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ON q-BASKAKOV-MASTROIANNI OPERATORS

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ABSTRACT. The aim of the paper is to investigate a q-analogue of a general class of linear positive operators defined by Baskakov and developed by Mastroianni. Our results are the following: the moments of the operators are explicitly expressed with the help of new q-analogues of Stirling numbers, the rate of convergence is established in different function spaces by using both modulus of continuity and a certain weighted modulus of smoothness, the identification, as particular cases, of q-analogues for two classical sequences of positive approximation processes.

1. Introduction. The approximation of functions by using linear positive operators introduced via q-calculus is currently under intensive research. The pioneer work has been done by Lupaş [**11**] and Phillips [**17**] who proposed generalizations of Bernstein polynomials based upon q-integers. The q-Bernstein polynomials quickly gained popularity, see [**6, 14, 15, 18, 21**]. A comprehensive review of the results on this class along with an extensive bibliography is given in [**16**]. Other important classes of discrete operators have been investigated by using q-calculus, for example q-Meyer-König operators [**7, 20**], q-Bleimann, Butzer and Hahn operators [**4, 12**], q-Szász-Mirakjan operators [**3**].

The starting point of this note is a general class of operators introduced by Baskakov [**5**]. Later on, Mastroianni's approach [**13**] developed this approximation process. For comparison, [**2**, pages 344, 350] may also be consulted. Our goal is to investigate a q-analogue of the original Baskakov and Mastroianni operators. The class has been introduced by the second author, and their weighted statistical approximation properties were studied [**19**]. As particular features of this construction, we mention the following: one requires the values of the approximated function on a flexible net described with the help of the powers of a parameter. For our investigation, we define and explore

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a new q-analogue of Stirling numbers. This way, we are able to get explicit formulae for all moments of our operators based on q-integers.

The paper is organized as follows. Section 2 includes basic facts regarding q-calculus and contains a new q-analogue of Stirling numbers of the second kind. The class of announced operators is presented in Section 3. The key result of this section consists of obtaining all moments of the operators in terms of q-Stirling numbers. Further on, in Section 4, the rate of convergence is established. The approach is developed in two cases: for bounded functions and for functions having a polynomial growth. As a main tool we use the ordinary modulus of smoothness and a certain weighed modulus, respectively. The last section centers around the order of approximation of two special cases of our general class.

2. On q-Stirling numbers of the second kind. To formulate our results we need the following definitions, see, e.g., [**9**, pages 7–13].

Let $q > 0$. For any $n \in \mathbf{N}_0 := \{0\} \cup \mathbf{N}$, the q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad n \in \mathbf{N},\ [0]_q := 0,$$

and the q-factorial $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbf{N},\ [0]_q! := 1.$$

Also, the q-binomial coefficients or the Gaussian coefficients are denoted by $\left[{n \atop k} \right]_q$ and are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \quad k = 0, 1, \ldots, n.$$

Clearly, for $q = 1$, $[n]_1 = n$, $[n]_1! = n!$ and $\left[{n \atop k} \right]_1 = \binom{n}{k}$ are the ordinary binomial coefficients. Moreover, for any real number a, we set

$$[a]_q := \frac{1 - q^a}{1 - q}, \quad q \neq 1.$$

In the sequel we will always assume that $q \in (0, 1)$.

The q-derivative of a function $f:\mathbf{R}\to\mathbf{R}$ is defined by

$$D_q f\,(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq 0,\qquad D_q f\,(0):=\lim_{x\to 0} D_q f\,(x)\,,$$

and the high q-derivatives $D_q^0 f:=f$, $D_q^n f:=D_q(D_q^{n-1}f)$, $n\in\mathbf{N}$.

It is said that a function f is q-differentiable on a real interval I if, for any $q\in(0,1)$, the q-derivative of f exists and is finite in every $x\in I$. The product rule is

$$D_q\left(f(x)g(x)\right)=D_q\left(f(x)\right)g(x)+f(qx)D_q\left(g(x)\right). \tag{1}$$

We recall the q-Taylor theorem as is given in [**8**, page 103].

