

(3.21)
$$\begin{aligned}
(t-\tau)^{\tilde{\beta}}&\|\nabla(\Phi(u)(t) - T(t)u_0)\|_{L^{m(\eta)}_\delta} \\
&\leq C\|a\|_{L^\alpha_\delta}(t-\tau)^{\tilde{\beta}} \\
&\quad \times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}}\|u(s)\|^q_{L^\eta_\delta}ds \\
&\quad + C\|b\|_{L^\beta_{\delta(x)}}(t-\tau)^{\tilde{\beta}} \\
&\quad \times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}(\|u(s)\|^\rho_{L^\eta_\delta}+1)\,ds \\
&\quad + C(t-\tau)^{(1/\nu')-(1/2)-[(N+1)/2]\max\{(1/r)-(1/m),0\}+\tilde{\beta}} \\
&\quad \times \|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^r_{\delta(x)}(\Omega))}
\end{aligned}$$

$$\leq C\|a\|_{L^{\alpha}_{\delta}}(t-\tau)^{(1/2)+\tilde{\beta}-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[1/m(\eta)],0\}-\tilde{\alpha}q}\sigma^{q}$$
$$+ C\|b\|_{L^{\beta}_{\delta}}\sigma^{\rho} + C\|b\|_{L^{\beta}_{\delta}}(t-\tau)^{1-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/r)]}$$
$$+ C(t-\tau)^{(1/\nu')-(1/2)-[(N+1)/2]\max\{(1/r)-(1/m),0\}+\tilde{\beta}}$$
$$\times \|g(x,t)\|_{L^{\nu}_{\mathrm{loc}}([\tau,T];L^{r}_{\delta(x)}(\Omega))}.$$

Letting $u, v \in X$, similar to the derivation of (3.18) and (3.19),

(3.22)
$$(t-\tau)^{\tilde{\alpha}}\|\Phi(u)(t)-\Phi(v)(t)\|_{L^{\eta}_{\delta(x)}}$$
$$\leq C\|a\|_{L^{\alpha}_{\delta(x)}}$$
$$\times (t-\tau)^{1-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})}d(u,v)$$
$$+ C\|b\|_{L^{\beta}_{\delta(x)}}[\sigma^{\rho-1}+(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}]\,d(u,v),$$
$$(t-\tau)^{\tilde{\beta}}\|\Phi(u)(t)-\Phi(v)(t)\|_{W^{1,m(\eta)}_{0,\delta(x)}}$$
$$\leq C\|a\|_{L^{\alpha}_{\delta(x)}}$$
$$\times (t-\tau)^{(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/m(\eta)],0\}-q(\tilde{\alpha}-\tilde{\beta})}d(u,v)$$
$$+ C\|b\|_{L^{\beta}_{\delta}}\left[\sigma^{\rho-1}+(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}\right]\,d(u,v).$$

Now, we can choose $\sigma$ small enough and $0 < T-\tau < 1$ such that

$$(t-\tau)^{\tilde{\alpha}}\|\Phi(u)(t)\|_{L^{\eta}_{\delta}} \leq \sigma,$$
$$(t-\tau)^{\tilde{\beta}}\|\nabla(\Phi(u)(t)-T(t)u_0)\|_{L^{m}_{\delta}} \leq 1, \tag{3.23}$$
$$\text{for all } t \in (\tau, T).$$

Therefore, $\Phi(X) \subset X$. By Lemma 2.4, the choice of $T$ depends only upon the compact set $\mathcal{K} \subset L^{r}_{\delta(x)}(\Omega)$ which contains $u_\tau$. Moreover, from (3.22) and (3.23), we can shrink $\sigma$ in the above if necessary such that $d(\Phi(u),\Phi(v)) \leq (1/2)\,d(u,v)$. Therefore, $\Phi$ has a unique fixed point in $X$.

**Part two: Regularity of solution.** Here, we use the idea as in [**1, 5, 8, 12**]. From the proof of the existence of the solution, we know that

$$(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta)]}\|u(t)\|_{L^{\eta}_{\delta}} \leq C, \tag{3.24}$$

for all $t \in (\tau, T)$, where $\eta$ is same as in Part one.

