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GAUSS'S THREE SQUARES THEOREM INVOLVING ALMOST-PRIMES

YINGCHUN CAI

ABSTRACT. Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. In this paper it is proved that, for every sufficiently large integer n satisfying the conditions $n \equiv 3 \pmod{24}$ and $5 \nmid n$, the equation $n = x_1^2 + x_2^2 + x_3^2$ is solvable, with solutions of the type $x_j = P_{106}$ $(j = 1, 2, 3)$, or of the type $x_1 x_2 x_3 = P_{304}$. These results constitute improvements upon the previous ones due to V. Blomer and to G.S. Lű, respectively.

1. Introduction. Gauss proved the classical three squares theorem, which states that all positive integers not of the form $4^k(8m+7)$ can be represented as the sum of three squares. Even more, the number of such representations can be given explicitly [**10**]. Up until now this result is still one of the most elegant in the circle of additive number theory.

It is conjectured that the three squares theorem still holds even if multiplicative structures are imposed on the variables. The strongest plausible conjecture in this respect concerns the sum of three squares of primes, as long as its validity is not precluded by local conditions. Here *local conditions* mean that

$$n \equiv 3 \pmod{24} \quad \text{and} \quad 5 \nmid n. \tag{1.1}$$

The local conditions are necessary here since, for prime $p > 5$, we have $p^2 \equiv 1 \pmod{24}$ and $p^2 \equiv \pm 1 \pmod{5}$.

This conjecture still remains open and is probably beyond the grasp of modern number theory. Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. Then the first approximation to this conjecture is due to Blomer and Brűdern [**2**].

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They showed that every sufficiently large integer n, which satisfies the local conditions (1.1), can be represented as the sum of three squares of P_r, with

$$r = \begin{cases} 371, & n \text{ is square-free,} \\ 521, & \text{otherwise.} \end{cases} \tag{1.2}$$

In their paper [**2**] Blomer and Brűdern combined the vector sieve in [**3**] with a mean value theorem which is deduced from the theory of theta-functions and modular forms.

In 2008 Blomer [**1**] refined the mean value theorem in [**2**] and showed that, for every sufficiently large n satisfying the conditions (1.1), the equation $n = x_1^2 + x_2^2 + x_3^2$ is solvable with x_1, x_2 and x_3 of the type P_{284}.

By a weighted sieve of dimension exceeding one and the mean value theorem in [**2**], Lű [**9**] proved that, for every sufficiently large integer n satisfying the local conditions (1.1), the equation $n = x_1^2 + x_2^2 + x_3^2$ is solvable, with $x_1x_2x_3 = P_r$, where

$$r = \begin{cases} 397, & n \text{ is square-free,} \\ 551, & \text{otherwise.} \end{cases} \tag{1.3}$$

Another topic about this conjecture involves the investigation of the exceptional set. Let $E(N)$ denote the number of positive integers not exceeding N, satisfying the local conditions (1.1) and not represented as the sum of three squares of primes. Then the first result in this direction goes to Hua [**6**], who proved in 1938 that $E(N) \ll N \log^{-A} N$ for some positive A, and the best result was obtained by Harman and Kumchev [**5**], $E(N) \ll N^{(6/7)+\varepsilon}$.

The aim of this paper is to show that the power of the vector sieve can be enhanced considerably by inserting a weighted process into it. By combing the weighted vector sieve with the mean value theorem developed by Blomer in [**1**], the following sharper results can be obtained, which constitute improvements upon that of Blomer and of Lű, respectively.

Theorem 1. *For every sufficiently large integer n satisfying the local conditions* (1.1)*, the equation*

$$n = x_1^2 + x_2^2 + x_3^2 \tag{1.4}$$

is solvable in square-free P_{106}, *and the number of solutions is* $\gg n^{(1/2)-\varepsilon}$ *for any* $\varepsilon > 0$.

Theorem 2. *Every sufficiently large integer* n, *which satisfies the local conditions* (1.1), *can be represented in the form*

$$n = x_1^2 + x_2^2 + x_3^2 \tag{1.5}$$

with $x_1x_2x_3 = P_{304}$, *and the number of representations is* $\gg n^{(1/2)-\varepsilon}$ *for any* $\varepsilon > 0$.

2. Some preliminary lemmas. In this paper, n denotes a sufficiently large integer satisfying the local condition (1.1). $\varepsilon \in (0, 10^{-10})$. The constants in O-terms and $\ll$-symbols depend at most upon ε. The letter p is reserved for prime numbers. Bold style letters denote vectors of dimension three. As usual, $\mu(n)$, $\varphi(n)$, $\tau(n)$, $\Omega(n)$ denote the Mőbius function, Euler's function, the number of divisors of n and the number of prime factors (counted according to multiplicity) of n, respectively. If $p^l \mid m$ but $p^{l+1} \nmid m$, then we write $p^l \| m$. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum_{x(q)}$ and $\sum_{x(q)*}$ sums with x running over a complete system and a reduced system of residues modulo q, respectively. If q is an odd integer, then by (l/q) we denote the Jacobi symbol. We denote by $\mathbf{N}$ the set of positive integers. For $\mathbf{d} = \langle d_1, d_2, d_3\rangle \in \mathbf{N}^3, \mathbf{l} = \langle l_1, l_2, l_3\rangle \in \mathbf{N}^3$, define $\mathbf{dl} = \langle d_1l_1, d_2l_2, d_3l_3\rangle$. The congruence $\mathbf{l} \equiv \mathbf{0} \pmod{\mathbf{d}}$ means that $l_j \equiv 0 \pmod{d_j}$, $j = 1, 2, 3$. Put

