

POSITIVE SOLUTIONS OF A SYSTEM OF COUPLED SECOND ORDER EQUATIONS WITH THREE POINT BOUNDARY CONDITIONS

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ABSTRACT. The aim of this paper is to present some existence results for a system of two coupled second order differential equations with three point nonlocal boundary conditions. The main tool used in the proofs is Guo-Krasnosel'skii's fixed point theorem. An example is included to illustrate the results.

1. Introduction. In this paper we are concerned with the existence of positive solutions of the system

$$(1.1) \quad \begin{cases} -u'' = \lambda_1 a_1(t) f_1(u, v) & t \in (0, 1), \\ -v'' = \lambda_2 a_2(t) f_2(u, v) & t \in (0, 1), \\ u(0) = \beta_1 u(\eta), \quad u(1) = \alpha_1 u(\eta), \\ v(0) = \beta_2 v(\eta), \quad v(1) = \alpha_2 v(\eta), \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ are positive parameters, $\alpha_1, \alpha_2 > 0, \beta_1, \beta_2 > 0, 0 < \eta < 1$ and $\alpha_i \eta < 1$ for $i = 1, 2$. By a positive solution to problem (1.1), we mean a vector-valued function $(u, v) \in C^1([0, 1], \mathbf{R}^2) := C^1([0, 1], \mathbf{R}) \times C^1([0, 1], \mathbf{R})$ satisfying (1.1), with $u, v \geq 0$ and $u + v > 0$ in $[0, 1]$.

The study of the existence of positive solutions of nonlinear differential equations under a variety of boundary conditions has generated a great deal of interest in the last several years. This is due in part to the fact that such boundary value problems (bvps) arise in applications (e.g., population dynamics, heat transfer) where only positive solutions have meaning. For example, Ma [13] considered the problem

$$\begin{cases} u'' + \lambda a(t) f(u) = 0 & t \in (0, 1), \\ u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \end{cases}$$

2010 AMS *Mathematics subject classification.* Primary 34B10.

Keywords and phrases. BVPs, Positive solution, nonlocal conditions, Guo-Krasnosel'skii's fixed point theorem.

Received by the editors on May 7, 2008, and in revised form on November 19, 2009.

DOI:10.1216/RMJ-2012-42-4-1169 Copyright ©2012 Rocky Mountain Mathematics Consortium

where $0 < \eta < 1$ and $0 < \alpha < 1/\eta$, and gave intervals on λ in which this problem had at least one positive solution $u(t)$. When $\beta = 0$, this problem was studied by Ma [14] and Raffoul [16]. For what has become the standard reference on such problems, we refer the reader to the monograph by Agarwal, O'Regan and Wong [1].

More recently, interest in obtaining the existence of positive solutions for systems has grown, and we cite as recent references the papers of Benchohra et al. [2], Henderson and Ntouyas [4, 5, 6], Henderson, Ntouyas and Purnaras [7], Henderson and Wang [8, 9], Hu and Wang [10], Liu, Liu and Wu [11], Ma [15], Wang [17] and Zhou and Xu [18]. For example, Henderson et al. [8] considered the four-point problem

$$\begin{cases} u'' + \lambda a(t)f(v) = 0 & t \in (0, 1), \\ v'' + \mu b(t)g(u) = 0 & t \in (0, 1), \\ u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta), \\ v(0) = \alpha v(\xi), \quad v(1) = \beta v(\eta), \end{cases}$$

with $0 < \xi < \eta < 1$ and $0 \leq \alpha, \beta < 1$. They too obtained the existence of at least one positive solution $(u(t), v(t))$ of this system.

Our interest here is to extend the results of Ma [13] to systems of the form (1.1).

We make the following assumptions.

(H₁) $a_1, a_2 \in C([0, 1], [0, +\infty))$ and there exists $x_0, x'_0 \in [0, 1]$, such that $a_1(x_0) > 0$ and $a_2(x'_0) > 0$.

(H₂) $f_1, f_2 \in C([0, +\infty) \times [0, +\infty), [0, +\infty))$ and there exist nonnegative constants f_1^0, f_1^∞, f_2^0 and f_2^∞ , such that

$$\begin{aligned} f_1^0 &= \lim_{u+v \rightarrow 0^+} \frac{f_1(u, v)}{u+v}, & f_1^\infty &= \lim_{u+v \rightarrow \infty} \frac{f_1(u, v)}{u+v}, \\ f_2^0 &= \lim_{u+v \rightarrow 0^+} \frac{f_2(u, v)}{u+v}, & f_2^\infty &= \lim_{u+v \rightarrow \infty} \frac{f_2(u, v)}{u+v}. \end{aligned}$$

Our approach is based on the following Guo-Krasnosel'skii's fixed point theorem in a cone.