Let $u$ be the solution obtained above, and write

$$
\begin{aligned}
u(t) &= T\bigg(\frac{t-\tau}{2}\bigg)u\bigg(\tau+\frac{t-\tau}{2}\bigg) \\
&\quad + \int_{\tau+[(t-\tau)/2]}^{t} T(t-s)[au^q(s)+f(x,u(s))+g(x,s)]\,ds.
\end{aligned} \tag{3.25}
$$

The assumption of $\alpha$ shows that $(1/\alpha)+[(q-1)/\eta] < (1/\alpha) < [2/(N+1)]$, which implies that $(1/\alpha)+(q/\eta)-[2/(N+1)] < (1/\eta)$. Furthermore, by Lemma 2.6 (iii), $(1/\beta)+(\rho/\eta)-[2/(N+1)] < (1/\eta)$. Therefore, there exists an $\eta' > \eta$ such that $(1/\alpha)+(q/\eta)-[2/(N+1)] < (1/\eta') < (1/\alpha)+(q/\eta) < 1$, $(1/\beta)+(\rho/\eta)-[2/(N+1)] < (1/\eta') < (1/\beta)+(\rho/\eta) < 1$, and $[(1/\eta)-(1/\eta')]\nu' < [2/(N+1)]$. Using (3.24), similar to the derivation of (3.5), from (3.25) we get that

(3.26)

$$
\begin{aligned}
&(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|u(t)\|_{L_\delta^{\eta'}} \\
&\le C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta)]}\Big\|u\bigg(\tau+\frac{t-\tau}{2}\bigg)\Big\|_{L_\delta^\eta} \\
&\quad + C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|a\|_{L_\delta^\alpha} \\
&\quad \times \int_{\tau+[(t-\tau)/2]}^{t} (t-s)^{-[(N+1)/2][(1/\alpha)+(q/\eta)-(1/\eta')]}\|u(s)\|_{L_\delta^\eta}^q ds \\
&\quad + C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|b\|_{L_\delta^\beta} \\
&\quad \times \int_{\tau+[(t-\tau)/2]}^{t} (t-s)^{-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/\eta')]}(\|u(\tau)\|_{L_\delta^\eta}^\rho+1)\,ds \\
&\quad + (t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]} \\
&\quad \times \int_{\tau+[(t-\tau)/2]}^{t} (t-s)^{-[(N+1)/2][(1/\eta)-(1/\eta')]}\|g(x,s)\|_{L_\delta^\eta} ds \\
&\le C + C(t-\tau)^{1-[(N+1)/2\alpha]+[(N+1)(1-q)]/2r}\|a\|_{L_\delta^\alpha} \\
&\quad \times \int_{1/2}^{1} (1-s)^{-[(N+1)/2][(1/\alpha)+(q/\eta)-(1/\eta')]} s^{-[(N+1)/2][(1/r)-(1/\eta)]q} d\tau \\
&\quad + C(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]}\|b\|_{L_\delta^\beta}
\end{aligned}
$$

$$\times \int_{1/2}^{1} (1-s)^{-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/\eta')]} s^{-[(N+1)/2][(1/r)-(1/\eta)]\rho} ds$$
$$+ C(t-\tau)^{1-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/r)]} \|b\|_{L^\beta_\delta}$$
$$\times \int_{1/2}^{1} (1-s)^{-[(N+1)/2][(1/\beta)+(\rho/\eta)-(1/\eta')]} ds$$
$$+ C(t-\tau)^{1+[(N+1)/2][(1/r)-(1/\eta)]} \|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^\infty(\Omega))}.$$

By Lemma 2.8 (ii), we get that $(1/\alpha) + [(q-1)/\eta] + [(1-q)/m(\eta)] < [1/(N+1)] - (1/\eta) + [1/m(\eta)]$, which implies $1 - [(N+1)/2][(1/\alpha) + [(q-1)/\eta] > 0$, and then $1 - [(N+1)/2\alpha] + [(N+1)(1-q)]/2r > 0$. Thus, (3.24) holds for some $\eta' > \eta$. We can bootstrap in a finite number steps to get that there exists a constant $C > 0$ such that

$$(t-\tau)^{(N+1)/2r} \|u(t)\|_{L^\infty} \le C. \tag{3.27}$$

Since $\|u(t)\|_{L^r_{\delta(x)}} \le M + 1$, using interpolation, we obtain that

$$(t-\tau)^{[(N+1)/2][(1/r)-(1/s)]} \|u(t)\|_{L^s_\delta} \le C, \quad r \le s \le \infty,\ \tau < t < T. \tag{3.28}$$