$$|\mathbf{d}| = \max_{1\le j\le 3} d_j,$$
$$\mu^2(\mathbf{d}) = \mu^2(d_1)\mu^2(d_3)\mu^2(d_3)$$

and

$$S(q, a) = \sum_{x(q)} e_q(ax^2),$$
$$S_{\mathbf{d}}(q, a) = \prod_{i=1}^{3} S(q, ad_i^2),$$

$$\begin{aligned}
\mathbf{d} &= \langle d_1, d_2, d_3 \rangle \in \mathbf{N}^3, \\
A(q, \mathbf{d}, n) &= \frac{1}{q^3} \sum_{a(q)*} S_{\mathbf{d}}(q, a) e_q(-an), \\
\mathfrak{S}(n, \mathbf{d}) &= \sum_{q=1}^{\infty} A(q, \mathbf{d}, n), \\
\mathfrak{S}(n) &= \mathfrak{S}(n, \langle 1, 1, 1 \rangle), \\
X &= \frac{\pi}{4} \mathfrak{S}(n) n^{1/2}.
\end{aligned}$$

By Siegel's theorem in [**10**] and the Hilfssätze 12 and 16 in Siegel [**11**], we have

$$\mathfrak{S}(n) \gg \frac{L(1, \chi_{-4n})}{\log\log n} \gg_{\varepsilon} n^{-\varepsilon} \tag{2.1}$$

for $n \equiv 3 \pmod 8$ and for all $\varepsilon > 0$. Hence, we may set

$$\omega(\mathbf{d}) = \omega(n, \mathbf{d}) = \frac{\mathfrak{S}(n, \mathbf{d})}{\mathfrak{S}(n)}.$$

For $p^{\theta} \| n$, $\theta \geq 1$, we define

$$f_{\theta}(p) = \begin{cases} p^{-1} - p^{-(1+\theta)/2} - p^{-(3+\theta)/2}, & \theta \equiv 1 \pmod 2, \\ p^{-1} - p^{-(2+\theta)/2} - \left(\frac{-np^{-\theta}}{p}\right) p^{-(2+\theta)/2}, & \theta \equiv 0 \pmod 2, \end{cases}$$

and

$$\omega_1(p) = \begin{cases} \frac{1+(-1/p)[(p-1)/p]+pf_{\theta}(p)}{1+f_{\theta}(p)}, & p \mid n, \\ \frac{p-(-1/p)}{p+(-n/p)}, & p \nmid n, \end{cases}$$

$$\omega_2(p) = \begin{cases} \frac{1+p^2 f_{\theta}(p)}{1+f_{\theta}(p)}, & p \mid n, \\ \frac{p(1+(n/p))}{p+(-n/p)}, & p \nmid n, \end{cases}$$

$$\omega_3(p) = \begin{cases} \frac{p+p^3 f_{\theta}(p)}{1+f_{\theta}(p)}, & p \mid n, \\ 0, & p \nmid n. \end{cases}$$

Lemma 1 (see [**2**]). *For* $\mathbf{d} \in \mathbf{N}^3$ *with* $\mu^2(\mathbf{d}) = 1$ *and* n *which satisfies the local condition* (1.1)*, we have*

$$\omega(\mathbf{d}) = \prod_{\substack{p^v \| d_1 d_2 d_3 \\ v \geq 1}} \omega_v(p).$$

Lemma 2 (see [**2**]). *For square-free* $d \in \mathbf{N}$ *and* n *satisfying the local condition* (1.1)*, set*

$$\omega(d) = \omega(d, n) = \prod_{p|d} \omega_1(p), \tag{2.2}$$

and, for $\mathbf{d} \in \mathbf{N}^3$ *with square-free components, put* $d_{i,j} = (d_i, d_j)$ *for* $1 \leq i < j \leq 3$*. Then the following statements hold.*

(i) *There exists a function* $g : \mathbf{N}^3 \to \mathbf{R}$ *such that, for any* $\mathbf{d} \in \mathbf{N}^3$ *with* $\mu^2(\mathbf{d}) = 1$*, we have*

$$\omega(\mathbf{d}) = \omega(d_1)\omega(d_2)\omega(d_3)g(d_{1,2}, d_{1,3}, d_{2,3}).$$

(ii) *There exists an absolute constant* $C > 0$ *such that, for any* $\mathbf{d} \in \mathbf{N}^3$ *such that* $\mu^2(\mathbf{d}) = 1$*, we have*

$$g(d_{1,2}, d_{1,3}, d_{2,3}) \leq \left(\max_{1 \leq i < j \leq 3} d_{i,j} \right)^C.$$

(iii) *For any* $\mathbf{d} \in \mathbf{N}^3$ *with* $\mu^2(\mathbf{d}) = 1$*, we have the inequality*

$$\omega(\mathbf{d}) \leq \widetilde{\omega}(d_1)\widetilde{\omega}(d_2)\widetilde{\omega}(d_3),$$

where $\widetilde{\omega}$ *denotes the multiplicative function defined on square-free integers by*

$$\widetilde{\omega}(p) = \begin{cases} p^{2/3}, & p \mid n, \\ 2, & p \nmid n. \end{cases}$$

