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ON G-(n,d)-RINGS

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ABSTRACT. The main aim of this paper is to investigate a new class of rings called, for positive integers n and d, $G-(n,d)-$rings, over which every n-presented module has a Gorenstein projective dimension at most d. We characterize n-coherent $G-(n,0)$-rings. We conclude with various examples of $G-(n,d)$-rings.

1. Introduction. Throughout this paper all rings are commutative with identity element and all modules are unital. If M is any R-module, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the usual projective, injective and flat dimensions of M. It is convenient to use "m-local" to refer to a (not necessarily Noetherian) ring with a unique maximal ideal m.

During 1967–69, Auslander and Bridger [**1, 2**] introduced the G-dimension for finitely generated modules over Noetherian rings. Several decades later, this homological dimension was extended, by Enochs and Jenda [**11, 12**], to the Gorenstein projective dimension of modules that are not necessarily finitely generated and over non-necessarily Noetherian rings. And, dually, they defined the Gorenstein injective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [**14**] introduced the Gorenstein flat dimension.

In the past few years, Gorenstein homological dimensions have become a vigorously active area of research (see [**4, 9, 11, 13, 17**] for more details). In 2004, Holm [**17**] generalized several results which had already been obtained over Noetherian rings.

The Gorenstein projective, injective and flat dimensions of a module are defined in terms of resolutions by Gorenstein projective, injective and flat modules, respectively.

Keywords and phrases. Gorenstein projective and flat modules, Gorenstein global dimension, trivial ring extension, subring retract, (n,d)-ring.
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Definition 1.1 [**17**]. 1. An R-module M is said to be Gorenstein projective if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that $M \cong \operatorname{Im}(P_0 \to P^0)$ and such that $\operatorname{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.

2. The Gorenstein injective modules are defined dually.

3. An R-module M is said to be Gorenstein flat if there exists an exact sequence of flat modules

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $M \cong \operatorname{Im}(F_0 \to F^0)$ and such that $- \otimes I$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective module.

Let R be a commutative ring, and let M be an R-module. For any positive integer n, we say that M is n-presented whenever there is an exact sequence:

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of R-modules in which each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules. We set $\lambda_R(M) = \sup\{n \mid M \text{ is } n\text{-presented}\}$, except that we set $\lambda_R(M) = -1$ if M is not finitely generated. Note that $\lambda_R(M) \geq n$ is a way to express the fact that M is n-presented.

Costa, [**10**], introduced a doubly filtered set of classes of rings in order to categorize the structure of non-Noetherian rings: for non-negative integers n and d, we say that a ring R is an (n, d)-ring if $\operatorname{pd}_R(M) \leq d$ for each n-presented R-module M. (n, d)-rings are known rings in some particular values of n and d. For example, R is a Noetherian (n, d)-ring, which means that R has global dimension $\leq d$. $(0, 0)$, $(1, 0)$, and $(0, 1)$-rings are, respectively, semi-simple, von Neumann regular and hereditary rings (see [**10**, Theorem 1.3]). According to Costa, [**10**], a ring R is called an n-coherent ring if every n-presented R-module is $(n+1)$-presented. For more results about (n, d)-rings see, for instance, [**10, 22, 23**].

The object of this paper is to extend the idea of Costa and introduce a doubly filtered set of classes of rings called $G-(n,d)-$rings and defined as follows:

Definition 1.2. Let n, $d \geq 0$ be integers. A ring R is called a $G-(n,d)$-ring if every n-presented R-module has a Gorenstein projective dimension at most d (i.e., $\lambda_R(M) \geq n$ implies $\mathrm{Gpd}_R(M) \leq d$).

In Section 2, we characterize some known rings by the $G-(n,d)$-property, for small values of n and d. Then, we study the transfer of this property into some particular ring extensions. In the main result of this section, we characterize n-coherent $G-(n,0)$-rings. Section 3 is devoted to examples. We give an example of a ring which is a $G-(n,d)$-ring but not an (n,d)-ring for any positive integers n and d. Also we give examples of $G-(n,0)$-rings which are not $G-(n-1,d)$-rings, for $n = 2, 3$ and for any positive integer d.

