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MULTIDIMENSIONAL WEIGHTED BOAS TYPE INEQUALITY AND RELATED RESULTS WITH APPLICATIONS

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ABSTRACT. An n-dimensional general, refined weighted Boas-type inequality for superquadratic functions and Hardy Littlewood averages is proved. Moreover, we apply this result to unify and refine the so-called strengthened general Hardy inequality with complement, deriving their new refinements as special cases of the general relation obtained. In particular, we get new refinements of the n-dimensional Hardy integral inequality from [**15**].

1. Introduction. To begin, we recall some well-known classical integral inequalities. If $p > 1$ and $f \in L^p(\mathbf{R}_+)$ is a non-negative function, then in [**5**] Hardy announced, and in [**6**] proved, a highly important classical integral inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx. \tag{1}$$

Inequality (1) has been discussed by several authors, who either gave its alternative proofs using different techniques, or applied, refined, generalized and reversed it in various ways [**3, 4, 8, 9, 17**]. In [**16**] Boas generalized (1) by proving a much more general inequality:

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \le \int_0^\infty \Phi\left(f(x)\right)\frac{dx}{x}, \tag{2}$$

for continuous convex functions $\Phi : [0,\infty) \to \mathbf{R}$, measurable non-negative functions $f : \mathbf{R}_+ \to \mathbf{R}$, and non-decreasing and bounded functions $m : [0,\infty) \to \mathbf{R}$ where $M = m(\infty) - m(0) > 0$ and the inner integral on the left hand side of (2) is a Lebesgue-Stieltjes integral with

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respect to m. After its author, relation (2) was named Boas's inequality [**16**, Chapter 8, Theorem 8.1]. In the case of a concave function Φ, (2) holds with a reversed sign of inequality. The weighted version of (1) with complement is discussed in [**14**]. For an historical background about Hardy's inequality see [**10, 11**]. In 1992 Pachpatte [**15**] discussed Hardy inequality in n independent variables having integrals over n-cells, that is, over an axis parallel to rectangular blocks in $\mathbf{R}_+$ as:

$$F(x_1,\dots,x_n)=\begin{cases}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n, & m>1,\\ \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\,dt_1\cdots dt_n, & m<1;\end{cases}$$

then

$$\begin{aligned}(3)\quad &\int_0^\infty\cdots\int_0^\infty (x_1\cdots x_n)^{-m}F^p\,dx_1\cdots dx_n\\ &\qquad\le\left[\frac{p}{|m-1|}\right]^{np}\int_0^\infty\cdots\int_0^\infty (x_1\cdots x_n)^{p-m}f^p dx_1\cdots dx_n.\end{aligned}$$

In [**12**] Oguntuase et al. gave some new multi-dimensional Hardy type inequalities generalizing the results in [**14**], and in [**7**] some new refinements and reverses of Hardy type inequalities are discussed. In this paper we prove a new n-dimensional weighted Boas type inequality and some related results for superquadratic and subquadratic functions, introduced by Abramovich et al. in [**1, 2**]. As a consequence, we observe that our results generalize and refine the results in [**15**]. Consider a general positive Borel measure λ on $\mathbf{R}_+$, such that:

$$(4)\qquad L=\lambda(\mathbf{R}_+)=\int_0^\infty d\lambda(t)<\infty.$$

On the other hand, for a finite Borel measure λ on $\mathbf{R}_+$, that is, having property (4), and a Borel measurable function $f:\mathbf{R}_+^n\to\mathbf{R}$, by Af we denote its Hardy-Littlewood average, defined in terms of the Lebesgue integrals as:

(5)

$$Af(x_1,\dots,x_n)=\frac{1}{L^n}\int_0^\infty\cdots\int_0^\infty f(t_1x_1,\dots,t_nx_n)\,d\lambda(t_1)\cdots d\lambda(t_n).$$

Definition 1 [**2**, Definition 2.1]. A function $\varphi:[0,\infty)\to\mathbf{R}$ is superquadratic provided that, for all $x\ge 0$, there exists a constant

$C_x \in \mathbf{R}$ such that:

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C_x(y - x)$$

for all $y \geq 0$. We say that f is *subquadratic* if $-f$ is *superquadratic.*

Lemma 1 [**2**, Lemma 3.1]. *Suppose $\varphi : [0, \infty)$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superquadratic or $(\varphi'(x))/x$ is non-decreasing, then φ is superquadratic.*

Lemma 2 [**2**, Theorem 2.3]. *Let (Ω, μ) be a probability measure space. The inequality*

$$\begin{aligned}\varphi\left(\int_\Omega f(s)\, d\mu(s)\right) \leq \int_\Omega \varphi(f(s))\, d\mu(s) \\ - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)\, d\mu(s)\right|\right) d\mu(s)\end{aligned} \tag{6}$$

holds for all probability measures μ and all non-negative μ-integrable functions f if and only if φ superquadratic. Moreover, (6) *holds in the reverse direction if and only if φ is subquadratic.*

This paper is organized in the following way. After the introduction, in Section 2 we prove a weighted multidimensional Boas-type inequality and general weighted multidimensional Hardy type inequality with complement, and in Section 3 we give applications of the results in Section 2 which generalize, unify and refine the results in [**15**].