(iv) *For the function* ω_1*, we have*

$$\omega_1(p) \le \begin{cases} 1 + (1/p), & \text{if } p \mid n \text{ and } p \equiv -1 \pmod 4, \\ 3, & \text{if } p \mid n \text{ and } p \equiv 1 \pmod 4, \\ [(p+1)/(p-1)], & \text{if } p \nmid n. \end{cases}$$

Lemma 3 (see [**1**]). *For a sufficiently large integer* n *satisfying the local condition* (1.1)*, let*

$$\mathscr{A} = \{\mathbf{x} \in \mathbf{N}^3 : x_1^2 + x_2^2 + x_3^2 = n\},$$

and for $\mathbf{d} \in \mathbf{N}^3$ *with square-free odd components, put*

$$\begin{aligned} \mathscr{A}_{\mathbf{d}} &= \{\mathbf{x} \in \mathscr{A} : \mathbf{x} \equiv \mathbf{0} \pmod{\mathbf{d}}\} \\ &= \{\mathbf{x} \in \mathbf{N}^3 : d_1^2 x_1^2 + d_2^2 x_2^2 + d_3^2 x_3^2 = n\} \\ &= \frac{\omega(\mathbf{d})}{d_1 d_2 d_3} X + R(n, \mathbf{d}), \\ \eta &= \frac{1}{192} - \varepsilon. \end{aligned}$$

Then we have

$$\sum_{|\mathbf{d}| \le n^\eta} \mu^2(\mathbf{d}) |R(n, \mathbf{d})| \ll n^{(1/2) - 4\varepsilon}, \tag{2.3}$$

$$X = \frac{\pi}{4} \mathfrak{S}(n) n^{1/2} \gg n^{(1/2) - \varepsilon}. \tag{2.4}$$

Lemma 4 (see [**2**]). *Let* $z_0 \ge 2$*. For* $\mathbf{l} \in \mathbf{N}^3$ *with square-free odd components and all prime factors of* $l_1 l_2 l_3$ *exceeding* z_0*, put*

$$\begin{aligned} S(\mathscr{A}_{\mathbf{l}}, z_0) &= \sharp\{\mathbf{x} \in \mathscr{A}_{\mathbf{l}} : p \mid x_1 x_2 x_3 \Rightarrow p \ge z_0\}, \\ \Omega'(p) &= 3\omega_1(p) - \frac{3\omega_2(p)}{p} + \frac{\omega_3(p)}{p^2}, \\ W(z) &= \prod_{p<z} \left(1 - \frac{\Omega'(p)}{p}\right), \\ H(n) &= \prod_{p \mid n} \left(1 + p^{-1/6}\right), \\ s_0 &= \frac{\log D_0}{\log z_0}, \\ E &= H^4(n) \Delta^{-1/2} \log^{19} D_0 + \Delta^c e^{-s_0} \log^L n, \end{aligned}$$

where c and L are some absolute constants. Then for $D_0 \geq z_0^2$ and $\Delta \geq 1$ we have

$$S(\mathscr{A}_{\mathbf{l}}, z_0) = (W(z_0) + O(E))\frac{\omega(\mathbf{l})}{l_1 l_2 l_3} X + O\Bigg(\sum_{\substack{|\mathbf{d}| \leq D_0 \\ p | d_1 d_2 d_3 \Rightarrow p < z_0}} \mu^2(\mathbf{d}) |R(n, \mathbf{dl})| \Bigg).$$

For a fixed $D \geq 1$ we define Rosser's weights $\lambda^{\pm}(d)$ of order D as follows: for $d = p_1 p_2 \cdots p_r$ with $p_1 > p_2 > \cdots > p_r$, let

$$\lambda^{+}(d) = \begin{cases} (-1)^r, & \text{if } p_1 p_2 \cdots p_{2l} p_{2l+1}^3 < D \\ & \text{whenever } 0 \leq l \leq (1/2)(r-1), \\ 0, & \text{otherwise,} \end{cases}$$

$$\lambda^{-}(d) = \begin{cases} (-1)^r, & \text{if } p_1 p_2 \cdots p_{2l} p_{2l}^3 < D \text{ whenever } 1 \leq l \leq (r/2), \\ 0, & \text{otherwise.} \end{cases}$$

Finally, put $\lambda^{\pm}(1) = 1$ and $\lambda^{\pm}(d) = 0$ if d is not square-free.

Lemma 5 (see [**3, 7, 8**]). *Let $\mathscr{P}$ denote a set of primes, and put*

$$P(z) = \prod_{\substack{p < z \\ p \in \mathscr{A}}} p.$$

Then, for Rosser's weights $\lambda^{\pm}(d)$ of order D, any integer $n \geq 1$ and real number $z \geq 2$, we have

$$\sum_{d | (n, P(z))} \lambda^{-}(d) \leq \sum_{d | (n, P(z))} \mu(d) \leq \sum_{d | (n, P(z))} \lambda^{+}(d). \tag{2.5}$$

For any multiplicative functions ω satisfying

$$\begin{cases} 0 < \omega(p) < p, & \textit{if } p \in \mathscr{P}, \\ \omega(p) = 0, & \textit{if } p \notin \mathscr{P}, \end{cases} \tag{2.6}$$

and

$$\prod_{w_1\le p<w_2}\left(1-\frac{\omega(p)}{p}\right)^{-1}\le\frac{\log w_2}{\log w_1}\left(1+\frac{L}{\log w_1}\right), \tag{2.7}$$