Theorem 1.1 [3, Theorem 2.3.4, page 94]. *Let E be a Banach space and $K \subset E$ a cone in E . Assume that Ω_1 and Ω_2 are two bounded open*

sets in E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 - \Omega_1) \rightarrow K$ be a completely continuous operator such that either:

(i) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$,

or

(ii) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 - \Omega_1)$.

2. Preliminary lemmas. In this section, we present several lemmas that will be used in the proofs of our results.

Lemma 2.1 [12, 13]. Let $\beta \neq (1 - \alpha\eta)/(1 - \eta)$. Then, for $y \in C([0, 1], \mathbf{R})$, the boundary-value problem

$$(2.1) \quad \begin{cases} u'' + y(t) = 0 & t \in (0, 1), \\ u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta) \end{cases}$$

has a unique solution

$$\begin{aligned} u(t) = & - \int_0^t (t-s)y(s) ds \\ & + \frac{(\beta - \alpha)t - \beta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^\eta (\eta - s)y(s) ds \\ & + \frac{(1 - \beta)t + \beta\eta}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 (1 - s)y(s) ds. \end{aligned}$$

Lemma 2.2 [12, 13]. Let $0 < \alpha < 1/\eta$, $0 < \beta < (1 - \alpha\eta)/(1 - \eta)$. Then, for $y \in C([0, 1], \mathbf{R})$, and $y \geq 0$, the unique solution of problem (2.1) satisfies

$$u(t) \geq 0, \quad t \in [0, 1].$$

In what follows, by $\|u\|$ we will mean the sup norm in $C([0, 1], \mathbf{R})$, i.e.,

$$\|u\| = \sup_{t \in [0, 1]} |u(t)|.$$

Lemma 2.3 [12, 13]. *Let $0 < \alpha < 1/\eta$, $0 < \beta < (1 - \alpha\eta)/(1 - \eta)$. Then, for $y \in C([0, 1], \mathbf{R})$, and $y \geq 0$, the unique solution of the problem (2.1) satisfies*

$$\min_{t \in [0,1]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \min \left\{ \frac{\alpha(1 - \eta)}{1 - \alpha\eta}, \alpha\eta, \beta\eta, \beta(1 - \eta) \right\}.$$

Note that (u, v) is a solution of (1.1), if and only if

$$\begin{aligned} u(t) &= \lambda_1 \left[- \int_0^t (t - s)a_1(s)f_1(u(s), v(s)) ds \right. \\ &\quad + \frac{(\beta_1 - \alpha_1)t - \beta_1}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \int_0^\eta (\eta - s)a_1(s)f_1(u(s), v(s)) ds \\ &\quad \left. + \frac{(1 - \beta_1)t + \beta_1\eta}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \int_0^1 (1 - s)a_1(s)f_1(u(s), v(s)) ds \right] \\ &:= T_1(u, v)(t) \end{aligned}$$

and

$$\begin{aligned} v(t) &= \lambda_2 \left[- \int_0^t (t - s)a_2(s)f_2(u(s), v(s)) ds \right. \\ &\quad + \frac{(\beta_2 - \alpha_2)t - \beta_2}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \int_0^\eta (\eta - s)a_2(s)f_2(u(s), v(s)) ds \\ &\quad \left. + \frac{(1 - \beta_2)t + \beta_2\eta}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \int_0^1 (1 - s)a_2(s)f_2(u(s), v(s)) ds \right] \\ &:= T_2(u, v)(t). \end{aligned}$$

We will take E to be the Banach space $C([0, 1], \mathbf{R}^2) := C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$ endowed with the norm

$$(2.2) \quad \|(u, v)\| = \|u\| + \|v\|.$$

Let

$$\gamma_3 = \min\{\gamma_1, \gamma_2\}$$

where

$$\gamma_i = \min \left\{ \frac{\alpha_i(1 - \eta)}{1 - \alpha_i\eta}, \alpha_i\eta, \beta_i\eta, \beta_i(1 - \eta) \right\}, \quad i = 1, 2,$$

and define a cone in the space E as $K_1 \times K_2$ where

$$K_i = \left\{ u : u \in C[0, 1], u \geq 0, \min_{t \in [0, 1]} u(t) \geq \gamma_i \|u\| \right\}, \quad i = 1, 2.$$