2. Main results. As in [**10**, Theorem 1.3], the $G-(n,d)$-property is used to characterize the rings of small Gorenstein global dimension. Recall, from [**5**], the Gorenstein global dimension of a ring R, denoted G-gldim(R), is defined as follows:

$$\text{G-gldim}(R) = \sup\{\mathrm{Gpd}_R(M) \mid M \ R\text{-module}\}.$$

Recall first the following rings:

Definition 2.1 [**7, 25, 26**]. Let R be a ring.

1. R is called G-semisimple if every R module is Gorenstein projective (= R is quasi-Frobenius).

2. R is called G-Von Neuman regular if every R-module is Gorenstein flat (= R is an **F**-ring).

3. R is called G-hereditary if G-gldim$(R) \leq 1$. Also R is called G-Dedekind if it is an integral domain G-hereditary.

4. R is called G-semi-hereditary if R is coherent and every submodule of a flat R-module is Gorenstein flat. Also R is called G-Prüfer if it is an integral domain G-semi-hereditary.

Recall that an R-module M is n-presented if there is an exact sequence:

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

such that each F_i is a finitely generated free R-module for $0 \leq i \leq n$. If $n = \infty$, we say that M is infinitely presented.

Theorem 2.2. *Let R be a ring. Then*:

1. *R is a $G-(0,0)$-ring if and only if R is* G-*semisimple.*
2. *R is a $G-(0,1)$-ring if and only if R is* G-*hereditary.*
3. *R is a $G-(0,d)$-ring if and only if* G-gldim$(R) \leq d$.
4. *R is a $G-(1,0)0$-ring if and only if R is* G-*von Neuman regular.*
5. *If R is coherent, then R is a $G-(1,1)$-ring if and only if R is* G-*semi-hereditary.*
6. *R is a $G-(0,1)$-domain if and only if R is* G-*Dedekind.*
7. *If R is coherent, then R is a $G-(1,1)$-domain if and only if R is* G-*Prüfer.*
8. *If R is Noetherian, then R is a $G-(n,d)$-ring if and only if* G-gldim$(R) \leq d$.

Proof. (1) Follows from [**7**, Proposition 2.1]. The assertions (2)–(6) and (7) follow respectively from [**25**, Proposition 2.3, Proposition 3.3, Definition 2.1 and Definition 3.1] and [**26**, Theorem 2.6]. Equation (3) follows from [**5**, Lemma 2.2]. Equation (8) follows from (3) and, since it is in a Noetherian ring R, every finitely generated R-module is infinitely presented. □

Remark 2.3. 1) An (n,d)-ring is a $G-(n,d)$-ring for any positive integers n and d. The converse is not true in general (see Example 3.2).

2) $G-(n,d)$-rings are $G-(n',d')$-rings for any $n' \geq n$ and $d \geq d'$. The converse is not true in general (see Theorem 3.1).

Recall that, for an extension of rings $A \subseteq B$, A is called a module retract of B if there exists an A-module homomorphism $f : B \rightarrow A$ such that $f_{/A} = \mathrm{id}_{/A}$. The homomorphism f is called a module retraction map. If such map f exists, B contains A as a direct summand A-module.

Proposition 2.4. *Let A be a subring retract of R, $R = A \oplus_A E$, such that E is a flat A-module and* G-gldim(A) *is finite. If R is a $G-(n, d)$-ring, then A is a $G-(n, d)$-ring.*

Proof. Let M be an n-presented A-module. Since R is a flat A-module, $M \otimes_A R$ is an n-presented R-module, and by hypothesis $\mathrm{Gpd}_R(M \otimes_A R) \leq d$. Then, $\mathrm{Gpd}_A(M) \leq d$ from [**24**, Proposition2.4]. ◻

Let A be a ring, and let E be an A-module. The trivial ring extension of A by E is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory (see, for instance, [**16, 18, 20, 22, 23**]).

A direct application of Proposition 2.4 is the following corollary.

Corollary 2.5. *Let A be a ring with* G-gldim$(A) < \infty$*, and let E be a flat A-module. If $R = A \propto E$ is a $G-(n, d)$-ring, then A is a $G-(n, d)$-ring.*

In the next result, we study the transfer of the $G-(n, d)$-property to the polynomial ring.