(*for all* $2\le w_1<w_2$, *where* L *is a positive constant*), *set*

$$V(z)=\prod_{p<z}\left(1-\frac{\omega(p)}{p}\right),\qquad s=\frac{\log D}{\log z}.$$

Then we have

(2.8)

$$V(z)\ge\sum_{d|P(z)}\lambda^-(d)\frac{\omega(d)}{d}\ge V(z)\left(f(s)+O(e^{\sqrt{L}-s}\log^{-(1/3)}D)\right),$$

for $2\le z\le D^{1/2}$, *and*

$$V(z)\le\sum_{d|P(z)}\lambda^+(d)\frac{\omega(d)}{d}\le V(z)\left(F(s)+O(e^{\sqrt{L}-s}\log^{-1/3}D)\right),$$

for $2\le z\le D$, *where* $f(s)$ *and* $F(s)$ *denote the classical functions in the linear sieve.*

Lemma 6 (see [**4**]). *For the functions* $f(s)$ *and* $F(s)$, *we have*

$$\begin{aligned}
sf(s)&=2e^{\gamma}\left(\log(s-1)+\int_2^{s-2}\frac{\log(t-1)}{t}\log\frac{s-1}{t+1}\,dt\right), && 4\le s\le 6;\\
sF(s)&=2e^{\gamma}, && 1\le s\le 3;\\
sF(s)&=2e^{\gamma}\left(1+\int_2^{s-1}\frac{\log(t-1)}{t}\,dt\right), && 3\le s\le 5;\\
sF(s)&=2e^{\gamma}\left(1+\int_2^{s-1}\frac{\log(t-1)}{t}\,dt\right.\\
&\qquad\left.+\int_2^{s-3}\frac{\log(t-1)}{t}\,dt\int_{t+2}^{s-1}\log\frac{u-1}{t+1}\frac{du}{u}\right), && 5\le s\le 7,
\end{aligned}$$

where $\gamma=0.577\ldots$ *denotes Euler's constant.*

3. Proof of the theorems. In the proof of the theorems we adopt the following notation. Let $\eta = 1/192$, and

$$D_0 = n^{\varepsilon}, \qquad D_1 = n^{\eta - 2\varepsilon}, \qquad D = D_0 D_1,$$
$$z_0 = \log^{1000} n, \qquad z_1 = D_1^{1/34}, \qquad z_2 = D_1^{33/34},$$
$$P_0 = \prod_{2<p<z_0} p, \qquad P_1 = \prod_{z_0 \le p < z_1} p, \qquad P = P_0 P_1,$$
$$g_0(p) = 1 - \frac{\log p}{\log z_2}, \qquad g(x) = \sum_{\substack{z_1 \le p < z_2 \\ p \mid x}} g_0(p),$$

$\lambda^{\pm}(d)$ Rosser's weights of order D_1,

$\lambda^{\pm(p)}(d)$ Rosser's weights of order $\dfrac{D_1}{p}$, $\qquad z_1 \le p < z_2$,

$$\Lambda_j = \sum_{l \mid (x_j, P_1)} \mu(l), \qquad \Lambda_j^{\pm} = \sum_{l \mid (x_j, P_1)} \lambda^{\pm}(l), \qquad j = 1, 2, 3,$$
$$\Lambda_j^{\pm(p)} = \sum_{l \mid (x_j, P_1)} \lambda^{\pm(p)}(l), \qquad z_1 \le p < z_2, \qquad j = 1, 2, 3.$$

Let $0 < \vartheta < 1$ denote a constant to be chosen later. For the proof of the theorems, we consider the sum

$$\begin{aligned} F &= \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3, P)=1}} \left(1 - \vartheta \sum_{j=1}^{3} g(x_j)\right) \\ &= \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3, P)=1}} 1 - \vartheta \sum_{j=1}^{3} \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3, P)=1}} g(x_j) \qquad (3.1) \\ &= F^{(0)} - \vartheta \sum_{j=1}^{3} F_j^{(1)} = F^{(0)} - \vartheta F^{(1)}. \end{aligned}$$

By the assumption $n \equiv 3 \pmod{24}$ we know that those solutions of (1.4) such that $2 \mid x_1x_2x_3$ are not counted in F. Next we show that for some $0 < \vartheta < 1$, F has a positive lower bound.

3.1. A lower bound for $F^{(0)}$. By the inequality

$$\Lambda_1\Lambda_2\Lambda_3 \ge \Lambda_1^-\Lambda_2^+\Lambda_3^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^- - 2\Lambda_1^+\Lambda_2^+\Lambda_3^+,$$

(see Lemma 4.2 in [**2**]), we have

$$F^{(0)} = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1}} \Lambda_1\Lambda_2\Lambda_3 \geq \sum_{j=1}^{3} F_j^{(0)} - 2F_4^{(0)}, \tag{3.2}$$

where

$$F_1^{(0)} = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1}} \Lambda_1^-\Lambda_2^+\Lambda_3^+,$$
$$F_2^{(0)} = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1}} \Lambda_1^+\Lambda_2^-\Lambda_3^+,$$
$$F_3^{(0)} = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1}} \Lambda_1^+\Lambda_2^+\Lambda_3^-,$$
$$F_4^{(0)} = \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1}} \Lambda_1^+\Lambda_2^+\Lambda_3^+.$$