2. Main results.

Theorem 1. *Let λ be a finite Borel measure on $\mathbf{R}_+$, let L defined by* (4) *be* 1 *and let $u, v, f : \mathbf{R}_+^n \to \mathbf{R}_+$ be measurable functions such that*:

$$v(x_1, \ldots, x_n) = \int_0^\infty \cdots \int_0^\infty u\left(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\right) d\lambda(t_1) \cdots d\lambda(t_n) < \infty.$$

If the real valued function Φ is superquadratic on $\mathbf{R}_+$, then

$$
\begin{aligned}
(7)\quad & \int_0^\infty \cdots \int_0^\infty u(x_1,\ldots,x_n) \\
& \Phi\left(\int_0^\infty \cdots \int_0^\infty f(t_1x_1,\ldots,t_nx_n)\, d\lambda(t_1)\cdots d\lambda(t_n)\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
& + \int_0^\infty \cdots \int_0^\infty u(x_1,\ldots,x_n) \int_0^\infty \cdots \int_0^\infty \Phi\left(\Big| f(t_1x_1,\ldots,t_nx_n)\right. \\
& \qquad \left. - \int_0^\infty \cdots \int_0^\infty f(t_1x_1,\ldots,t_nx_n)\, d\lambda(t_1)\cdots d\lambda(t_n)\Big|\right) \\
& \qquad\qquad \times d\lambda(t_1)\cdots d\lambda(t_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
& \le \int_0^\infty \cdots \int_0^\infty \Phi(f(x_1,\ldots,x_n))\, v(x_1,\ldots,x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.
\end{aligned}
$$

For the subquadratic function Φ, the sign of inequality in relation (7) *is reversed.*

Proof. By the refined Jensen's inequality (6)

$$
\begin{aligned}
& \int_0^\infty \cdots \int_0^\infty u(x_1,\ldots,x_n)\, \Phi(Af(x_1,\ldots,x_n))\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
& \le \int_0^\infty \cdots \int_0^\infty u(x_1,\ldots,x_n) \\
& \times \left[\int_0^\infty \cdots \int_0^\infty \Phi(f(t_1x_1,\ldots,t_nx_n))\, d\lambda(t_1)\ldots d\lambda(t_n)\right. \\
& \qquad - \int_0^\infty \cdots \int_0^\infty \Phi\left(\Big| f(t_1x_1,\ldots,t_nx_n) - \int_0^\infty \cdots\right. \\
& \times \left.\left. \int_0^\infty f(t_1x_1,\ldots,t_nx_n)\, d\lambda(t_1)\cdots d\lambda(t_n)\Big|\right) d\lambda(t_1)\cdots d\lambda(t_n)\right]\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.
\end{aligned}
$$

And

$$
\begin{aligned}
I_1 &= \int_0^\infty \cdots \int_0^\infty \left[\int_0^\infty \cdots \int_0^\infty \Phi(f(t_1x_1,\ldots,t_nx_n)) d\lambda(t_1)\cdots d\lambda(t_n)\right] \\
&\qquad \times u(x_1,\ldots,x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}. \\
&= \int_0^\infty \cdots \int_0^\infty \Phi(f(y_1,\ldots,y_n))\, v(y_1,\ldots,y_n)\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}.
\end{aligned}
$$

From here we get (7). ◻

Corollary 1. *Suppose* $b_i \in \mathbf{R}_+$, $u : \prod_{i=1}^n (0, b_i) \to \mathbf{R}_+$ *and* v_1 *is non-negative measurable function such that*:

$$v_1(x_1, \dots, x_n) = \int_0^{b_1} \cdots \int_0^{b_n} u\left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}\right) dt_1 \cdots dt_n < \infty.$$