Then, (u, v) is a positive solution of (1.1) if and only if it is a fixed point of the operator

$$(2.3) \quad T : K_1 \times K_2 \longrightarrow E = C([0, 1], \mathbf{R}^2), \quad T = (T_1, T_2).$$

By Lemmas 2.2 and 2.3, we know that

$$T(K_1 \times K_2) = \left(T_1(K_1 \times K_2), T_2(K_1 \times K_2) \right) \subset K_1 \times K_2,$$

and it is easy to verify that $T : K_1 \times K_2 \rightarrow K_1 \times K_2$ is completely continuous since $T_1 : K_1 \times K_2 \rightarrow K_1$ and $T_2 : K_1 \times K_2 \rightarrow K_2$ are completely continuous.

3. Main results. Throughout this paper, we shall use the following notation:

$$\begin{aligned} A_1 &= \frac{1 + \beta_1(1 + \eta)}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \int_0^1 (1 - s)a_1(s) ds, \\ B_1 &= \frac{\beta_1(1 - \eta)}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \int_0^\eta sa_1(s) ds, \\ A_2 &= \frac{1 + \beta_2(1 + \eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \int_0^1 (1 - s)a_2(s) ds, \\ B_2 &= \frac{\beta_2(1 - \eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \int_0^\eta sa_2(s) ds. \end{aligned}$$

Theorem 3.1. *Suppose that (H_1) – (H_2) hold,*

$$(3.1) \quad 0 < \alpha_i < 1/\eta$$

and

$$0 < \beta_i < (1 - \alpha_i \eta) / (1 - \eta) \quad \text{for } i = 1, 2,$$

and let p and q be positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(i) If $A_1 f_1^0 < \gamma_3 B_1 f_1^\infty$ and $A_2 f_2^0 < \gamma_3 B_2 f_2^\infty$, then for each

$$(\lambda_1, \lambda_2) \in \left(\frac{1}{p\gamma_3 B_1 f_1^\infty}, \frac{1}{pA_1 f_1^0} \right) \times \left(\frac{1}{q\gamma_3 B_2 f_2^\infty}, \frac{1}{qA_2 f_2^0} \right),$$

problem (1.1) has at least one positive solution.

(ii) If $f_1^0 = f_2^0 = 0$ and $f_1^\infty = f_2^\infty = \infty$, then for any

$$(\lambda_1, \lambda_2) \in (0, \infty) \times (0, \infty),$$

problem (1.1) has at least one positive solution.

(iii) If $f_1^\infty = f_2^\infty = \infty$ and $0 < f_1^0, f_2^0 < \infty$, then for each

$$(\lambda_1, \lambda_2) \in \left(0, \frac{1}{pA_1 f_1^0} \right) \times \left(0, \frac{1}{qA_2 f_2^0} \right),$$

problem (1.1) has at least one positive solution.

(iv) If $f_1^0 = f_2^0 = 0$ and $0 < f_1^\infty, f_2^\infty < \infty$, then for each

$$(\lambda_1, \lambda_2) \in \left(\frac{1}{p\gamma_3 B_1 f_1^\infty}, \infty \right) \times \left(\frac{1}{q\gamma_3 B_2 f_2^\infty}, \infty \right),$$

problem (1.1) has at least one positive solution.

Proof. Since the proof of (ii)–(iv) is similar to the proof of (i), we only prove (i). Let

$$(\lambda_1, \lambda_2) \in \left(\frac{1}{p\gamma_3 B_1 f_1^\infty}, \frac{1}{pA_1 f_1^0} \right) \times \left(\frac{1}{q\gamma_3 B_2 f_2^\infty}, \frac{1}{qA_2 f_2^0} \right)$$

and choose $\varepsilon > 0$ such that

$$(3.2) \quad \frac{1}{p\gamma_3 B_1(f_1^\infty - \varepsilon)} < \lambda_1 < \frac{1}{pA_1(f_1^0 + \varepsilon)}$$

and

$$(3.3) \quad \frac{1}{q\gamma_3 B_2(f_2^\infty - \varepsilon)} < \lambda_2 < \frac{1}{qA_2(f_2^0 + \varepsilon)}.$$

By the definition of f_1^0, f_2^0 , there exists $H_1 > 0$ such that

$$f_1(x, y) \leq (f_1^0 + \varepsilon)(x + y) \quad \text{for } x, y \geq 0 \text{ with } x + y \in [0, H_1],$$

and

$$f_2(x, y) \leq (f_2^0 + \varepsilon)(x + y) \quad \text{for } x, y \geq 0 \text{ with } x + y \in [0, H_1].$$