Some trivial arrangements lead to

$$\begin{aligned} F_1^{(0)} &= \sum_{l_1,l_2,l_3|P_1} \lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3) \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ (x_1x_2x_3,P_0)=1 \\ \mathbf{x}\equiv\mathbf{0} \pmod{\mathbf{l}}}} 1 \\ &= \sum_{l_1,l_2,l_3|P_1} \lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3) S(\mathscr{A}_{\mathbf{l}}, z_0). \end{aligned} \tag{3.3}$$

Take

$$\Delta = H^8(n)\log^{240} n, \qquad s_0 = \frac{\log D_0}{\log z_0} = \frac{\varepsilon \log n}{1000 \log\log n},$$

in Lemma 4. Then we obtain

$$E = H^4(n)\Delta^{-1/2}\log^{19} D_0 + \Delta^c e^{-s_0}\log^L n = O\left(\frac{1}{\log^{100} n}\right), \tag{3.4}$$

where the bound $\log H(n) \ll \log^{5/6} n$ is used. By (3.4) and Lemma 4, we have

$$S(\mathscr{A}_{\mathbf{l}}, z_0) = \left(W(z_0) + O\left(\frac{1}{\log^{100} n}\right)\right)\frac{\omega(\mathbf{l})}{l_1 l_2 l_3}X + O\Bigg(\sum_{\substack{|\mathbf{d}|\le D_0 \\ p|d_1d_2d_3 \Rightarrow p<z_0}} \mu^2(\mathbf{d})|R(n,\mathbf{dl})|\Bigg). \tag{3.5}$$

By (3.3) and (3.5) we find that

$$\begin{aligned} F_1^{(0)} = {} & \left(W(z_0) + O\left(\frac{1}{\log^{100} n}\right)\right)X \\ & + \sum_{l_1,l_2,l_3|P_1} \lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(\mathbf{l})}{l_1l_2l_3} \\ & + O\Bigg(\sum_{\substack{|\mathbf{l}|\le D_1 \\ p|l_1l_2l_3 \Rightarrow z_0\le p<z_1}} \mu^2(\mathbf{l}) \sum_{\substack{|\mathbf{d}|\le D_0 \\ p|d_1d_2d_3\Rightarrow p<z_0}} \mu^2(\mathbf{d})|R(n,\mathbf{dl})|\Bigg). \end{aligned} \tag{3.6}$$

Since any positive integer m with the property $p|m \Rightarrow p < z_1$ can be decomposed into the form $m = m_1m_2$ with $p|m_1 \Rightarrow p < z_0$ and $p|m_2 \Rightarrow z_0 \le p < z_1$ uniquely, we have

$$\begin{aligned} & \sum_{\substack{|\mathbf{l}|\le D_1 \\ p|l_1l_2l_3 \Rightarrow z_0\le p<z_1}} \mu^2(\mathbf{l}) \sum_{\substack{|\mathbf{d}|\le D_0 \\ p|d_1d_2d_3\Rightarrow p<z_0}} \mu^2(\mathbf{d})|R(n,\mathbf{dl})| \\ & \qquad \ll \sum_{|\mathbf{d}|\le D} \mu^2(\mathbf{d})|R(n,\mathbf{d})| \ll n^{(1/2)-4\varepsilon}, \end{aligned} \tag{3.7}$$

in the last step Lemma 3 is used.

Write

$$\begin{aligned} G &= \sum_{l_1,l_2,l_3|P_1} \lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(\mathbf{l})}{l_1l_2l_3} \\ &= \Bigg(\sum_{\substack{l_1,l_2,l_3|P_1 \\ \mu^2(l_1l_2l_3)=1}} + \sum_{\substack{l_1,l_2,l_3|P_1 \\ \mu^2(l_1l_2l_3)=0}}\Bigg)\lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(\mathbf{l})}{l_1l_2l_3} \\ &= G_1 + G_2. \end{aligned} \tag{3.8}$$

By Lemma 2 (iii), we get

(3.9)
$$G_2 \ll \sum_{\substack{l_1,l_2,l_3|P_1\\(l_1,l_2)>1}} \frac{\widetilde{\omega}(l_1)\widetilde{\omega}(l_2)\widetilde{\omega}(l_3)}{l_1l_2l_3} \ll \sum_{\substack{d|P_1\\d\ge z_0}} \frac{\widetilde{\omega}^2(d)}{d^2} \sum_{l_1,l_2,l_3|P_1} \frac{\widetilde{\omega}(l_1)\widetilde{\omega}(l_2)\widetilde{\omega}(l_3)}{l_1l_2l_3}.$$

By Rankin's trick and Lemma 2 (iii), we find that

(3.10)
$$\begin{aligned}\sum_{\substack{d|P_1\\d\ge z_0}} \frac{\widetilde{\omega}^2(d)}{d^2} &\ll \sum_{d|P_1} \left(\frac{d}{z_0}\right)^{1/3} \frac{\widetilde{\omega}^2(d)}{d^2}\\ &\ll z_0^{-1/3} \prod_{p<z_1}\left(1+\frac{4}{p^{5/3}}\right) \prod_{p|n,p\ge z_0}\left(1+\frac{1}{p^{1/3}}\right) \ll z_0^{-1/3},\end{aligned}$$

and

$$\begin{aligned}\sum_{l_1,l_2,l_3|P_1} &\frac{\widetilde{\omega}(l_1)\widetilde{\omega}(l_2)\widetilde{\omega}(l_3)}{l_1l_2l_3}\\ &\ll \prod_{p<z_1}\left(1+\frac{2}{p}\right)^3 \prod_{p|n,z_0\le p<z_1}\left(1+\frac{1}{p^{1/3}}\right)^3 \ll \log^6 z_1.\end{aligned} \tag{3.11}$$