Similarly, from the proof of the existence of the solution, we get that

$$(t-\tau)^{[(N+1)/2][(1/r)+[1/(N+1)]-[1/m(\eta)]]} \|\nabla(u(t) - T(t)u_\tau)\|_{L^{m(\eta)}_\delta} \le 1, \quad \tau < t < T. \tag{3.29}$$

From (3.25), we get

$$u(t) - T(t)u_\tau = T\left(\frac{t-\tau}{2}\right)\left[u\left(\tau + \frac{t-\tau}{2}\right) - T\left(\tau + \frac{t-\tau}{2}\right)u_\tau\right] + \int_{\tau+[(t-\tau)/2]}^{t} T(t-s)[au^q(s) + f(x,u(s)) + g(x,s)]\, ds. \tag{3.30}$$

For $N > 1$, by Lemma 2.8, choose $\eta' > \eta$ such that $(1/\alpha) + (q/\eta) - [1/m(\eta')] < [1/(N+1)]$ and $0 \le (1/\beta) + (\rho/\eta) - [1/m(\eta')] <$

$[1/(N+1)]$. Since $m(\eta') > m(\eta)$, from (3.30) we get

$$
\begin{aligned}
(3.31)\quad & (t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|\nabla(u(t)-T(t)u_\tau)\|_{L^{m(\eta')}_{\delta(x)}} \\
& \le C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')-[1/m(\eta)]+[1/m(\eta')]]} \\
& \quad \times \left\|\nabla\left(u\left(\tau+\frac{t-\tau}{2}\right)-T\left(\tau+\frac{t-\tau}{2}\right)u_\tau\right)\right\|_{L^{m(\eta)}_\delta} \\
& \quad + C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|a\|_{L^\alpha_\delta} \\
& \quad \times \int_{\tau+[(t-\tau)/2]}^t (t-s)^{-(1/2)-[(N+1)/2][(1/\alpha)+(q/\eta)-[1/m(\eta')]]} \\
& \qquad \times \|u(s)\|^q_{L^\eta_\delta}\,ds \\
& \quad + C(t-\tau)^{[(N+1)/2][(1/r)-(1/\eta')]}\|b\|_{L^\beta_\delta} \\
& \quad \times \int_{\tau+[(t-\tau)/2]}^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta')]]} \\
& \quad \times (\|u(s)\|^\rho_{L^\eta_\delta}+1)\,ds + C\|g(x,t)\|_{L^\nu_{\mathrm{loc}}([\tau,T];L^\infty(\Omega))} \\
& \le C + C\|a\|_{L^\alpha_\delta}(t-\tau)^{1+[(N+1)/2][-(1/\alpha)+[(1-q)/r]]} \\
& \quad + C\|b\|_{L^\beta_\delta}(t-\tau)^{1-[(N+1)/2][(1/\beta)+[(\rho-1)/r]]} \le C.
\end{aligned}
$$

Thus, (3.29) holds for some $\eta' > \eta$. We can bootstrap in a finite number steps to get that there exists a constant $C > 0$ such that

$$(3.32)\qquad (t-\tau)^{[(N+1)/2r]}\|\nabla(u(t)-T(t)u_\tau)\|_{L^{N+1}_\delta} \le C.$$

**Part three: Uniqueness of solution.** We first show that the solution generated by the variation of constants formula is unique.

**Proposition 3.2.** *Suppose the assumptions of Theorem* 3.1 *hold, and let* $\eta$ *be the same as in Lemma* 2.6. *Then there exists a unique solution of problem*

$$(3.33)\quad u(t) = T(t)u_\tau + \int_\tau^t T(t-s)[au(\tau)^q + f(x,u(s)) + g(x,s)]\,ds$$

*in the class*

$$(3.34)\qquad L^\infty((\tau,T),L^\eta_{\delta(x)}(\Omega)) \bigcap L^\infty_{\mathrm{loc}}((\tau,T),W^{1,m(\eta)}_{0,\delta(x)}(\Omega))$$

*such that*

$$\operatorname*{sup\,ess}_{t\in(\tau,T)} (t-\tau)^{(1/2)+[(N+1)/2\eta]-[(N+1)/2m(\eta)]} \times \|u(t)-T(t)u_\tau\|_{W^{1,m(\eta)}_{0,\delta(x)}(\Omega)} < \infty, \tag{3.35}$$

*and* $u(t) \geq \kappa\delta(x)$ *for some* $\kappa > 0$.