Let $(u, v) \in K_1 \times K_2$ with $\|(u, v)\| = H_1$; from (3.2) and (3.3), we obtain

$$\begin{aligned} T_1(u, v)(t) &\leq \frac{\lambda_1 \beta_1 t}{(1 - \alpha_1 \eta) - \beta_1 (1 - \eta)} \\ &\quad \times \int_0^\eta (\eta - s) a_1(s) f_1(u(s), v(s)) ds \\ &\quad + \frac{\lambda_1 (t + \beta_1 \eta)}{(1 - \alpha_1 \eta) - \beta_1 (1 - \eta)} \\ &\quad \times \int_0^1 (1 - s) a_1(s) f_1(u(s), v(s)) ds \\ &\leq \frac{\lambda_1 \beta_1}{(1 - \alpha_1 \eta) - \beta_1 (1 - \eta)} \\ &\quad \times \int_0^1 (1 - s) a_1(s) f_1(u(s), v(s)) ds \\ &\quad + \frac{\lambda_1 (1 + \beta_1 \eta)}{(1 - \alpha_1 \eta) - \beta_1 (1 - \eta)} \\ &\quad \times \int_0^1 (1 - s) a_1(s) f_1(u(s), v(s)) ds \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda_1(1 + \beta_1 + \beta_1\eta)}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_1(s)f_1(u(s), v(s)) ds \\
&\leq \frac{\lambda_1(1 + \beta_1 + \beta_1\eta)}{(1 - \alpha_1\eta) - \beta_1(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_1(s)(f_1^0 + \varepsilon)(u(s) + v(s)) ds \\
&\leq \lambda_1 A_1 (f_1^0 + \varepsilon) \|(u, v)\| \leq \frac{1}{p} \|(u, v)\|.
\end{aligned}$$

As a result, $\|T_1(u, v)\| \leq (1/p)\|(u, v)\|$.

Similarly, we have

$$\begin{aligned}
T_2(u, v)(t) &\leq \frac{\lambda_2\beta_2 t}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
&\quad \times \int_0^\eta (\eta - s)a_2(s)f_2(u(s), v(s)) ds \\
&\quad + \frac{\lambda_2(t + \beta_2\eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_2(s)f_2(u(s), v(s)) ds \\
&\leq \frac{\lambda_2\beta_2}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_2(s)f_2(u(s), v(s)) ds \\
&\quad + \frac{\lambda_2(1 + \beta_2\eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_2(s)f_2(u(s), v(s)) ds \\
&= \frac{\lambda_2(1 + \beta_2 + \beta_2\eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
&\quad \times \int_0^1 (1 - s)a_2(s)f_2(u(s), v(s)) ds \\
&\leq \frac{\lambda_2(1 + \beta_2 + \beta_2\eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)}
\end{aligned}$$

$$\begin{aligned} & \times \int_0^1 (1-s)a_2(s)(f_2^0 + \varepsilon)(u(s) + v(s)) ds \\ & \leq \lambda_2 A_2 (f_2^0 + \varepsilon) \|(u, v)\| \\ & \leq \frac{1}{q} \|(u, v)\|. \end{aligned}$$

Therefore, $\|T_2(u, v)\| \leq (1/q)\|(u, v)\|$.

Combining the above two inequalities, we obtain

$$\begin{aligned} \|T(u, v)\| &= \|T_1(u, v)\| + \|T_2(u, v)\| \\ &\leq \left(\frac{1}{p} + \frac{1}{q}\right) \|(u, v)\| \\ &= \|(u, v)\|. \end{aligned}$$

Let $\Omega_1 = \{(u, v) \in C([0, 1], \mathbf{R}^2) : \|(u, v)\| < H_1\}$; then

$$(3.4) \quad \|T(u, v)\| \leq \|(u, v)\|, \quad \text{for } u \in K \cap \partial\Omega_1.$$

Also, from the definitions of f_1^∞ and f_2^∞ , there exists a $\widehat{H}_2 > 0$ such that

$$f_1(x, y) \geq (f_1^\infty - \varepsilon)(x + y) \quad \text{for } x, y \geq 0 \text{ with } x + y \in [\widehat{H}_2, \infty),$$

and

$$f_2(x, y) \geq (f_2^\infty - \varepsilon)(x + y) \quad \text{for } x, y \geq 0 \text{ with } x + y \in [\widehat{H}_2, \infty).$$