Theorem A. *If the function $g(x)$ is capable of expansion as a convergent power series and q is not a root of unity, then*

$$g(x)=\sum_{r=0}^{\infty}\frac{(x-a)_q^r}{[r]_q!}D_q^r g(a),$$

where

$$(x-a)_q^r=\prod_{s=0}^{r-1}(x-q^s a)=\sum_{k=0}^{r}\begin{bmatrix} r\\ k\end{bmatrix}_q q^{[k(k-1)]/2}x^{r-k}(-a)^k.$$

Since our aim is to approximate functions defined on $\mathbf{R}_+$, we need meshes of this interval. Taking into account the nets $\Delta_n=([k]_q/([n]_q q^{k-1}))_{k\geq 0}$, $n\in\mathbf{N}$, we define a suitable q-difference operator as follows

$$\Delta_q^0 f_{k,s}=f_{k,s}, \tag{2}$$

$$\Delta_q^{r+1}f_{k,s}=q^r\Delta_q^r f_{k+1,s}-\Delta_q^r f_{k,s-1},\quad r\in\mathbf{N}_0, \tag{3}$$

where $f_{k,s}=f([k]_q/(q^s[n]_q))$, $k\in\mathbf{N}_0$, $s\in\mathbf{Z}$.

The following lemma gives an expression for the rth q-differences $\Delta_q^r f_{k,s}$ as a sum of multiples of values of f.

Lemma 1. *The q-difference operator Δ_q^r defined by* (2), (3) *satisfies*

$$\Delta_q^r f_{k,s} = \sum_{j=0}^{r} (-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q f_{k+j,j+s-r} \quad \text{for } r,k \in \mathbf{N}_0, \quad s \in \mathbf{Z}. \tag{4}$$

Proof. We apply mathematical induction with respect to $r \in \mathbf{N}_0$. By using (2) it is easy to see that (4) is satisfied for $r = 0$.

Further on, we assume that (4) is true for an arbitrary $r \in \mathbf{N}$. From the recurrence relation (3) we obtain

$$\begin{aligned}
\Delta_q^{r+1} f_{k,s} &= q^r \Delta_q^r f_{k+1,s} - \Delta_q^r f_{k,s-1} \\
&= q^r \sum_{j=0}^{r} (-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q f_{k+j+1,j+s-r} \\
&\quad - \sum_{j=0}^{r} (-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q f_{k+j,j+s-r-1} \\
&= q^{r(r+1)/2} f_{k+r+1,s} + (-1)^{r+1} f_{k,s-r-1} \\
&\quad + q^r \sum_{j=0}^{r-1} (-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q f_{k+j+1,j+s-r} \\
&\quad - \sum_{j=0}^{r-1} (-1)^{r-j-1} q^{j(j+1)/2} \begin{bmatrix} r \\ j+1 \end{bmatrix}_q f_{k+j+1,j+s-r} \\
&= q^{r(r+1)/2} f_{k+r+1,s} + (-1)^{r+1} f_{k,s-r-1} \\
&\quad + \sum_{j=0}^{r-1} (-1)^{r-j} q^{j(j-1)/2} \\
&\qquad \times f_{k+j+1,j+s-r} \left(q^r \begin{bmatrix} r \\ j \end{bmatrix}_q + q^j \begin{bmatrix} r \\ j+1 \end{bmatrix}_q \right).
\end{aligned}$$