*Proof.* Let $u, v$ be two solution of (3.33) in class (3.34), which satisfy (3.35) and $u(t), v(t) \geq \kappa\delta(x)$. Define

$$\psi(t) = \sup_{s\in[\tau,t]} \|u(s)-v(s)\|_{L^\eta_\delta} + \operatorname*{sup\,ess}_{s\in[\tau,t]} (s-\tau)^{\tilde\beta}\|u(s)-v(s)\|_{W^{1,m(\eta)}_{0,\delta(x)}},$$

and let $M = \operatorname*{sup\,ess}_{t\in[\tau,T]} \{\|u(t)\|_{L^\eta_{\delta(x)}}, \|v(t)\|_{L^\eta_{\delta(x)}}\}$.

From (3.33), we get

$$u(t)-v(t) = \int_\tau^t T(t-s)a[u^q(s)-v^q(s)]\,ds + \int_\tau^t T(t-s)[f(x,u(s))-f(x,v(s))]\,ds. \tag{3.36}$$

Similarly as in the derivation of (3.13), (3.15), (3.16) and (3.17), we have

(3.37)

$$\begin{aligned}\|I_1(t)\|_{L^\eta_\delta} &\leq C\|a\|_{L^\alpha_\delta}\int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \\ &\qquad \times \|u(s)-v(s))\|^q_{L^\eta_\delta}\|\nabla(u(s)-v(s))\|^{1-q}_{L^{m(\eta)}_\delta}\,ds \\ &\leq C\int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \\ &\qquad \times (s-\tau)^{-\tilde\beta(1-q)}\psi(s)\,ds,\end{aligned}$$

(3.38)
$$
\begin{aligned}
(t-\tau)^{\tilde{\beta}}\|I_1(t)\|_{W^{1,m(\eta)}_{0,\delta(x)}}
&\le C(t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/m(\eta)],0\}} \\
&\qquad\qquad \times \|u(s)-v(s)\|^q_{L^\eta_\delta}\|u(s)-v(s)\|^{1-q}_{W^{1,m(\eta)}_{0,\delta(x)}}\,ds \\
&\le C(t-\tau)^{\tilde{\beta}} \\
&\quad \times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/(m(\eta))],0\}} \\
&\qquad\qquad \times (s-\tau)^{-\tilde{\beta}(1-q)}\psi(s)\,ds,
\end{aligned}
$$

(3.39)
$$
\begin{aligned}
\|I_2(t)\|_{L^\eta_\delta} &\le C\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}\|u(s)-v(s))\|_{L^\eta_\delta} \\
&\qquad\qquad \times (\|u(s)\|^{\rho-1}_{L^\eta_\delta}+\|v(s)\|^{\rho-1}_{L^\eta_\delta}+1)\,ds \\
&\le C\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+(\rho-1)/\eta]} \\
&\qquad\qquad \times (s-\tau)^{-[(N+1)/2][(1/r)-(1/\eta)](\rho-1)}\psi(s)\,ds \\
&\quad + C\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}\psi(s)\,ds,
\end{aligned}
$$

(3.40)
$$
\begin{aligned}
(t-\tau)^{\tilde{\beta}}\|I_2(t)\|_{W^{1,m(\eta)}_{0,\delta(x)}} &\le C(t-\tau)^{\tilde{\beta}} \\
&\quad \times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}\|u(s)-v(s)\|_{L^\eta_\delta} \\
&\qquad\qquad \times (\|u(s)\|^{\rho-1}_{L^\eta_\delta}+\|v(s)\|^{\rho-1}_{L^\eta_\delta}+1)\,ds \\
&\le C(t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]} \\
&\qquad\qquad \times (s-\tau)^{-[(N+1)/2][(1/r)-(1/\eta)](\rho-1)}\psi(s)\,ds \\
&\quad + C(t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}\psi(s)\,ds.
\end{aligned}
$$