If the real valued function Φ *is superquadratic on* $\mathbf{R}_+$ *and* $f : \mathbf{R}_+^n \to \mathbf{R}_+$ *is a measurable function, then*

$$\begin{aligned} (8) \quad & \int_0^{b_1} \cdots \int_0^{b_n} u(x_1, \dots, x_n) \\ & \times \Phi\left(\int_0^\infty \cdots \int_0^\infty f\Big(t_1 x_1, \dots, t_n x_n\Big) \, d\lambda(t_1) \cdots d\lambda(t_n) \right) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\ & + \int_0^{b_1} \cdots \int_0^{b_n} \cdots (x_1, \dots, x_n) \int_0^\infty \cdots \int_0^\infty \Phi\Bigg(\bigg| f(t_1 x_1, \dots, t_n x_n) \\ & \qquad - \int_0^\infty \cdots \int_0^\infty f(t_1 x_1, \dots, t_n x_n) \, d\lambda(t_1) \cdots d\lambda(t_n) \bigg| \Bigg) \\ & \qquad \qquad \times d\lambda(t_1) \cdots d\lambda(t_n) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\ & \le \int_0^{b_1} \cdots \int_0^{b_n} \Phi(f(x_1, \dots, x_n)) \, v_1(x_1, \dots, x_n) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}. \end{aligned}$$

For subquadratic function Φ, *the sign of inequality in relation* (8) *is reversed.*

Theorem 2. *Let* u *be a non-negative function on* $\mathbf{R}_+^n$ *such that the function* $(t_1, \dots, t_n) \mapsto [u(t_1, \dots, t_n)]/(t_1^2 \cdots t_n^2)$ *is locally integrable in* $\mathbf{R}_+^n$, *and let*

$$w_1(x_1, \dots, x_n) = x_1 \cdots x_n \int_{x_1}^\infty \cdots \int_{x_n}^\infty u(t_1, \dots, t_n) \frac{dt_1}{t_1^2} \cdots \frac{dt_n}{t_n^2} < \infty, \quad t_i \in \mathbf{R}_+, \quad 1 \le i \le n.$$

If the real valued function Φ *is superquadratic on* $\mathbf{R}_+$ *and* $f:\mathbf{R}_+^n\to\mathbf{R}_+$ *is a measurable function, then*

$$\begin{aligned}(9)\quad &\int_0^\infty\cdots\int_0^\infty u(x_1,\dots,x_n)\Phi(Hf(x_1,\dots,x_n))\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &\quad+\int_0^\infty\cdots\int_0^\infty \frac{u(x_1,\dots,x_n)}{x_1\cdots x_n}\\ &\quad\times\int_0^1\cdots\int_0^1\Phi\Big(|f(t_1x_1,\dots,t_nx_n)\\ &\qquad\qquad -Hf(x_1,\dots,x_n)|\Big)\,dt_1\cdots dt_n\,dx_1\cdots dx_n\\ &\leq\int_0^\infty\cdots\int_0^\infty\Phi\left(f(x_1,\dots,x_n)\right)w_1(x_1,\dots,x_n)\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\end{aligned}$$

holds for $Hf(x_1,\dots,x_n)$ *defined by*

$$Hf(x_1,\dots,x_n)=\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1,\dots,t_n)\,dt_1\cdots dt_n.$$

For a subquadratic function Φ, *the sign of inequality in relation* (9) *is reversed.*

Proof. Follows directly from Theorem 1, rewritten with the measure $d\lambda(t_i)=\chi_{[0,1]}(t_i)\,dt_i$, for all i. In this setting, we have

$$\begin{aligned}Af(x_1,\dots,x_n)&=\int_0^1\cdots\int_0^1 f(t_1x_1,\dots,t_nx_n)\,dt_1\cdots dt_n\\ &=Hf(x_1,\dots,x_n)\end{aligned}$$

and

$$\begin{aligned}v(x_1,\dots,x_n)&=\int_0^1\cdots\int_0^1 u\left(\frac{x_1}{t_1},\dots,\frac{x_n}{t_n}\right)dt_1\cdots dt_n\\ &=w_1(x_1,\dots,x_n)\end{aligned}$$

so (9) follows. ◻

Corollary 2. *Suppose* $0<b_i\leq\infty$, $1\leq i\leq n$ *and* $u:\prod_{i=1}^n(0,b_i)\to\mathbf{R}_+$ *is such that* $(s_1,\dots,s_n)\mapsto[u(s_1,\dots,s_n)]/(s_1^2\cdots s_n^2)$ *is locally integrable in* $\prod_{i=1}^n(0,b_i)$ *and* w_3 *is defined by*

$$w_3(x_1,\dots,x_n)=x_1\cdots x_n\int_{x_1}^{b_1}\cdots\int_{x_n}^{b_n}u(t_1,\dots,t_n)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2}<\infty.$$