By using the formula

$$\begin{bmatrix} r+1 \\ j+1 \end{bmatrix}_q = q^{r-j} \begin{bmatrix} r \\ j \end{bmatrix}_q + \begin{bmatrix} r \\ j+1 \end{bmatrix}_q,$$

we can write

$$\begin{aligned}\Delta_q^{r+1} f_{k,s} &= q^{r(r+1)/2} f_{k+r+1,s} + (-1)^{r+1} f_{k,s-r-1} \\ &\quad + \sum_{j=0}^{r-1} (-1)^{r-j} q^{j(j+1)/2} \begin{bmatrix} r+1 \\ j+1 \end{bmatrix}_q f_{k+j+1,j+s-r} \\ &= \sum_{j=0}^{r+1} (-1)^{r-j+1} q^{j(j-1)/2} \begin{bmatrix} r+1 \\ j \end{bmatrix}_q f_{k+j,j+s-r-1},\end{aligned}$$

and the proof is finished. □

For each $(m,r) \in \mathbf{N}_0 \times \mathbf{N}_0$, let us consider the number $\sigma_q(m,r)$ defined by

$$\sigma_q(m,r) = \frac{1}{[r]_q!} \sum_{j=0}^{r} (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}. \tag{5}$$

Lemma 2. *The numbers $\sigma_q(m,r)$, $(m,r) \in \mathbf{N}_0 \times \mathbf{N}_0$, given by* (5) *enjoy the following properties.*

$$\sigma_q(m,0) = 0, \quad (m \in \mathbf{N}) \quad \text{and} \quad \sigma_q(0,0) = 1, \tag{6}$$

$$q^r \sigma_q(m+1,r) = [r]_q \sigma_q(m,r) + \sigma_q(m,r-1), \quad m \in \mathbf{N}_0,\ r \in \mathbf{N}, \tag{7}$$

$$\sigma_q(m,r) = 0, \quad r > m. \tag{8}$$

Proof. Identity (6) follows immediately from (5). In order to prove (7) we recall the formula

$$\begin{bmatrix} r-1 \\ j \end{bmatrix}_q = \frac{1}{q^j} \left(\begin{bmatrix} r \\ j \end{bmatrix}_q - \begin{bmatrix} r-1 \\ j-1 \end{bmatrix}_q \right), \quad r,j \in \mathbf{N}. \tag{9}$$

By using definition (5), for each $m \in \mathbf{N}_0$, $r \in \mathbf{N}$, we can write

$$\sigma_q(m, r-1) = \frac{1}{[r-1]_q!} \times \sum_{j=1}^{r} (-1)^{j-1} q^{(r-j)(r-j-1)/2} \begin{bmatrix} r-1 \\ j-1 \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}. \tag{10}$$

On the other hand,

$$\begin{aligned} &q^r \sigma_q(m+1, r) - [r]_q \sigma_q(m, r) \\ &= \frac{[r]_q}{[r]_q!} q^{r-1} \\ &\qquad \times \sum_{j=0}^{r} (-1)^j q^{(r-j-1)(r-j-2)/2} \begin{bmatrix} r-1 \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}} \\ &\quad - \frac{[r]_q}{[r]_q!} \sum_{j=0}^{r} (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}} \\ &= \frac{1}{[r-1]_q!} \sum_{j=0}^{r} (-1)^j \\ &\qquad \times q^{(r-j-1)(r-j-2)/2} \frac{[r-j]_q^m}{q^{(r-j)m}} \left(q^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_q - q^{r-j-1} \begin{bmatrix} r \\ j \end{bmatrix}_q \right) \\ &= \frac{1}{[r-1]_q!} \sum_{j=1}^{r} (-1)^j \\ &\qquad \times q^{(r-j-1)(r-j-2)/2} \frac{[r-j]_q^m}{q^{(r-j)m}} q^{r-1} \left(\begin{bmatrix} r-1 \\ j \end{bmatrix}_q - \frac{1}{q^j} \begin{bmatrix} r \\ j \end{bmatrix}_q \right). \end{aligned}$$