Theorem 2.6. *Let R be a ring and let X be an indeterminate over R.*

1. *Suppose that* G-gldim(R) *is finite. If $R[X]$ is a $G-(n, d)$-ring, then R is a $G-(n, d)$-ring.*

2. *If R is a $G-(n, d)$-ring which is not a $G-(n, d-1)$-ring, then $R[X]$ is not a $G-(n, d)$-ring.*

3. *Suppose that* G-gldim(R) *is finite. If $R[X]$ is a $G-(n, d)$-ring, then R is a $G-(n, d-1)$-ring.*

Proof. 1. Let M be an R-module such that $\lambda_R(M) \geq n$. Since $R[X]$ is a free R-module, we have $\lambda_{R[X]}(M[X]) \geq n$ and, by hypothesis, $\mathrm{Gpd}_{R[X]}(M[X]) \leq d$. From [**6**, Lemma 2.8], $\mathrm{Gpd}_R(M) \leq d$, and R is a $G-(n, d)-$ring as desired.

2. Since R is a $G-(n,d)$-ring which is not a $G-(n,d-1)$-ring, there exists an R-module M such that $\lambda_R(M) \geq n$ and $\mathrm{Gpd}_R(M) = d$. Then, from [**17**, Theorem], it is easy to see that there exists a free R-module F such that $\mathrm{Ext}^d_R(M,F) \neq 0$. on the other hand, M is also an $R[X]$-module via the canonical morphism: $R[X] \to R$. Hence, from [**28**, Lemma 9.29], there exists an exact sequence of $R[X]$-modules:

$$0 \longrightarrow M[X] \longrightarrow M[X] \longrightarrow M \longrightarrow 0,$$

from which we conclude that $\lambda_{R[X]}(M) \geq \lambda_{R[X]}(M[X])$. But since $R[X]$ is a flat R-module we see that $\lambda_{R[X]}(M[X]) \geq \lambda_R(M) \geq n$, and we have $\lambda_{R[X]}(M) \geq n$. Then, [**28**, Theorem 9.37] shows that:

$$\mathrm{Ext}^{d+1}_{R[X]}(M,F[X]) \cong \mathrm{Ext}^{d+1}_R(M,F) \neq 0.$$

It follows from [**17**, Theorem 2.20], that $\mathrm{Gpd}_{R[X]}(M) \geq d$. Finally, $R[X]$ is not a $G-(n,d)-$ring as desired.

3. Follows from (1) and (2) of the same theorem. □

In the next theorem we study the transfer of the $G-(n,d)$-property to the finite direct product of rings.

Theorem 2.7. *Let $R = R_1 \times R_2 \cdots \times R_m$ be a finite direct product of rings. If R is a $G-(n,d)$-ring, then R_i is a $G-(n,d)$-ring for each $i = 1,\dots,m$. The converse is true if* $\sup\{\text{G-gldim}(R_i) \mid i = 1,\dots,m\}$ *is finite.*

To prove this theorem we need the following lemma.

Lemma 2.8. *Let $R = R_1 \times R_2 \cdots \times R_m$ be a finite direct product of rings, and let $n \geq 0$ be an integer. Then, $M = \oplus_i M_i$ is an n-presented R-module if and only if M_i is an n-presented R_i-module for each $i = 1,\dots,m$.*

Proof. Follows from [**8**, Corollary 2.6.9]. □

Proof of Theorem 2.7. Let M_i be an R_i-module such that $\lambda_{R_i}(M_i) \geq n$; then, from Lemma 2.8 above, we have $\lambda_R(\oplus_i M_i) \geq n$ and by hypothesis $\mathrm{Gpd}_R(\oplus_i M_i) \leq d$. Hence, from [**6**, Lemma 3.2], $\mathrm{Gpd}_{R_i}(M_i) \leq d$.

Conversely, suppose that $\sup\{\text{G-gldim}(R_i) \mid i = 1, \dots, m\}$ is finite, and let $M = M_1 \oplus \cdots \oplus M_m$ be an n-presented R-module. Then, for each i, M_i is an n-presented R_i-module by Lemma 2.8. And, by hypothesis, we have $\text{Gpd}_{R_i}(M_i) \leq d$. Hence, from [**6**, Lemma 3.3], $\text{Gpd}_R(M) \leq \sup\{\text{Gpd}_{R_i}(M_i) \mid i = 1, \dots, m\} \leq d$. $\square$

The next result shows that a $G-(n,d)$-ring has grade at most d. This theorem is a generalization of [**10**, Theorem 1.4].