From (3.9)–(3.11), we get

$$G_2 = O(z_0^{-1/3}\log^6 z_1). \tag{3.12}$$

By Lemma 1 and (2.2), we have

(3.13)
$$\begin{aligned}G_1 &= \sum_{\substack{l_1,l_2,l_3|P_1\\ \mu^2(l_1l_2l_3)=1}} \lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(l_1)\omega(l_2)\omega(l_3)}{l_1l_2l_3}\\ &= \left(\sum_{l_1,l_2,l_3|P_1} - \sum_{\substack{l_1,l_2,l_3|P_1\\ \mu^2(l_1l_2l_3)=0}}\right)\lambda^-(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(l_1)\omega(l_2)\omega(l_3)}{l_1l_2l_3}\\ &= G_3 - G_4.\end{aligned}$$

By arguments similar to the estimation of G_2, we get

$$G_4 = O(z_0^{-1/3}\log^6 z_1). \tag{3.14}$$

It is easy to see that

$$G_3 = (I^+)^2 I^-, \tag{3.15}$$

where

$$I^{\pm} = \sum_{d|P_1} \frac{\lambda^{\pm}(d)\omega(d)}{d}. \tag{3.16}$$

It follows from (3.8) and (3.12)–(3.15) that

$$G = (I^+)^2 I^- + O(z_0^{-1/3} \log^6 z_1). \tag{3.17}$$

By Lemma 2 iv), it is easy to verify that assumptions (2.6)–(2.7) are satisfied by the function $\omega(p) = \omega_1(p)$ for $z_0 \le p < z_1$, so if we set

$$V(z_0, z_1) = \prod_{z_0 \le p < z_1} \left(1 - \frac{\omega_1(p)}{p}\right).$$

Then by (2.8)–(2.9) in Lemma 5, we have

$$V(z_0, z_1) \le I^+ \le V(z_0, z_1)(F(34) + O(\log^{-1/3} n)), \tag{3.18}$$

$$V(z_0, z_1) \ge I^- \ge V(z_0, z_1)(f(34) + O(\log^{-1/3} n)). \tag{3.19}$$

By the definitions of $\omega_v(p)$ and Lemma 2 iv), it is easy to verify that

$$\Omega'(p) \le \begin{cases} 3, & p \nmid n, \\ 7, & p \mid n,\ p \equiv 1 \pmod 4 \\ 1, & p \mid n,\ p \equiv -1 \pmod 4), \end{cases} \tag{3.20}$$

and hence $0 \le \Omega'(p) < p$ for n satisfying (1.1). Therefore, by Mertens prime formula and (3.20), we have

$$W(z_0) \gg \log^{-7} z_0 \gg \frac{1}{(\log\log n)^7}. \tag{3.21}$$

In a similar manner, by Lemma 2 iv) and Mertens prime formula, we find that

$$V(z_0, z_1) \gg \frac{\log z_0}{\log z_1} \gg \frac{\log\log n}{\log n}. \tag{3.22}$$

It follows from (3.18)–(3.19) and (3.22) that

$$I^{\pm} \gg V(z_0, z_1) \gg \frac{\log\log n}{\log n}. \tag{3.23}$$

Now, by (3.6)–(3.7), (3.17) and (3.21)–(3.23), we obtain

$$F_1^{(0)} = (1 + o(1))W(z_0)(I^+)^2 I^- X, \tag{3.24}$$

where (2.4) is employed. By symmetry, we get

$$F_j^{(0)} = (1 + o(1))W(z_0)(I^+)^2 I^- X, \quad j = 2, 3. \tag{3.25}$$

The same method leads to

$$F_4^{(0)} = (1 + o(1))W(z_0)(I^+)^3 X. \tag{3.26}$$

By (3.2) and (3.24)–(3.26), we get

$$F^{(0)} \geq (1 + o(1))W(z_0)(I^+)^2(3I^- - 2I^+)X. \tag{3.27}$$

From (3.18)–(3.19) and (3.27), we get

$$F^{(0)} \geq (1 + o(1))W(z_0)V(z_0, z_1)^3(3f(34) - 2F(34))X. \tag{3.28}$$

3.2. An upper bound for $F^{(1)}$. Since the arguments about $F^{(1)}$ are similar to those about $F_1^{(0)}$, we therefore present it in a sketchy manner. Let

$$\beta(l) = \sum_{\substack{k|P_1 \\ z_1 \leq p < z_2 \\ kp = l}} g_0(p)\lambda^{+(p)}(k). \tag{3.29}$$