Taking into account (9) and (10) the proof is complete. □

For establishing relation (8) on the basis of (7), it is enough to prove $\sigma_q(m, m+1) = 0$, for any $m \in \mathbf{N}_0$. Since

$$[m+1-j]_q^m = \frac{1}{(1-q)^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{k(m+1)-kj}, \quad \text{for } 0 \le j \le m+1,$$

and

$$\sum_{j=0}^{m+1} q^{j(j-1)/2} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q \left(-\frac{1}{q^k} \right)^j = \left(1 - \frac{1}{q^k} \right)_q^{m+1} = 0,$$

$$\text{for } 0 \le k < m+1,$$

we can write

$$\begin{aligned} &\sigma_q(m, m+1) \\ &= \frac{1}{[m+1]_q! q^{[m(m+1)]/2}} \sum_{j=0}^{m+1} (-1)^j q^{[j(j-1)]/2} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q [m+1-j]_q^m \\ &= \frac{(1-q)^{-m}}{[m+1]_q! q^{[m(m+1)]/2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{k(m+1)} \left(1 - \frac{1}{q^k} \right)_q^{m+1} = 0. \quad \square \end{aligned}$$

In what follows, the monomial of m degree is denoted by e_m, $m \in \mathbf{N}_0$.

Lemma 3. *For $f = e_m$, $m \in \mathbf{N}_0$, one has*

(11) $$\Delta_q^r f_{0,r-1} = \frac{q^m [r]_q!}{[n]_q^m} \sigma_q(m, r), \quad r \in \mathbf{N}_0.$$

Proof. For $r = 0$ identity (11) follows immediately from (2) and (6). Let $r \in \mathbf{N}$, $m \in \mathbf{N}_0$ and $f = e_m$. By using (4), we obtain

$$\begin{aligned} \Delta_q^r f_{0,r-1} &= \sum_{j=0}^{r} (-1)^{r-j} q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \left(\frac{[j]_q}{q^{j-1}[n]_q} \right)^m \\ &= \frac{q^m}{[n]_q^m} \sum_{j=0}^{r} (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}} \\ &= \frac{q^m [r]_q!}{[n]_q^m} \sigma_q(m, r). \quad \square \end{aligned}$$

Remark 1. It can easily be proved that

$$\lim_{q \to 1^-} \sigma_q(m, r) = \frac{1}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^m, \quad m \in \mathbf{N}_0, \ r \in \mathbf{N}_0,$$

the limit representing $S(m,r)$, the classical Stirling numbers of the second kind, see [**1**, page 824]. Thus, $\sigma_q(m,r)$ can be considered to be a q-analogue of Stirling numbers. We point out that our numbers are distinct from the known q-analogue $S_q(m,r)$ used in [**3**].

3. The $T_{n,q}$ operators and their moments. Inspired by the general class introduced in [**5, 13**] and following [**19**, Section 3], let $(\phi_n)_{n\geq 1}$ be a sequence of real valued functions defined on $\mathbf{R}_+$, continuously infinitely q-differentiable on $\mathbf{R}_+$, $q \in (0,1)$, and satisfying the following conditions.

(12)
(P1) $\phi_n(0) = 1, \quad n \in \mathbf{N},$

(13)
(P2) $(-1)^k D_q^k \phi_n(x) \geq 0, \quad n \in \mathbf{N},\ k \in \mathbf{N}_0,\ x \geq 0,$

(P3) For all $(x,k) \in \mathbf{R}_+ \times \mathbf{N}_0$ there exists a positive integer i_k, $0 \leq i_k \leq k,$ such that

(14) $$D_q^{k+1}\phi_n(x) = (-1)^{i_k+1} D_q^{k-i_k}\phi_n(q^{i_k+1}x)\beta_{n,k,i_k,q}(x),$$
where

(15) $$\lim_n \frac{\beta_{n,k,i_k,q}(0)}{[n]_q^{i_k+1}\, q^{k-i_k}} = 1.$$

We consider the operators
(16)
$$T_{n,q}\,(f;x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{[k(k-1)]/2} D_q^k \phi_n(x) f\left(\frac{[k]_q}{[n]_q\, q^{k-1}}\right), \quad x \geq 0,$$
where $f \in \mathcal{F}(\mathbf{R}_+) := \{f : \mathbf{R}_+ \to \mathbf{R}$, the series in (16) is convergent$\}$.