If the real valued function Φ *is superquadratic on* (a,c), $0 \le a < c < \infty$, *then*

$$\begin{aligned}(10)\quad &\int_0^{b_1}\cdots\int_0^{b_n} u(x_1,\dots,x_n)\\ &\times \Phi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &+\int_0^{b_1}\cdots\int_0^{b_n}\cdots\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\frac{u(x_1,\dots,x_n)}{x_1^2\cdots x_n^2}\\ &\times\Phi\left(\left|f(t_1,\dots,t_n)-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right|\right)\\ &\qquad\times dx_1\cdots dx_n\,dt_1\cdots dt_n\\ &\le\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(x_1,\dots,x_n)\right)w_3(x_1,\dots,x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\end{aligned}$$

holds for all f *with* $a \le f(x_1,\dots,x_n) \le c$, $0 \le x_i \le b_i$, $1 \le i \le n$. *Moreover,* (10) *holds in reverse order for subquadratic function* Φ.

Remark 1. In particular, if the weight function u is chosen to be $u(x_1,\dots,x_n) \equiv 1$, in Corollary 2, then

$$w_3(x_1,\dots,x_n)=\begin{cases}\prod_{i=1}^n\left(1-(x_i/b_i)\right) & b_i<\infty;\\ 1 & b_i=\infty,\end{cases}$$

so for $b_i < \infty$, $1 \le i \le n$, inequality (10) reads

$$\begin{aligned}(11)\quad &\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\right.\\ &\qquad\left.\times\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &+\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\\ &\times\int_{t_n}^{b_n}\Phi\left(\left|f(t_1,\dots,t_n)-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\right.\right.\\ &\qquad\left.\left.\times\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right|\right)\frac{dx_1\cdots dx_n}{x_1^2\cdots x_n^2}\end{aligned}$$

$$\times\, dt_1 \cdots dt_n \le \int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(f(x_1, \ldots, x_n)\right)$$
$$\times \left[1 - \frac{x_1}{b_1}\right] \cdots \left[1 - \frac{x_n}{b_n}\right] \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

and, for $b_i = \infty$, $1 \le i \le n$,

$$\begin{aligned} (12)\quad & \int_0^{\infty} \cdots \int_0^{\infty} \Phi\Bigg(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \\ & \qquad \times \int_0^{x_n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n\Bigg) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\ & + \int_0^{\infty} \cdots \int_0^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} \int_0^{x_n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n \Bigg|\Bigg) \\ & \qquad \times \frac{dx_1 \cdots dx_n}{x_1^2 \cdots x_n^2}\, dt_1 \cdots dt_n \\ & \le \int_0^{\infty} \cdots \int_0^{\infty} \Phi\left(f(x_1, \ldots, x_n)\right) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}. \end{aligned}$$

Theorem 3. *Suppose $u : \mathbf{R}_+^n \to \mathbf{R}_+$ is a locally integrable function and w_2 is defined on $\mathbf{R}_+^n$ by*

$$w_2(x_1, \ldots, x_n) = \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} u\left(t_1, \ldots, t_n\right) dt_1 \cdots dt_n < \infty.$$

If the real valued function Φ is superquadratic on $\mathbf{R}_+$ and $f : \mathbf{R}_+^n \to \mathbf{R}$ is a measurable function, then

$$\begin{aligned} (13)\quad & \int_0^{\infty} \cdots \int_0^{\infty} u(x_1, \ldots, x_n) \Phi\left(\widetilde{H} f(x_1, \ldots, x_n)\right) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\ & + \int_0^{\infty} \cdots \int_0^{\infty} u(x_1, \ldots, x_n) \int_1^{\infty} \cdots \\ & \times \int_1^{\infty} \Phi\left(\left|f(t_1 x_1, \ldots, t_n x_n) - \widetilde{H} f(x_1, \ldots, x_n)\right|\right) \\ & \qquad \times \frac{dt_1}{t_1^2} \cdots \frac{dt_n}{t_n^2} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\ & \le \int_0^{\infty} \cdots \int_0^{\infty} \Phi\left(f(x_1, \ldots, x_n)\right) w_2(x_1, \ldots, x_n) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \end{aligned}$$

holds for $\widetilde{H} f(x_1, \ldots, x_n)$ defined by

$$\widetilde{H}f(x_1,\dots,x_n) = x_1\cdots x_n \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2}$$
$$\text{for } x_i > 0, \quad 1 \le i \le n.$$

For subquadratic function Φ*, the sign of inequality in relation* (13) *is reversed.*

Proof. Consider $d\lambda(t_i) = \chi_{[1,\infty)}(t_i)(dt_i/t^2)$, for all i in Theorem 1 so that

$$\begin{aligned} Af(x_1,\dots,x_n) &= \int_1^{\infty}\cdots\int_1^{\infty} f(t_1x_1,\dots,t_nx_n)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2} \\ &= \widetilde{H}f(x_1,\dots,x_n). \end{aligned}$$