Theorem 2.9. *Let R be a $G-(n,d)$-ring. Then R contains no regular sequence of length $d+1$.*

Proof. Let $x_1, \dots, x_t$ be a regular sequence in R, where $I = \sum_{i=1}^t Rx_i \neq R$. Then, the Koszul complex defined by $\{x_1, \dots, x_t\}$ is a finite free resolution of R/I and hence R/I is n-presented for every n. Then, since R is a $G-(n,d)$-ring, we have $\text{Gpd}_R(R/I) \leq d$. But $\text{Gpd}_R(R/I) = t$ from [**21**, Exercise 1, page 127]. Therefore, $t \leq d$. $\square$

In the next result we study the locality of the $G-(n,d)$-property.

Proposition 2.10. *Let R be a ring with* G-gldim(R) *finite, and let n and d be positive integers such that $d \leq n-1$. If R is locally a $G-(n,d)$-ring, then R is also a $G-(n,d)$-ring.*

To prove this theorem we need the following result.

Lemma 2.11 ([**10**, Lemma 3.1]). *Let M be an R-module, and let S be a multiplicative subset of a system in R. If M has a finite n-presentation, then*:

$$S^{-1}\text{Ext}_R^i(M,N) \cong \text{Ext}_{S^{-1}R}^i(S^{-1}M, S^{-1}N)$$

for all $0 \leq i \leq n-1$, and $S^{-1}\text{Ext}_R^n(M,N)$ is isomorphic to a submodule of $\text{Ext}_{S^{-1}R}^n(S^{-1}M, S^{-1}N)$.

Proof of Proposition 2.10. Let M be an n-presented R-module, and let m be a maximal ideal of R. Then, M_m is an n-presented R_m-

module. Let P be a projective R-module; then $(\mathrm{Ext}^i_R(M,P))_m = \mathrm{Ext}_{R_m}(M_m,P_m) = 0$, and from [**28**, Theorem3.80], $\mathrm{Ext}^i_R(M,P) = 0$ for all $0 \leq i \leq n$. Therefore, $\mathrm{Gpd}_R(M) \leq d$. □

Now we give our main result of this section in which we give a characterization of n-coherent and a $G-(n,0)$-ring.

Theorem 2.12. *Let R be an n-coherent ring. Then the following conditions are equivalent.*

A) *R is a $G-(n,0)-$ring.*

B) *The following conditions hold*:

1. *Every finitely generated ideal of R has a nonzero annihilator.*

2. *For each infinitely presented R-module M,* $\mathrm{Gpd}_R(M) < \infty$.

3. *For every finitely generated Gorenstein projective submodule G of a finitely generated projective R-module P, P/G is Gorenstein projective.*

To prove this theorem we need the following lemma.

Lemma 2.13 ([**3**, Theorem 5.4]). *The following assertions are equivalent for a ring R*:

1. *Every finitely generated projective submodule of a projective R-module P is a direct summand of P.*

2. *Every finitely generated proper ideal of R has a nonzero annihilator.*

Proof of Theorem 2.12. $(A) \Rightarrow (B)$. Condition (2) is obvious.

We prove (1). Let P be a finitely generated submodule of Q, and both P and Q are projective. Let Q' be a projective R-module such that $Q \oplus_R Q' = F_0$ is a free R-module. Then there exists an exact sequence:

$$(*) \qquad 0 \longrightarrow P \longrightarrow F_0 \longrightarrow Q/P \oplus_R Q' \longrightarrow 0$$

On the other hand, since P is a finitely generated projective R-module, there exists a finitely generated free submodule F_1 of F_0 such that $P \subseteq F_1$ and $F_0 = F_1 \oplus_R F_2$. Thus, we see easily that P is infinitely presented and from the exact sequence:

$$0 \longrightarrow P \longrightarrow F_1 \longrightarrow F_1/P \longrightarrow 0;$$

$\mathrm{pd}_R(F_1/P) \leq 1$ and F_1/P is also infinitely presented, and by hypothesis F_1/P is Gorenstein projective, then it is projective. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & F_1 & \longrightarrow & F_1/P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & F_0 & \longrightarrow & Q/P \bigoplus_R Q' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_2 & = & F_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since F_2 and F_1/P are projective, the exact sequence $(*)$ splits and $Q \oplus_R Q' \cong P \oplus_R Q/P \oplus_R Q'$. Then $Q \cong P \oplus_R Q/P$ as desired.