For each $n \in \mathbf{N}$, $T_{n,q}$ is a linear positive operator satisfying the interpolating property $T_{n,q}(f;0) = f(0)$.

With the help of the q-Stirling numbers introduced in the previous section, our main aim is to indicate all moments of our operators.

Lemma 4. *Let $T_{n,q}$, $n \in \mathbf{N}$, be defined by* (16)*. One has*

(17) $$T_{n,q}(e_m;x) = \sum_{r=0}^{m} \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0)\sigma_q(m,r), \quad x \geq 0.$$

Proof. Let $f \in \mathcal{F}(\mathbf{R}_+)$.

By using (2), the operator $T_{n,q}$ can be expressed as follows

$$T_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k-1)/2} D_q^k \phi_n(x) \Delta_q^0 f_{k,k-1}.$$

On the basis of (1), the q-derivative of $T_{n,q}f$ is given by

$$\begin{aligned}
&D_q T_{n,q}(f;x) \\
&\quad = -\sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k+1)/2} D_q^{k+1} \phi_n(x) \Delta_q^0 f_{k+1,k} \\
&\qquad + \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^k q^{k(k-1)/2} D_q^{k+1} \phi_n(x) \Delta_q^0 f_{k,k-1} \\
&\quad = -\sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k-1)/2} q^k D_q^{k+1} \phi_n(x) \left(\Delta_q^0 f_{k+1,k} - \Delta_q^0 f_{k,k-1} \right) \\
&\quad = -\sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k-1)/2} q^k D_q^{k+1} \phi_n(x) \Delta_q^1 f_{k,k}.
\end{aligned}$$

For $n \in \mathbf{N}$ and $x \in \mathbf{R}_+$, by induction with respect to $r \in \mathbf{N}$ we can prove

$$D_q^r T_{n,q}(f;x) = (-1)^r \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k-1)/2} q^{rk} D_q^{k+r} \phi_n(x) \Delta_q^r f_{k,k+r-1}.$$

Choosing $x = 0$, we deduce $D_q^r T_{n,q}(f;0) = (-1)^r D_q^r \phi_n(0) \Delta_q^r f_{0,r-1}$. Consequently, taking into account Lemma 3, we get

$$D_q^r T_{n,q}(e_m;0) = (-1)^r q^m \frac{[r]_q!}{[n]_q^m} D_q^r \phi_n(0) \sigma_q(m,r).$$

Choosing $a = 0$ in Theorem A, we obtain

$$T_{n,q}(e_m;x) = \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!} D_q^r T_{n,q}(e_m;0) = \sum_{r=0}^{\infty} \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0) \sigma_q(m,r).$$

Taking into account (8), the proof is complete. $\square$

Formula (17) gives the explicit form of the first three moments of $T_{n,q}$, $n \in \mathbf{N}$, operators which will be read as follows

$$T_{n,q}(e_0;x) = 1, \tag{18}$$

$$T_{n,q}(e_1;x) = -x\frac{D_q\phi_n(0)}{[n]_q}, \tag{19}$$

$$T_{n,q}(e_2;x) = x^2\frac{D_q^2\phi_n(0)}{q[n]_q^2} - x\frac{D_q\phi_n(0)}{[n]_q^2}. \tag{20}$$

We point out that these three particular moments have been obtained by a straightforward calculation in [**19**, Lemma 1], without involving q-Stirling numbers. Taking advantage of the above relations, the second central moment of $T_{n,q}$, $n \in \mathbf{N}$, is given by the formula

$$T_{n,q}(\varphi_x^2;x) = a_{n,q}x^2 + b_{n,q}x, \quad x \geq 0, \tag{21}$$

where $\varphi_x(t) = t - x$, $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$, and

$$a_{n,q} = 1 + 2\frac{D_q\phi_n(0)}{[n]_q} + \frac{D_q^2\phi_n(0)}{q[n]_q^2}, \qquad b_{n,q} = -\frac{D_q\phi_n(0)}{[n]_q^2}. \tag{22}$$