And

$$\begin{aligned} v(x_1,\dots,x_n) &= \int_1^{\infty}\cdots\int_1^{\infty} u\left(\frac{x_1}{t_1},\dots,\frac{x_n}{t_n}\right)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2} \\ &= w_2(x_1,...,x_n) \end{aligned}$$

so (13) follows. □

Corollary 3. *Suppose* $0 \le b_i < \infty$, $1 \le i \le n$ *and* $u : \prod_{i=1}^{n}(b_i,\infty) \to \mathbf{R}_+$ *a locally integrable function in* $\prod_{i=1}^{n}(b_i,\infty)$ *and the function* w_4 *is given by*

$$w_4(x_1,\dots,x_n) = \frac{1}{x_1\cdots_n}\int_{b_1}^{x_1}\cdots\int_{b_n}^{x_n} u(t_1,\dots,t_n)\,dt_1\cdots dt_n < \infty.$$

If the real valued function Φ *is a superquadratic function on* (b,d)*,* $0 \le b < d < \infty$*, then*

$$\begin{aligned}
(14)\quad &\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(x_1,\dots,x_n)\\
&\times\Phi\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\
&+\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n} u(x_1,\dots,x_n)\\
&\times\Phi\left(\left|f(t_1,\dots,t_n)-x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\frac{dt_1}{t_1^2}\cdots\frac{dt_n}{t_n^2}\right|\right)\\
&\times\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\,dx_1\cdots dx_n\\
&\qquad\le\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\left(f(x_1,\dots,x_n)\right)w_4(x_1,\dots,x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}
\end{aligned}$$

holds for all f *with* $b\le f(x_1,\dots,x_n)\le d$, $0<x_i\le b_i$, $1\le i\le n$. *Moreover,* (14) *holds in reverse order if* Φ *is a subquadratic function.*

Remark 2. Similarly, by setting $u(x_1,\dots,x_n)\equiv 1$, in Corollary 3 we obtain

$$w_4(x_1,\dots,x_n)=\begin{cases}\prod_{i=1}^{n}\left(1-(b_i/x_i)\right) & b_i>0;\\ 1 & b_i=0.\end{cases}$$

In this setting relation (14) for $b_i>0$, $1\le i\le n$, has the form

(15)

$$\begin{aligned}
&\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\
&\quad+\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\Phi\Bigg(\bigg|f(t_1,\dots,t_n)\\
&\quad-x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\bigg|\Bigg)\\
&\qquad\qquad\qquad\times\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\,dx_1\cdots dx_n\\
&\le\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Phi\left(f(x_1,\dots,x_n)\right)\left[1-\frac{b_1}{x_1}\right]\cdots\left[1-\frac{b_n}{x_n}\right]\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},
\end{aligned}$$

while for $b_i = 0$, $1 \le i \le n$, (14) becomes

(16)
$$\int_0^\infty \cdots \int_0^\infty \Phi\left(x_1 \cdots x_n \int_{x_1}^\infty \cdots \int_{x_n}^\infty f(t_1, \dots, t_n) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \right) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$
$$+ \int_0^\infty \cdots \int_0^\infty \int_0^{t_1} \cdots \int_0^{t_n} \Phi\left(\left| f(t_1, \dots, t_n) \right.\right.$$
$$\left.\left. - x_1 \cdots x_n \int_{x_1}^\infty \cdots \int_{x_n}^\infty f(t_1, \dots, t_n) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \right| \right) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \, dx_1 \cdots dx_n$$
$$\le \int_0^\infty \cdots \int_0^\infty \Phi\left(f(x_1, \dots, x_n)\right) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

3. Applications.

Theorem 4. *Let $p > 1$, $m > 1$, $0 < b_i \le \infty$, and let the function f be locally integrable on $\prod_{i=1}^n (0, b_i)$ such that:*

$$0 < \int_0^{b_1} \cdots \int_0^{b_n} f^p(x_1, \dots, x_n) \frac{dx_1 \cdots dx_n}{(x_1 \cdots x_n)^{m-p}} < \infty.$$