Then by (2.5) and some routine arrangements, we have

(3.30)
$$
\begin{aligned}
F_1^{(1)} &= \sum_{\substack{x_1^2+x_2^2+x_3^2=n\\(x_1x_2x_3,P)=1}} g(x_1) = \sum_{z_1\le p<z_2} g_0(p) \sum_{\substack{x_1^2+x_2^2+x_3^2=n\\(x_1x_2x_3,P)=1\\x_1\equiv 0(\bmod p)}} 1\\
&\le \sum_{z_1\le p<z_2} g_0(p) \sum_{\substack{x_1^2+x_2^2+x_3^2=n\\(x_1x_2x_3,P_0)=1\\x_1\equiv 0(\bmod p)}} \Lambda_1^{+(p)}\Lambda_2^+\Lambda_3^+\\
&= \sum_{\substack{|\mathbf{l}|\le D_1\\ l_2,l_3|P_1}} \beta(l_1)\lambda^+(l_2)\lambda^+(l_3) \sum_{\substack{x_1^2+x_2^2+x_3^2=n\\(x_1x_2x_3,P_0)=1\\ \mathbf{x}\equiv\mathbf{0}\pmod{\mathbf{l}}}} 1\\
&= \sum_{\substack{|\mathbf{l}|\le D_1\\ l_2,l_3|P_1}} \beta(l_1)\lambda^+(l_2)\lambda^+(l_3) S(\mathscr{A}_{\mathbf{l}}, z_0)\\
&= \left(W(z_0) + O\left(\frac{1}{\log^{100} n}\right)\right)X\\
&\quad + \sum_{\substack{|\mathbf{l}|\le D_1\\ l_2,l_3|P_1}} \beta(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(\mathbf{l})}{l_1l_2l_3} + O(n^{(1/2)-4\varepsilon}),
\end{aligned}
$$

in the last step (3.5) and the argument leading to (3.7) which are applied. Let

$$I = \sum \beta(l)\frac{\omega(l)}{l}.$$

Then, by arguments similar to those about G, we have

$$\sum_{\substack{|\mathbf{l}|\le D_1\\ l_2,l_3|P_1}} \beta(l_1)\lambda^+(l_2)\lambda^+(l_3)\frac{\omega(\mathbf{l})}{l_1l_2l_3} = (I^+)^2 I + O(z_0^{-1/3}\log^6 z_1). \tag{3.31}$$

By the definition of $\beta(l)$ and (2.2) in Lemma 2, we find that

$$I = \sum_{\substack{z_1 \le p < z_2 \\ k | P_1}} \frac{g_0(p)\lambda^{+(p)}(k)\omega(pk)}{pk}$$

$$(3.32) \qquad = \sum_{\substack{z_1 \le p < z_2 \\ k | P_1}} \frac{g_0(p)\lambda^{+(p)}(k)\omega(p)\omega(k)}{pk}$$

$$= \sum_{z_1 \le p < z_2} \frac{g_0(p)\omega(p)}{p} I^{+(p)},$$

where we have set

$$(3.33) \qquad I^{+(p)} = \sum_{k | P_1} \frac{\lambda^{+(p)}(k)\omega(k)}{k}.$$

By arguments similar to those for $I^{\pm}$ and (2.9) in Lemma 5, we deduce that

$$(3.34) \qquad I^{+(p)} \le V(z_0, z_1)\left(F\left(\frac{\log D_1 p^{-1}}{\log z_1}\right) + O(\log^{-1/3} n)\right).$$

By (2.2), (3.34) and Lemma 2 iv), we get

$$I = \left(\sum_{\substack{z_1 \le p < z_2 \\ (p,n)=1}} + \sum_{\substack{z_1 \le p < z_2 \\ p | n}}\right) \frac{g_0(p)\omega(p)}{p} I^{+(p)}$$

$$(3.35) \qquad = \sum_{\substack{z_1 \le p < z_2 \\ (p,n)=1}} \frac{g_0(p)\omega_1(p)}{p} I^{+(p)} + O(z_1^{-1} V(z_0, z_1) \log n)$$

$$\le (1 + o(1)) V(z_0, z_1) \int_{1/34}^{33/34} \left(1 - \frac{34}{33} t\right) \frac{F(34(1-t))\, dt}{t},$$

in the last step the prime number theorem and summation by parts are employed.

By (2.4), (3.30), (3.31), (3.18) and (3.35), we conclude that

$$F_1^{(1)} \le (1 + o(1)) W(z_0) V(z_0, z_1)^3 X$$
$$\times F^2(34) \int_{1/34}^{33/34} \left(1 - \frac{34}{33} t\right) \frac{F(34(1-t))\, dt}{t},$$

and

$$
\begin{aligned}
F^{(1)} &= \sum_{j=1}^{3} F_j^{(1)} = 3F_1^{(1)} \\
&\le 3(1+o(1))W(z_0)V(z_0,z_1)^3 X \\
&\qquad \times F^2(34)\int_{1/34}^{33/34}\left(1-\frac{34}{33}t\right)\frac{F(34(1-t))\,dt}{t},
\end{aligned} \tag{3.36}
$$

where the symmetry between $F_1^{(1)}$, $F_2^{(1)}$ and $F_3^{(1)}$ is used.

3.3. Proof of the theorems. By Lemma 6 and numerical integration, we have

$$F(6) \le 1.00011, \qquad f(6) \ge 0.99989. \tag{3.37}$$

From (3.37) and the well-known monotonic properties of $F(s)$ and $f(s)$, we get

$$3f(34) - 2F(34) \ge 3f(6) - 2F(6) = 0.99945, \tag{3.38}$$

and

$$
\begin{aligned}
F^2(34)&\int_{1/34}^{33/34}\left(1-\frac{34}{33}t\right)\frac{F(34(1-t))dt}{t} \\
&\le 1.00011^2 \times \int_{28/34}^{33/34}\left(1-\frac{34}{33}t\right)\frac{F(34(1-t))\,dt}{t} \\
&\quad + 1.00011^3 \times \int_{1/34}^{28/34}\left(1-\frac{34}{33}t\right)\frac{dt}{t} \\
&\le 2.52902,
\end{aligned} \tag{3.39}
$$

where Lemma 6 and numerical integration are used.