Relation (13) implies $b_{n,q} \geq 0$, $n \in \mathbf{N}$. Examining properties (P1)–(P3), we also deduce

$$\lim_n \frac{D_q\phi_n(0)}{[n]_q} = -1, \qquad \lim_n \frac{D_q^2\phi_n(0)}{q[n]_q^2} = \frac{1}{q^{i_1}}, \quad i_1 \in \{0,1\}. \tag{23}$$

The second limit was established keeping in view that (14)–(15) imply two variants for $D_q^2\phi_n(0)$: $i_1 = 0$ and $i_1 = 1$, respectively.

Since $\lim_n [n]_q = (1-q)^{-1}$, by using the above limits, relation (20) implies $\lim_n T_{n,q}(e_2;x) = q^{-i_1}x^2 + (1-q)x$. On the basis of the Bohman-Korovkin theorem, it is clear that $(T_{n,q})_{n\geq 1}$ does not form an approximation process. The next step is to transform it for satisfying this property. For each $n \in \mathbf{N}$, constant q will be replaced by a number $q_n \in (0,1)$ such that $\lim_n q_n = 1$. An immediate result will be seen as follows.

Theorem 1. *Let* $(q_n)_{n\ge 1}$, $0<q_n<1$, *be a sequence, and let* T_{n,q_n}, $n\in\mathbf{N}$, *be defined as in* (16). *If* $\lim_n q_n=1$, *for any compact* $\mathcal{K}\subset\mathbf{R}_+$ *and for each* $f\in\mathcal{F}(\mathbf{R}_+)\cap C(\mathbf{R}_+)$, *one has*

$$\lim_n T_{n,q_n}(f;x)=f(x),\quad \textit{uniformly in } x\in\mathcal{K}.$$

Proof. We get $\lim_n q_n^{i_1}=1$, $0\le i_1\le 1$, and $\lim_n[n]_{q_n}=\infty$. On the basis of relations (18)–(20), see also (23), the Bohman-Korovkin theorem implies our statement. □

4. Error estimations. We explore the rate of convergence of T_{n,q_n}, $n\in\mathbf{N}$ and $q_n\in(0,1)$, in terms of both the modulus of continuity and of a certain weighted modulus.

Denoting by $C_B(\mathbf{R}_+)$ the space of all bounded continuous real-valued functions on $\mathbf{R}_+$, we state the following general estimate.

Theorem 2. *Let* $(q_n)_{n\ge 1}$, $0<q_n<1$, *be a sequence such that* $\lim_n q_n=1$ *and* T_{n,q_n}, $n\in\mathbf{N}$, *are defined as in* (16). *For each* $n\in\mathbf{N}$ *and every* $f\in C_B(\mathbf{R}_+)$, *one has*

$$|T_{n,q_n}(f;x)-f(x)|\le\left(1+\sqrt{\max\{x,x^2\}}\right)\omega\left(f;\sqrt{c_{n,q_n}}\right),\quad x\ge 0,$$

and

$$\lim_n c_{n,q_n}=0. \tag{24}$$

Here $\omega(f;\cdot)$ *stands for the modulus of continuity associated to* f *and*

$$c_{n,q_n}:=|a_{n,q_n}|+b_{n,q_n}, \tag{25}$$

the sequences $(a_{n,q_n})_n$, $(b_{n,q_n})_n$ *being defined by* (22).

Proof. For any positive linear operator L, see e.g. the monograph [**2**, Theorem 5.1.2], quantitative error estimates can be expressed in terms of the test functions as follows

$$\begin{aligned}|Lf-f|&\le|f|\,|Le_0-1|\\&\quad+\left(Le_0+\frac{1}{\delta}\sqrt{(Le_0)\,(L\varphi_x^2)}\right)\omega\,(f;\delta),\quad \delta>0.\end{aligned} \tag{26}$$

On the other hand, (21) implies

$$0 \le T_{n,q_n}(\varphi_x^2;x) \le |a_{n,q_n}|\, x^2 + b_{n,q_n} x \le \max\left\{x, x^2\right\} c_{n,q_n}.$$

Also taking into account (18), from (26) with $\delta = \sqrt{c_{n,q_n}}$, our first statement follows.