(i) *If $p \ge 2$, then*

(17)
$$\int_0^{b_1} \cdots \int_0^{b_n} (x_1 \cdots x_n)^{-m} \left(\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \dots, t_n) dt_1 \cdots dt_n \right)^p dx_1 \cdots dx_n$$
$$+ \left(\frac{m-1}{p} \right)^n \int_0^{b_1} \cdots \int_0^{b_n} \int_{t_1}^{b_1} \cdots$$
$$\int_{t_n}^{b_n} \left| \left(\frac{p}{m-1} \right)^n f(t_1, \dots, t_n) \left(\frac{t_1 \cdots t_n}{x_1 \cdots x_n} \right)^{(p-m+1)/p} \right.$$
$$\left. - \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \dots, t_n) \, dt_1 \cdots dt_n \right|^p$$
$$\times \frac{dx_1 \cdots dx_n}{(x_1 \cdots x_n)^{(m-p^2+pm-1)/p}} \frac{dt_1 \cdots dt_n}{(t_1 \cdots t_n)^{(1-m+p)/p}}$$
$$\le \left[\frac{p}{m-1} \right]^{np} \int_0^{b_1} \cdots \int_0^{b_n} f^p(x_1, \dots, x_n)$$

$$\times\left[1-\left(\frac{x_1}{b_1}\right)^{(m-1)/p}\right]\cdots\left[1-\left(\frac{x_n}{b_n}\right)^{(m-1)/p}\right]\frac{dx_1\cdots dx_n}{(x_1\cdots x_n)^{m-p}}.$$

(ii) *If* $1<p<2$, *then the inequality in* (17) *holds in the reversed order.*

Proof. By setting the function $\Phi = x^p$, $p\geq 2$, inequality (11) has form

(18)
$$\begin{aligned}&\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right)^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &\quad+\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|f(t_1,\dots,t_n)-\frac{1}{x_1\cdots x_n}\right.\\ &\quad\times\left.\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n\right|^p\frac{dx_1\cdots dx_n}{x_1^2\cdots x_n^2}\,dt_1\cdots dt_n\\ &\leq\int_0^{b_1}\cdots\int_0^{b_n} f^p(x_1,\dots,x_n)\left[1-\frac{x_1}{b_1}\right]\cdots\left[1-\frac{x_n}{b_n}\right]\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.\end{aligned}$$

Let I_1 and I_2 be the first and second terms of the left hand side of inequality (18) and I_3 the term on the right hand side of (18), respectively. Replace b_i by $a_i(=b_i^{(m-1)/p})$ and choose for f the function $(x_1,\dots,x_n)\mapsto f(x_1^{p/(m-1)},\dots,x_n^{p/(m-1)})x_1^{(p/m-1)-1}\cdots x_n^{(p/m-1)-1}$. Thereafter, by using substitutions $s_i=t_i^{p/(m-1)}$ and $y_i=x_i^{p/(m-1)}$, $1\leq i\leq n$, we have

(19)
$$\begin{aligned}I_1&=\int_0^{a_1}\cdots\int_0^{a_n}\left[\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f\left(t_1^{p/(m-1)},\dots,t_n^{p/(m-1)}\right)\right.\\ &\qquad\left.\times\,(t_1\cdots t_n)^{(p/m-1)-1}dt_1\cdots dt_n\right]^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &=\left(\frac{m-1}{p}\right)^{np}\int_0^{a_1}\cdots\int_0^{a_n}(x_1\dots x_n)^{-p}\\ &\quad\times\left(\int_0^{x_1^{p/(m-1)}}\cdots\int_0^{x_n^{p/(m-1)}} f(s_1,\dots,s_n)\,ds_1\cdots ds_n\right)^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\end{aligned}$$

$$= \left(\frac{m-1}{p}\right)^{n(p+1)} \int_0^{b_1} \cdots \int_0^{b_n} (y_1 \cdots y_n)^{-m} \times \left(\int_0^{y_1} \cdots \int_0^{y_n} f(s_1, \dots, s_n)\, ds_1 \cdots ds_n \right)^p dy_1 \cdots dy_n,$$

(20)