By (3.28), (3.36), (3.38) and (3.39), we have

$$F^{(0)} \ge 0.99940W(z_0)V(z_0,z_1)^3 X, \tag{3.40}$$

$$F^{(1)} \le 7.58708W(z_0)V(z_0,z_1)^3 X. \tag{3.41}$$

Let $\vartheta = 0.1315$. Then (3.1), (3.40), (3.41), (3.21), (3.22) and (2.4) imply that

$$\begin{aligned} F &= F^{(0)} - \vartheta F^{(1)} \\ &> (0.99940 - 0.99756)W(z_0)V(z_0, z_1)^3 X \\ &\geq 0.0016 W(z_0)V(z_0, z_1)^3 X \\ &\gg n^{(1/2)-2\varepsilon}. \end{aligned} \tag{3.42}$$

Let F^+ denote the sub-sum of F which is composed of those terms such that

$$1 - \vartheta \sum_{j=1}^{3} g(x_j) > 0.$$

Then, by (3.42), we have

$$F^+ \geq F \gg n^{(1/2)-2\varepsilon}. \tag{3.43}$$

Let F_2^+ be that part of F^+ which consists of all terms such that $x_j \equiv 0(p^2)$ for some p and j, where $z_1 \leq p < n^{1/4}$, $1 \leq j \leq 3$. Then we find that

$$\begin{aligned} F_2^+ &\ll \sum_{z_1 \leq p < n^{1/4}} \sum_{\substack{x_1^2+x_2^2+x_3^2=n \\ x_1 \equiv 0(p^2)}} 1 \\ &\leq \sum_{z_1 \leq p < n^{1/4}} \sum_{\substack{x_1 \leq n^{1/2} \\ x_1 \equiv 0(p^2)}} \sum_{x_2^2+x_3^2=n-x_1^2} 1 \\ &\ll n^{\varepsilon} \sum_{z_1 \leq p < n^{1/4}} \sum_{\substack{x_1 \leq n^{1/2} \\ x_1 \equiv 0(p^2)}} 1 \\ &\ll n^{\varepsilon}(n^{1/2} z_1^{-1} + n^{1/4}) \\ &\ll n^{(1/2)-10\varepsilon}. \end{aligned} \tag{3.44}$$

By (3.43) and (3.44), we deduce that $\gg n^{(1/2)-2\varepsilon}$ triples (x_1, x_2, x_3) exist such that

$$\mu^2(\mathbf{x}) = \mu^2(x_1)\mu^2(x_2)\mu^2(x_3) = 1, \tag{3.45}$$

$$(x_1 x_2 x_3, P) = 1, \tag{3.46}$$

$$x_1^2 + x_2^2 + x_3^2 = n, \tag{3.47}$$

$$1 - \vartheta \sum_{j=1}^{3} g(x_j) > 0. \tag{3.48}$$

For any triples (x_1, x_2, x_3) satisfying (3.45)–(3.48), we have

$$\Omega(x_j) = \sum_{\substack{p \geq z_1 \\ p | x_j}} 1, \quad j = 1, 2, 3. \tag{3.49}$$

3.3.1. Proof of Theorem 1. For a triple (x_1, x_2, x_3) satisfying (3.45)–(3.48), it follows from (3.48) that

$$1 - \vartheta g(x_j) > 0, \quad j = 1, 2, 3,$$

and this implies that

$$\sum_{\substack{p \geq z_1 \\ p | x_j}} 1 < \frac{1}{\vartheta} + \frac{17}{33}(\eta - 2\varepsilon)^{-1}, \quad j = 1, 2, 3. \tag{3.50}$$

By (3.49) and (3.50), we find that, for any triples (x_1, x_2, x_3) which satisfy (3.45)–(3.48), we have

$$\Omega(x_j) = \sum_{\substack{p \geq z_1 \\ p | x_j}} 1 \leq 106, \quad j = 1, 2, 3. \tag{3.51}$$

Since $\gg n^{(1/2)-2\varepsilon}$, such triples (x_1, x_2, x_3) exist, by (3.51), and the proof of Theorem 1 is completed. □

3.3.2. Proof of Theorem 2. For a triple (x_1, x_2, x_3) satisfying (3.45)–(3.48), from (3.48), we find that

$$\sum_{j=1}^{3} \sum_{\substack{p \geq z_1 \\ p | x_j}} 1 < \frac{1}{\vartheta} + 3 \times \frac{17}{33}(\eta - 2\varepsilon)^{-1}. \tag{3.52}$$

By (3.49) and (3.52), we conclude that, for any triples (x_1, x_2, x_3) which satisfy (3.45)–(3.48), we have

$$\Omega(x_1 x_2 x_3) = \sum_{j=1}^{3} \Omega(x_j) = \sum_{j=1}^{3} \sum_{\substack{p \geq z_1 \\ p | x_j}} 1 \leq 304. \tag{3.53}$$

By (3.53), the Proof of Theorem 2 is completed. □

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Department of Mathematics, Tongji University, Shanghai 200092, China
Email address: yingchuncai@tongji.edu.cn