To finish the proof of the first implication, it remains to prove that condition (3) holds. Let G be a finitely generated Gorenstein projective submodule of a finitely generated projective R-module P. Consider the exact sequence:

$$0 \longrightarrow G \longrightarrow P \longrightarrow P/G \longrightarrow 0.$$

It follows that $\mathrm{Gpd}_R(P/G) \leq 1$, and from [**17**, Theorem 2.10], there exists an exact sequence of R-modules:

$$(\star) \qquad 0 \longrightarrow K \longrightarrow H \longrightarrow P/G \longrightarrow 0$$

where K is projective and H is Gorenstein projective. But, by the proof of [**17**, Theorem 2.10], we may assume that K and H are finitely generated since G and P are finitely generated. Hence, combining (1) of this theorem with [**25**, Lemma 2.10], we conclude that K is a direct summand of H and the exact sequence $(\star)$ splits. Then P/G is Gorenstein projective as a direct summand of H.

$(B) \Rightarrow (A)$. Let M be an n-presented R-module. Since R is n-coherent, M is infinitely presented and $\mathrm{Gpd}_R(M)$ is finite. Let $\mathrm{Gpd}_R(M) = d$; then we have the exact sequence of R-modules:

$$0 \longrightarrow G \overset{u_{d-1}}{\longrightarrow} P_{d-1} \overset{u_{d-2}}{\longrightarrow} P_{d-2} \cdots \longrightarrow P_1 \overset{u_1}{\longrightarrow} P_0 \overset{u_0}{\longrightarrow} M \longrightarrow 0,$$

where P_i is a finitely generated projective for each i and G is a Gorenstein projective. Then we have the exact sequences of R-modules:

$$\begin{gathered} 0 \longrightarrow G(= \ker(u_{d-1})) \longrightarrow P_{d-1} \longrightarrow \mathrm{Im}\,(u_{d-1}) \longrightarrow 0, \\ 0 \longrightarrow \mathrm{Im}\,(u_i)(= \ker(u_{i-1})) \longrightarrow P_{i-1} \longrightarrow \mathrm{Im}\,(u_{i-1}) \longrightarrow 0 \\ \text{for } i = 2, \ldots, d-1, \\ 0 \longrightarrow \mathrm{Im}\,(u_1)(= \ker(u_0)) \longrightarrow P_0 \longrightarrow \mathrm{Im}\,(u_0) = M \longrightarrow 0. \end{gathered}$$

Then, by hypothesis and since G is a finitely generated Gorenstein projective submodule of a projective R-module P_{d-1}, we have $textIm\,(u_{d-1}) \cong P_{d-1}/G$ is a finitely generated Gorenstein projective R-module. Thus, by induction, we conclude that $M = \mathrm{Im}\,(u_0)$ is a finitely generated Gorenstein projective R-module, and this completes the proof. □

In the next proposition we study the relation between $G-(n,d)$-rings and $G-(n,0)$-rings.

Proposition 2.14. *Let R be a $G-(n,d)$-ring. Then R is a $G-(n,0)$-ring if and only if* $\mathrm{Ext}_R(M,K) = 0$ *for every n-presented R-module M and every R-module K with* $\mathrm{pd}_R(K) = \mathrm{Gpd}_R(M) - 1$.

Proof. $\Rightarrow$). Obvious.