$$\begin{aligned} I_2 = &\int_0^{a_1} \cdots \int_0^{a_n} \Bigg[\int_{t_1}^{a_1} \cdots \int_{t_n}^{a_n} \Bigg| f\left(t_1^{p/(m-1)}, \dots, t_n^{p/(m-1)}\right) (t_1 \cdots t_n)^{(p/m-1)-1} \\ &- \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} \\ &\times f\left(t_1^{\frac{p}{m-1}}, \dots, t_n^{\frac{p}{m-1}}\right) (t_1 \cdots t_n)^{\frac{p}{m-1}-1} dt_1 \cdots dt_n \Bigg|^p \frac{dx_1}{x_1^2} \cdots \frac{dx_n}{x_n^2} \Bigg] dt_1 \cdots dt_n \\ = &\left(\frac{m-1}{p}\right)^{2n} \int_0^{b_1} \cdots \int_0^{b_n} \Bigg[\int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \Bigg| f(s_1, \dots, s_n)(s_1 \cdots s_n)^{(p-m+1)/p} \\ &- \left(\frac{m-1}{p}\right)^n (y_1 \cdots y_n)^{\frac{1-m}{p}} \int_0^{y_1} \cdots \int_0^{y_n} f(s_1, \dots, s_n)\, ds_1 \cdots \\ &ds_n \Bigg|^p \frac{dy_1 \cdots dy_n}{(y_1 \cdots y_n)^{(m-1/p)+1}} \Bigg] (s_1 \cdots s_n)^{(m-1-p)/p}\, ds_1 \cdots ds_n \\ = &\left(\frac{m-1}{p}\right)^{n(p+2)} \int_0^{b_1} \cdots \int_0^{b_n} \int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \Bigg| \left(\frac{p}{m-1}\right)^n \\ &\times f(s_1, \dots, s_n) \left(\frac{s_1 \cdots s_n}{y_1 \cdots y_n}\right)^{(p-m+1)/p} \\ &- \frac{1}{y_1 \cdots y_n} \int_0^{y_1} \cdots \int_0^{y_n} f(s_1, \dots, s_n)\, ds_1 \cdots ds_n \Bigg|^p \\ &\times \frac{(y_1 \cdots y_n)^{(p^2-pm+1-m)/p}}{(s_1 \cdots s_n)^{(1-m+p)/p}}\, dy_1 \cdots dy_n\, ds_1 \cdots ds_n. \end{aligned}$$

(21)

$$I_3 = \int_0^{a_1} \cdots \int_0^{a_n} \left[f\left(x_1^{p/(m-1)}, \dots, x_n^{p/(m-1)}\right) (x_1 \cdots x_n)^{(p/m-1)-1} \right]^p \times \left[1 - \frac{x_1}{a_1}\right] \cdots \left[1 - \frac{x_n}{a_n}\right] \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$= \left[\frac{m-1}{p}\right]^n \int_0^{b_1} \cdots \int_0^{b_n} f^p(y_1, \ldots, y_n) \left[1 - \left(\frac{y_1}{b_1}\right)^{(m-1)/p}\right] \cdots \left[1 - \left(\frac{y_n}{b_n}\right)^{(m-1)/p}\right] \frac{dy_1 \cdots dy_n}{(y_1 \cdots y_n)^{m-p}}.$$

By combining inequalities (18)–(21) we get (17). □

Theorem 5. *Let $p > 1$, $m < 1$, $0 \le b_i < \infty$, and let the function f be locally integrable on $\prod_{i=1}^n (b_i, \infty)$ such that:*

$$0 < \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(x_1, \ldots, x_n)(x_1 \cdots x_n)^{p-m} dx_1 \cdots dx_n < \infty.$$

(i) *If $p \ge 2$, then*

(22)
$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} (x_1 \cdots x_n)^{-m} \left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n\right)^p dx_1 \cdots dx_n$$
$$+ \left(\frac{1-m}{p}\right)^n \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \int_{b_1}^{t_1} \cdots \int_{b_n}^{t_n} \left| \left(\frac{p}{1-m}\right)^n \times f(t_1, \ldots, t_n) \left(\frac{t_1 \cdots t_n}{x_1 \cdots x_n}\right)^{(p-m+1)/p} - x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, \ldots, t_n) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \right|^p \frac{dx_1 \cdots dx_n}{(x_1 \cdots x_n)^{(m-p^2+pm-1)/p}} \frac{dt_1 \cdots dt_n}{(t_1 \cdots t_n)^{(1-m+p)/p}}$$
$$\le \left(\frac{p}{1-m}\right)^{np} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(x_1, \ldots, x_n) \left[1 - \left(\frac{b_1}{x_1}\right)^{(1-m)/p}\right] \cdots \left[1 - \left(\frac{b_n}{x_n}\right)^{(1-m)/p}\right] \frac{dx_1 \cdots dx_n}{(x_1 \cdots x_n)^{m-p}}.$$

(ii) *If $1 < p < 2$, then the inequality in (22) holds in the reversed order.*

Proof. Set the function $\Phi = x^p$, $p \geq 2$, in (15) so that

(23)
$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}f(t_1,\ldots,t_n)\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\right)^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$
$$+\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\Bigg|f(t_1,\ldots,t_n)-x_1\cdots x_n\int_{x_1}^{\infty}\cdots$$
$$\int_{x_n}^{\infty}f(t_1,\ldots,t_n)\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}\Bigg|^p dx_1\cdots dx_n\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}$$
$$\leq\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}f^p(x_1,\ldots,x_n)\left[1-\frac{b_1}{x_1}\right]\cdots\left[1-\frac{b_n}{x_n}\right]\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