Thus, from (3.37)–(3.40) we get that

(3.41)

$$\begin{aligned}\psi(t) \le C\int_\tau^t (t-s)^{-[(N+1)/2]\max\{(1/\alpha)+[(q-1)/\eta]+[(1-q)/m(\eta)],0\}} \\ \times (s-\tau)^{-\tilde{\beta}(1-q)}\psi(s)\,ds \\ + C\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]} \\ \times (s-\tau)^{-[(N+1)/2][(1/r)-(1/\eta)](\rho-1)}\psi(s)\,ds \\ + C\int_\tau^t (t-s)^{-[(N+1)/2][(1/\beta)+[(\rho-1)/\eta]]}\psi(s)\,ds \\ + C(t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2]\max\{(1/\alpha)+(q/\eta)-[q/m(\eta)],0\}} \\ \times (s-\tau)^{-\tilde{\beta}(1-q)}\psi(s)\,ds + C(t-\tau)^{\tilde{\beta}} \\ \times \int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]}\psi(s)\,ds \\ + C(t-\tau)^{\tilde{\beta}}\int_\tau^t (t-s)^{-(1/2)-[(N+1)/2][(1/\beta)+(\rho/\eta)-[1/m(\eta)]]} \\ \times (s-\tau)^{-[(N+1)/2][(1/r)-(1/\eta)](\rho-1)}\psi(s)\,ds.\end{aligned}$$

By Lemma 2.5, from (3.41) we get that $\psi(t) = 0$, that is, $u(t) = v(t)$ for all $t \in [\tau, T]$. □

We now return to the proof of uniqueness of the solution. Here, we give the proof only in the critical case $(1/\beta) + [(p-1)/r] = [2/(N+1)]$, $r > 1, 0 < q < 1$; the other case is similar. Let $v(t) \in C([\tau,T], L^r_{\delta(x)}(\Omega)) \cap L^\infty_{\text{loc}}((\tau,T), L^\infty(\Omega))$ be a solution of (1.1) in the sense of (3.1) and $v(t) \ge \kappa\delta(x)$ for some $\kappa > 0$. We first show that $v(t) = u(t)$ on some interval $[\tau, T']$.

Let

$$\mathcal{K} = v([\tau, T]),$$

and

$$M = \sup_{\tau<t<T} \|v(t)\|_{L^r_\delta}.$$

Since $\mathcal{K}$ is a compact set in $L^r_{\delta(x)}(\Omega)$, the proof of the existence part shows that there exists a uniform $T_1 > \tau$ such that, for every $s \in (\tau, T)$, there exists a solution $v_s \in C([\tau, T_1], L^r_{\delta(x)}(\Omega))$ of (3.33) such that

$$v_s \in C((\tau, T_1), L^\eta_{\delta(x)}(\Omega)) \bigcap C((\tau, T_1), W^{1,m(\eta)}_{0,\delta(x)}(\Omega))$$

with $v_s(\tau) = v(s)$, $v_s \in X(T_1)$.

On the other hand, from (3.1), for $s \in (\tau, T)$ and $\tau < t < \tau + T - s$, we have

$$\begin{aligned} v(t) &= T(t-s)v(s) \\ &\quad + \int_{\tau+s}^{t+s} T(t+s-\theta)[av^q(\theta) + f(x, v(\theta)) + g(x,\theta)]\, d\theta. \end{aligned} \tag{3.42}$$

Proceeding as in the above, $v(t+s) \geq \kappa\delta(x)$ for $t \in (\tau, \tau + T - s)$. Similarly to the derivation of (3.20) and (3.21), using Lemma 2.4, we get that there exists a $T_s > 0$ such that

$$\begin{aligned} (t-s)^{\tilde{\alpha}} \|v(t)\|_{L^\eta_\delta} &\leq \sigma, \\ (t-s)^{\tilde{\beta}} \|\nabla(v(t) - T(t)v(s))\|_{L^{m(\eta)}_\delta} &\leq 1 \end{aligned} \tag{3.43}$$

hold for all $t \in [\tau, \min\{T_s, \tau + T - s\}]$. By uniqueness and Proposition 3.2, we get that $v_s(t) = v(t+s)$ for all $t \in [0, \min\{T_s, \tau+T-s, T_1\}]$. Arguing as (3.9) and (3.10), we get that $v$ is continuous. Let $s \to \tau^+$, and we deduce that