Since $q_n \to 1^-$, on the basis of (22) and (23), we get (24). □

In what follows, we give estimates of the errors $|T_{n,q_n} f - f|$, $n \in \mathbf{N}$, involving functions with polynomial growth. For a given $N \in \mathbf{N}$, $N \ge 2$, considering the weight w_N, $w_N(x) = (1 + x^N)^{-1}$, $x \ge 0$, we consider the space

$$C_N(\mathbf{R}_+) := \{f \in C(\mathbf{R}_+) : \text{ a constant } M_f \text{ exists such that } |w_N f| \le M_f\},$$

endowed with the norm $\|\cdot\|_N^*$, $\|f\|_N^* := \sup_{x\ge 0} |w_N(x) f(x)|$. Since $N \ge 2$, the Korovkin test functions e_j, $j \in \{0, 1, 2\}$, belong to $C_N(\mathbf{R}_+)$.

For our purposes, we need some preliminary results regarding $T_{n,q}$ operator.

Setting $\gamma_{n,r,q} := (-1)^r D_q^r \phi_n(0)$, (13) guarantees $\gamma_{n,r,q} \ge 0$, $(n,r) \in \mathbf{N} \times \mathbf{N}_0$.

For any $m \in \mathbf{N}$, $\sigma_q(m,0) = 0$ and formula (17) can be written

$$T_{n,q}(e_m;x) = \frac{q^m}{[n]_q^m} \sum_{r=1}^{m} \gamma_{n,r,q}\sigma_q(m,r)x^r, \quad x \ge 0, \tag{27}$$

a polynomial of degree almost m having all coefficients positive.

Setting

$$A_{n,m,q} := q^m \sum_{r=1}^{m} \gamma_{n,r,q}\sigma_q(m,r), \tag{28}$$

relation (27) implies

$$0 \le T_{n,q}(e_m;x) \le \frac{1}{[n]_q^m} A_{n,m,q}(1 + x^m), \quad x \ge 0. \tag{29}$$

Lemma 5. *Let* $T_{n,q}$, $n \in \mathbf{N}$, *be defined by* (16). *The operators map* $C_N(\mathbf{R}_+)$ *into* $C_N(\mathbf{R}_+)$.

Proof. Let $f \in C_N(\mathbf{R}_+)$, meaning $|f(x)| \le M_f(1+x^N)$, $x \ge 0$. Since $T_{n,q}$ is a linear positive operator, consequently monotone, we can write successively

$$\begin{aligned}|T_{n,q}(f;x)| &\le T_{n,q}\left(|f|;x\right) \le M_f T_{n,q}(1+e_N;x)\\ &\le M_f\left(1+\frac{A_{n,N,q}}{[n]_q^N}\right)(1+x^N), \quad x \ge 0.\end{aligned}$$

We used (18) and (29). Clearly, $T_{n,q}f \in C_N(\mathbf{R}_+)$. ∎

Lemma 6. *Let* $T_{n,q}$, $n \in \mathbf{N}$, *be defined by* (16). *If*

$$\theta_x(t) = 1 + (x + |t-x|)^N \quad t \ge 0,\ x \ge 0, \tag{30}$$

then

$$\begin{aligned}T_{n,q}\left(\theta_x;x\right) &\le B_{n,N,q}(1+x^N),\\ \sqrt{T_{n,q}\left(\theta_x^2;x\right)} &\le \sqrt{2B_{n,2N,q}}(1+x^N), \quad x \ge 0,\end{aligned} \tag{31}$$

where

$$B_{n,m,q} := 2^{m-1}\left(2^m + \frac{A_{n,m,q}}{[n]_q^m}\right) \tag{32}$$

and $A_{n,m,q}$ *is given by* (28).