Let I_1 and I_2 be the first and second terms of the left hand side of inequality (23) and I_3 the term on the right hand side of (23), respectively. Replace b_i by $a_i (= b_i^{(1-m)/p})$ and choose for f the function $(x_1,\ldots,x_n) \mapsto f(x_1^{p/(1-m)},\ldots,x_n^{p/(1-m)})x_1^{(p/1-m)+1}\cdots x_n^{(p/1-m)+1}$. Thereafter, by using substitutions $s_i = t_i^{p/(m-1)}$ and $y_i = x_i^{p/(m-1)}$ for $1 \leq i \leq n$, we have

(24)
$$I_1=\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\Bigg(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}f\left(t_1^{p/(1-m)},\ldots,t_n^{p/(1-m)}\right)$$
$$\times(t_1\cdots t_n)^{(p/1-m)-1}dt_1\cdots dt_n\Bigg)^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$
$$=\left(\frac{1-m}{p}\right)^{np}\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}(x_1\cdots x_n)^{-p}$$
$$\times\left(\int_{x_1^{p/(1-m)}}^{\infty}\cdots\int_{x_n^{p/(1-m)}}^{\infty}f(s_1,\ldots,s_n)\,ds_1\cdots ds_n\right)^p\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$
$$=\left(\frac{1-m}{p}\right)^{\frac{n}{p+1}}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\left(\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty}f(s_1,\ldots,s_n)ds_1\cdots ds_n\right)^p\frac{dy_1\cdots dy_n}{(y_1\cdots y_n)^m}$$

(25)
$$I_2=\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\left[\int_{a_1}^{t_1}\cdots\int_{a_n}^{t_n}\left|f\left(t_1^{\frac{p}{1-m}},\ldots,t_n^{\frac{p}{1-m}}\right)(t_1\cdots t_n)^{\frac{p}{1-m}+1}\right.\right.$$

$$- x_1\cdots x_n \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f\left(t_1^{\frac{p}{1-m}},\ldots,t_n^{\frac{p}{1-m}}\right)(t_1\cdots t_n)^{\frac{p}{1-m}-1} dt_1\cdots dt_n\Big|^p$$

$$dx_1\cdots dx_n\Big]\frac{dt_1\cdots dt_n}{t_1^2\cdots t_n^2}$$

$$= \left(\frac{1-m}{p}\right)^{2n}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\Big[\int_{b_1}^{s_1}\cdots\int_{b_n}^{s_n}\Big|$$

$$\times f(s_1,\ldots,s_n)(s_1\cdots s_n)^{\frac{p-m+1}{p}}\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty} f(s_1,\ldots,s_n)\,ds_1\cdots ds_n\Big|^p$$

$$-\left(\frac{1-m}{p}\right)^{n}(y_1\cdots y_n)^{\frac{1-m}{p}}$$

$$\times\frac{dy_1\cdots dy_n}{(y_1\cdots y_n)^{(m-1/p)+1}}\Big](s_1\cdots s_n)^{(m-1-p)/p} ds_1\cdots ds_n$$

$$= \left(\frac{1-m}{p}\right)^{n(p+2)}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{s_1}\cdots\int_{b_n}^{s_n}\Big|\left(\frac{p}{1-m}\right)^{n}$$

$$\times f(s_1,\ldots,s_n)\left(\frac{s_1\cdots s_n}{y_1\cdots y_n}\right)^{(p-m+1)/p} - \frac{1}{y_1\cdots y_n}\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty}$$

$$\times f(s_1,\ldots,s_n)\,ds_1\cdots ds_n\Big|^p \frac{dy_1\cdots dy_n}{(y_1\cdots y_n)^{(m-p^2+pm-1)/p}}\frac{ds_1\cdots ds_n}{(s_1\cdots s_n)^{(1-m+p)/p}}$$

(26)

$$I_3 = \int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\left[f\left(x_1^{p/(1-m)},\ldots,x_n^{p/(1-m)}\right)(x_1\ldots x_n)^{(p/1-m)+1}\right]^p$$

$$\times\left[1-\frac{a_1}{x_1}\right]\cdots\left[1-\frac{a_n}{x_n}\right]\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$= \left[\frac{1-m}{p}\right]^n\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} f^p(y_1,\ldots,y_n)\left[1-\left(\frac{b_1}{y_1}\right)^{(1-m)/p}\right]\cdots$$

$$\times\left[1-\left(\frac{b_n}{y_n}\right)^{(1-m)/p}\right]\frac{dy_1\cdots dy_n}{(y_1\cdots y_n)^{m-p}}.$$

By combining inequalities (23)–(26) we get (22). □

Note 1. For $b_i = 0, \infty, 1 \le i \le n$, inequalities (22) and (17) together give improvement in inequality (3).

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