Set $H_2 = \max\{2H_1, \widehat{H}_2/\gamma_3\}$, and let $\Omega_2 = \{(u, v) \in C([0, 1], \mathbf{R}^2) : \|(u, v)\| < H_2\}$. If $(u, v) \in K_1 \times K_2$ with $\|(u, v)\| = H_2$, then $\min_{t \in [0, 1]}(u(t) + v(t)) \geq \gamma_1\|u\| + \gamma_2\|v\| \geq \gamma_3\|(u, v)\| \geq \widehat{H}_2$. We then have

$$\begin{aligned} T_1(u, v)(0) &= -\frac{\lambda_1 \beta_1}{(1 - \alpha_1 \eta) - \beta_1(1 - \eta)} \\ &\quad \times \int_0^\eta (\eta - s)a_1(s)f_1(u(s), v(s)) ds \\ &\quad + \frac{\lambda_1 \beta_1 \eta}{(1 - \alpha_1 \eta) - \beta_1(1 - \eta)} \end{aligned}$$

$$\begin{aligned}
& \times \int_0^1 (1-s)a_1(s)f_1(u(s), v(s)) ds \\
& \geq -\frac{\lambda_1\beta_1}{(1-\alpha_1\eta) - \beta_1(1-\eta)} \\
& \times \int_0^\eta (\eta-s)a_1(s)f_1(u(s), v(s)) ds \\
& + \frac{\lambda_1\beta_1\eta}{(1-\alpha_1\eta) - \beta_1(1-\eta)} \\
& \times \int_0^\eta (1-s)a_1(s)f_1(u(s), v(s)) ds \\
& = \frac{\lambda_1\beta_1(1-\eta)}{(1-\alpha_1\eta) - \beta_1(1-\eta)} \\
& \times \int_0^\eta sa_1(s)f_1(u(s), v(s)) ds \\
& \geq \frac{\lambda_1\beta_1(1-\eta)}{(1-\alpha_1\eta) - \beta_1(1-\eta)} \\
& \times \int_0^\eta sa_1(s)(f_1^\infty - \varepsilon)(u(s) + v(s)) ds \\
& \geq \lambda_1\gamma_3B_1(f_1^\infty - \varepsilon)\|(u, v)\| \\
& \geq \frac{1}{p}\|(u, v)\|.
\end{aligned}$$

Consequently, $\|T_1(u, v)\| \geq (1/p)\|(u, v)\|$ for $(u, v) \in K_1 \times K_2 \cap \partial\Omega_2$.

Similarly, we have

$$\begin{aligned}
T_2(u, v)(0) & = -\frac{\lambda_2\beta_2}{(1-\alpha_2\eta) - \beta_2(1-\eta)} \\
& \times \int_0^\eta (\eta-s)a_2(s)f_2(u(s), v(s)) ds \\
& + \frac{\lambda_2\beta_2\eta}{(1-\alpha_2\eta) - \beta_2(1-\eta)} \\
& \times \int_0^1 (1-s)a_2(s)f_2(u(s), v(s)) ds \\
& \geq -\frac{\lambda_2\beta_2}{(1-\alpha_2\eta) - \beta_2(1-\eta)}
\end{aligned}$$

$$\begin{aligned}
 & \times \int_0^\eta (\eta - s)a_2(s)f_2(u(s), v(s)) ds \\
 & + \frac{\lambda_2\beta_2\eta}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
 & \times \int_0^\eta (1 - s)a_2(s)f_2(u(s), v(s)) ds \\
 & = \frac{\lambda_2\beta_2(1 - \eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
 & \times \int_0^\eta sa_2(s)f_2(u(s), v(s)) ds \\
 & \geq \frac{\lambda_2\beta_2(1 - \eta)}{(1 - \alpha_2\eta) - \beta_2(1 - \eta)} \\
 & \times \int_0^\eta sa_2(s)(f_2^\infty - \varepsilon)(u(s) + v(s)) ds \\
 & \geq \lambda_2\gamma_3B_2(f_2^\infty - \varepsilon)\|(u, v)\| \\
 & \geq \frac{1}{q}\|(u, v)\|.
 \end{aligned}$$

Thus, $\|T_2(u, v)\| \geq (1/q)\|(u, v)\|$ for $(u, v) \in K_1 \times K_2 \cap \partial\Omega_2$.