$$\begin{aligned} (t-\tau)^{\tilde{\alpha}} \|v(t)\|_{L^\eta_{\delta(x)}} &\leq \sigma, \\ (t-\tau)^{\tilde{\beta}} \|\nabla(v(t) - T(t)v(\tau))\|_{L^{m(\eta)}_{\delta(x)}} &\leq 1 \end{aligned} \tag{3.44}$$

for $t \in [\tau, \min\{T, T_1\}]$. Thus, $v(t) = u(t)$ for all $t \in [\tau, T']$ with $T' = \min\{T, T_1\}$. By the standard $L^\infty$ estimate and using Proposition 3.2, we get that $v(t)$ is a unique solution after $T'$. Therefore, $v(t) = u(t)$ in $[\tau, T]$ □

For $q = 1$, we have the following results.

**Theorem 3.3.** *Suppose that* $(\mathrm{P}_1)$–$(\mathrm{P}_3)$ *hold. Let* $a(x) \in L^{\alpha}_{\delta(x)}(\Omega)$, $b(x) \in L^{\beta}_{\delta(x)}(\Omega)$ *with* $1 < \alpha,\ \beta \le +\infty$. *Let* $1 \le r < +\infty$, $\alpha > (N+1)/2$, $(1/\alpha) + (1/r) \le 1 + [2/(N+1)\rho]$ *and* $(1/\beta) + [(\rho-1)/r] < [2/(N+1)]$ (*respectively,* $(1/\beta) + [(\rho-1)/r] = [2/(N+1)]$, $r > 1$). *Then, if* $u_\tau \in L^{r}_{\delta(x)}(\Omega)$, *then there exists a* $T = T(u_\tau) > 0$ *such that there is a unique solution* $u \in C([\tau, T], L^{r}_{\delta(x)}(\Omega))$ *of* (1.1).

*Moreover, there exists a* $C > 0$ *such that*

$$u(t) \in C((\tau, T), L^{s}_{\delta(x)}(\Omega)), \quad r \le s \le +\infty, \tag{3.45}$$

$$(t-\tau)^{[(N+1)/2][(1/r)-(1/s)]} \|u(t)\|_{L^{s}_{\delta(x)}} \le C, \quad \tau < t < T,\ r \le s \le +\infty. \tag{3.46}$$

*This solution is unique in the class*

$$C([\tau, T], L^{r}_{\delta(x)}(\Omega)) \bigcap L^{\infty}_{\mathrm{loc}}((\tau, T), L^{\infty}(\Omega)).$$

*Furthermore, for any bounded set* (*respectively, compact set*) $\mathcal{K}$ *in* $L^{r}_{\delta(x)}(\Omega)$, *there is a* (*uniform*) *time* $T = T(\mathcal{K})$ *such that, for any* $u_\tau \in \mathcal{K}$, *the solution of* (1.1) *exists on* $[\tau, T]$.

*Proof.* For the subcritical case $(1/\beta) + [(p-1)/r] < [2/(N+1)]$, let $X = \{C((\tau, T), L^{\eta}_{\delta(x)}(\Omega)) : \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t)\|_{L^{\eta}_{\delta}} \le M + 1\}$, where $\eta$ is given by Lemma 2.6 with $q = 1$, $\tilde{\alpha} = [(N+1)/2][(1/r) - (1/\eta)]$, and $M$ is a positive constant such that $\|u_\tau\|_{L^{r}_{\delta}} \le M$. For the critical case $(1/\beta) + [(p-1)/r] = [2/(N+1)]$, let $X = \{C((\tau, T), L^{\eta}_{\delta(x)}(\Omega)) : \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t)\|_{L^{\eta}_{\delta}} \le \sigma\}$, where $0 < \sigma < 1$. In both cases, equip $X$ with the distance $d(u, v) = \sup_{\tau<t<T} (t-\tau)^{\tilde{\alpha}} \|u(t) - v(t)\|_{L^{\eta}_{\delta}}$ for all $u(t), v(t) \in X$. The proof obtained is similar to that of Theorem 3.1, so we omit the details. ◻

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