$\Leftarrow$). Let M be an n-presented R-module. Since R is a $G-(n,d)$-ring, we have $\mathrm{Gpd}_R(M) \leq d$. And, from [**17**, Theorem 2.10], there exists an exact sequence:

$$(\star) \qquad 0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

where G is Gorenstein projective and $\mathrm{pd}_R(K) = \mathrm{Gpd}_R(M) - 1$. By hypothesis $\mathrm{Ext}_R(M, K) = 0$, and the exact sequence ($\star$) splits. Then, by [**17**, Theorem 2.5], M is Gorenstein projective as a direct summand of G. □

3. Examples. In this section, we construct a class of $G-(2,0)$-rings (respectively, $G-(3,0)$-rings) which are not $(1,d)$-rings (respectively, not $G-(2,d)$-rings) for every integer $d \geq 1$. Also we give an example of a $G-(n,d)$-ring which is not an (n,d)-ring for every integer $n, d \geq 0$.

In the next result we give an example of a $G-(2,0)$-ring which is not a $G-(1,d)$-ring. Also, we give an example of a $G-(2,d)$-ring which is neither a $G-(2,d-1)$-ring nor a $G-(1,d)$-ring for any integer $d \geq 0$. This theorem is a generalization [**22**, Theorem 3.4].

Theorem 3.1. *Let K be a field, and let $E(\cong K^\infty)$ be a K-vector space with infinite rank. Let $R := K \propto E$ be the trivial ring extension of K by E. Then*:

1. *R is a $G-(2,0)$-ring.*
2. *R is not a $G-(1,d)$-ring for every positive integer d.*
3. *Let S be a Noetherian ring with* G-gldim$(S) = d$. *Then, $T = R \times S$ is a $G-(2,d)$-ring but neither a $G-(1,d)$-ring nor a $G-(2,d-1)$-ring.*

Proof. 1. R is a $G-(2,0)$-ring since it is a $(2,0)$-ring from [**22**, Theorem 3.4].

2. Let d be a positive integer. We have to prove that R is not a $G-(1,d)$-ring. $M = 0 \propto E$ is the maximal ideal of R, and let $(0, e_i)_{i\in}$ be a set of generators of M. Consider the exact sequence of R-modules:

$$0 \longrightarrow M^{(I)} \longrightarrow R^{(I)} \longrightarrow M \longrightarrow 0;$$

from this exact sequence, we deduce that $\mathrm{Gpd}_R(M) = 0$ or $\mathrm{Gpd}_R(M) = \infty$. Suppose that $\mathrm{Gpd}_R(M) = 0$, and let $J = R(0,f) \cong 0 \propto K$ be a principal ideal of R. J is a direct summand of M; then $\mathrm{Gpd}_R(J) = 0$. Consider the exact sequence of R-modules:

$$0 \longrightarrow \ker(u) \longrightarrow R \xrightarrow{u} J \longrightarrow 0,$$

where $u((a,e)) = (a,e)(0,f) = (0,af)$. Then, $\ker(u) = \{(a,e) \in R \mid af = 0\}$. We can easily see that $\ker(u) = M$. Then, $M \cong R/J \cong K$; hence, K is a Gorenstein projective R-module. In particular $\mathrm{Ext}_R(K,R) = 0$, and R is self-injective $(0 = \mathrm{id}_K(E) = \mathrm{id}_R(R))$ from [**15**, Proposition 4.35], a contradiction. Indeed, R is not self-injective since $\mathrm{Ann}_R(\mathrm{ann}_R((J)) = M \neq J$ and from [**27**, Corollary 1.38]. Then $\mathrm{Gpd}_R(M) = \infty$. On the other hand, R/J is a 1-presented R-module and $\mathrm{Gpd}_R(M) = \mathrm{Gpd}_R(R/J) = \infty$. Finally, R is not a $G-(1,d)$-ring for each positive integer d.

3. Follows from Theorem 2.7 and Theorem 2.2 (8). □

Next we give an example of a $G-(n,d)$-ring which is not an (n,d)-ring for positive integers n and d.

Example 3.2. Let K be a field and $R = K \propto K$ the trivial ring extension of K by K. Then R is a $G-(n,d)$-ring but not an (n,d)-ring for positive integers n and d.

Proof. From [**7**, Theorem 3.7], R is a $G-(0,0)$-ring (=quasi-Frobenius); then, from Remark 2.3, R is a $G-(n,d)$-ring for positive integers n and d. And it follows from [**23**, Example 3.4] that R is not a (n,d)-ring. □

The next result generates an example of a $G-(3,0)$-ring which is not a $G-(2,d)$-ring for every integer $d \geq 0$. Also we give an example of a $G-(3,d)$-ring which is neither a $G-(3,d-1)$-ring nor a $G-(2,d)$-ring.