Combining the above two inequalities yields

$$\begin{aligned}
 \|T(u, v)\| &= \|T_1(u, v)\| + \|T_2(u, v)\| \\
 &\geq \left(\frac{1}{p} + \frac{1}{q}\right)\|(u, v)\| = \|(u, v)\|.
 \end{aligned}$$

It follows from part (i) of Theorem 1.1 that T has a fixed point (u, v) with $H_1 \leq \|(u, v)\| \leq H_2$ in $K_1 \times K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)$. \square

The proof of the following theorem is similar to that of Theorem 3.1, and so we omit the details.

Theorem 3.2. *Suppose that (H₁)–(H₂) hold,*

$$0 < \alpha_i < 1/\eta \quad \text{and} \quad 0 < \beta_i < (1 - \alpha_i\eta)/(1 - \eta) \quad \text{for } i = 1, 2,$$

and let p and q be positive numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(i) If $A_1 f_1^\infty < \gamma_3 B_1 f_1^0$ and $A_2 f_2^\infty < \gamma_3 B_2 f_2^0$, then for each

$$(\lambda_1, \lambda_2) \in \left(\frac{1}{p\gamma_3 B_1 f_1^0}, \frac{1}{pA_1 f_1^\infty} \right) \times \left(\frac{1}{q\gamma_3 B_2 f_2^0}, \frac{1}{qA_2 f_2^\infty} \right),$$

problem (1.1) has at least one positive solution.

(ii) If $f_1^0 = f_2^0 = \infty$ and $f_1^\infty = f_2^\infty = 0$, then for any

$$(\lambda_1, \lambda_2) \in (0, \infty) \times (0, \infty),$$

problem (1.1) has at least one positive solution.

(iii) If $f_1^0 = f_2^0 = \infty$ and $0 < f_1^\infty, f_2^\infty < \infty$, then for each

$$(\lambda_1, \lambda_2) \in \left(0, \frac{1}{pA_1 f_1^\infty} \right) \times \left(0, \frac{1}{qA_2 f_2^\infty} \right),$$

problem (1.1) has at least one positive solution.

(iv) If $f_1^\infty = f_2^\infty = 0$ and $0 < f_1^0, f_2^0 < \infty$, then for each

$$(\lambda_1, \lambda_2) \in \left(\frac{1}{p\gamma_3 B_1 f_1^0}, \infty \right) \times \left(\frac{1}{q\gamma_3 B_2 f_2^0}, \infty \right),$$

problem (1.1) has at least one positive solution.

Remark 3.1. There appears to be a misprint in the statement of parts (3) and (4) of Theorem 3.2 of Ma [13]. As written there, the resulting eigenvalue intervals are degenerate. In view of Theorem 3.2 above, it is easy to see what the correct statements should be.

Remark 3.2. It would be interesting to extend the results here to four-point problems like the one considered by Henderson et al. in [7], or to systems of higher order equations such as those considered by Henderson and Ntouyas [5], or to problems on time scales like the one considered by Luo and Ma [12].

4. Example. In this final section we present an example to illustrate the applicability of our results.

Example 4.1. Consider the boundary value problem

$$\begin{cases} -u'' = \lambda_1 (2 + 4t^2) \frac{(u+v)[1+270(u+v)]}{1+u+v} & t \in (0, 1), \\ -v'' = \lambda_2 (1 + t) \frac{(u+v)[567(u+v)^2 + e^{-(u+v)}]}{(u+v)^2 + 1} & t \in (0, 1), \\ u(0) = \frac{1}{2}u(\frac{2}{3}), \quad u(1) = \frac{1}{4}u(\frac{2}{3}), \\ v(0) = \frac{1}{3}v(\frac{2}{3}), \quad v(1) = \frac{3}{4}v(\frac{2}{3}), \end{cases}$$

where $\lambda_1, \lambda_2 > 0$ are positive parameters. We take $p = q = 2$. It is easy to check that (3.1) holds, $\gamma_1 = 1/10$, $\gamma_2 = 1/9$, $f_1^0 = 1$, $f_1^\infty = 270$, $f_2^0 = 1$ and $f_2^\infty = 567$. A simple calculation shows that $A_1 = 11/3$, $B_1 = 13/81$, $A_2 = 8/3$ and $B_2 = 52/567$. By part (i) of Theorem 3.1, the boundary value problem (4.1) has at least one positive solution if

$$(\lambda_1, \lambda_2) \in \left(\frac{3}{26}, \frac{3}{22} \right) \times \left(\frac{5}{52}, \frac{3}{16} \right).$$

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