Theorem 3.3. *Let (A,M) be a local ring, and let $R = A \propto A/M$ be the trivial ring extension of A by A/M. Then*:

1. *If M is not finitely generated, then R is a $G-(3,0)$-ring.*

2. *If M contains a regular element, then R is not a $G-(2,d)$-ring, for every integer $d \leq 0$.*

3. *Let S be a Noetherian ring with* G-gldim$(S) = d$ *for some integer $d \geq 0$. Then, $T = R \times S$ is a $G-(3,d)$-ring which is neither a $G-(2,d)$ nor a $G-(3,d-1)$-ring.*

Proof. 1. Follows from [**19**, Theorem 1.1].

2. Suppose that M contains a regular element. Consider the exact sequence of R-modules:

$$(\star) \qquad 0 \longrightarrow M \propto A/M \longrightarrow R \longrightarrow R/(M \propto A/M) \longrightarrow 0.$$

We prove that $\mathrm{Gpd}_R(R/(M \propto A/M)) = \infty$. If not, $\mathrm{Gpd}_R(R/(M \propto A/M))$ is finite. From the exact sequence $(\star)$ and [**17**, Proposition 2.18], we have:

$$\mathrm{Gpd}_R(M \propto A/M) + 1 = \mathrm{Gpd}_R(R/(M \propto A/M)) \tag{1}$$

Let $(x_i)_{i\in I}$ be a set of generators of M, and let $R^{(I)}$ be a free R module. Consider the exact sequence of R-modules:

$$0 \longrightarrow \ker(u) \longrightarrow R^{(I)} \oplus_R R \xrightarrow{u} M \propto A/M \longrightarrow 0,$$

where

$$u((a_i,e_i)_{i\in I},(b_0,f_0)) = \sum_{i\in I}(a_i,e_i)(x_i,0) + (b_0,f_0)(0,1) = \sum_{i\in I}(a_i x_i, b_0),$$

since $x_i \in M$ for each $i \in I$. Hence,

$$\ker(u) = (U \propto (A/M)^{(I)}) \oplus_R (M \propto A/M),$$

where $U = \{(a_i)_{i\in I} \in A^{(I)} \mid \sum_{i\in I} a_i x_i = 0\}$. Therefore, we have the isomorphism of R-modules:

$$M \propto A/M \cong [R^{(I)}/(U \propto (A/M)^{(I)})] \oplus_R [R/(M \propto A/M)].$$

Hence, from [**17**, Proposition 2.19], we have:

$$\mathrm{Gpd}(R/(M \propto A/M)) \le \mathrm{Gpd}_R(M \propto A/M). \tag{2}$$

It follows from (1) and (2) that $\mathrm{Gpd}(R/(M \propto A/M)) = \mathrm{Gpd}_R(M \propto A/M) = \infty$. Now, from the exact sequence of R-modules:

$$0 \longrightarrow M \propto A/M \longrightarrow R \longrightarrow 0 \propto A/M \longrightarrow 0,$$

we conclude that $\mathrm{Gpd}_R(0 \propto A/M) = \infty$. On the other hand, let $m \in M$ be a regular element and $J = R(m,0)$ an ideal of R. Consider the following exact sequence of R modules:

$$0 \longrightarrow \ker(v) \longrightarrow R \xrightarrow{v} J \longrightarrow 0,$$

where $v((b,f)) = (b,f)(m,0) = (bm, mf)$. Since m is a regular element, we have $\ker(v) = 0 \propto A/M$. Therefore, it follows that $\mathrm{Gpd}_R(J) = \mathrm{Gpd}_R(0 \propto A/M) = \infty$. On the other hand, $0 \propto A/M$ is a finitely generated ideal of R; hence, J is a finitely presented ideal of R. Finally, the exact sequence of R-modules:

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0,$$

shows that $\lambda_R(R/J) \geq 2$ and $\mathrm{Gpd}_R(R/J) = \infty$. Then R is not a $(2,d)$-ring for each positive integer d.

3. Follows from Theorem 2.7 and Theorem 2.2 (8). □

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