Proof. Since $\theta_x(t) \le 1 + (2x+t)^N \le 1 + 2^{N-1}((2x)^N + t^N)$, by using (18) and (29), we can write

$$\begin{aligned}T_{n,q}\left(\theta_x;x\right) &\le 1 + 2^{N-1}\left((2x)^N + T_{n,q}\left(e_N;x\right)\right)\\ &\le 2^{2N-1}(1+x^N) + 2^{N-1}\frac{A_{n,N,q}}{[n]_q^N}(1+x^N),\end{aligned}$$

and the first inequality from (31) is proved.

Further on,

$$\begin{aligned}\theta_x^2(t) &\le \left(1+(2x+t)^N\right)^2\\ &\le 2\left(1+(2x+t)^{2N}\right)\\ &\le 2\left(1+2^{4N-1}x^{2N}+2^{2N-1}t^{2N}\right).\end{aligned}$$

By applying $T_{n,q}$ and by using again (18), (29), the second inequality follows. □

For obtaining bounds on the approximation error by $T_{n,q}$, we use a modulus of smoothness Ω_N defined as follows

(33)

$$\Omega_N(f;\delta) = \sup_{\substack{x\ge 0\\ 0<h\le\delta}} w_N(x+h)\,|f(x+h)-f(x)|, \quad \delta>0,\ f\in C_N(\mathbf{R}_+).$$

The usefulness of this modulus and some basic properties of it may be found in [**10**]. Among them, $\Omega_N(f;m\delta)\le m\Omega_N(f;\delta)$, $m\in\mathbf{N}$, is mentioned. Since $\Omega_N(f;\delta)$ is monotonically increasing with respect to δ ($\delta>0$) and $\alpha<[\alpha]+1\le\alpha+1$ holds, the above property implies

$$\Omega_N(f;\alpha\delta)\le(\alpha+1)\Omega_N(f;\delta), \quad \delta>0, \tag{34}$$

for any $\alpha>0$.

Theorem 3. *Let* $(q_n)_{n\ge1}$, $0<q_n<1$, *be a sequence such that* $\lim_n q_n=1$ *and* T_{n,q_n}, $n\in\mathbf{N}$, *are defined as in* (16). *For each* $n\in\mathbf{N}$ *and every* $f\in C_N(\mathbf{R}_+)$, $N\ge2$, *one has*

(35)

$$\begin{aligned}&|T_{n,q_n}(f;x)-f(x)|\\ &\qquad\le 2\left(B_{n,N,q_n}+\sqrt{2B_{n,2N,q_n}}\right)(1+x^{N+1})\Omega_N\left(f;\sqrt{c_{n,q_n}}\right),\end{aligned}$$

$x\ge0$, *where* B_{n,m,q_n} *and* c_{n,q_n} *are described by* (32) *and* (25), *respectively.*

Proof. Let $n\in\mathbf{N}$ and $f\in C_N(\mathbf{R}_+)$ be fixed. For each $t\ge0$ and $\delta>0$, on the basis of (33) and (34) with $\alpha:=|t-x|\delta^{-1}$, we get

$$\begin{aligned}|f(t)-f(x)| &\le \left(1+(x+|t-x|)^N\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_N(f;\delta)\\ &= \left(\theta_x(t)+\frac{1}{\delta}\theta_x(t)\,|t-x|\right)\Omega_N(f;\delta),\end{aligned}$$

where θ_x was introduced in (30). Taking into account that T_{n,q_n} is a linear positive operator preserving the constants, we can write

$$\begin{aligned}|T_{n,q_n}(f;x)-f(x)| &= |T_{n,q_n}(f-f(x);x)| \le T_{n,q_n}\left(|f-f(x)|\,;x\right)\\ &\le T_{n,q_n}\left(\theta_x+\frac{1}{\delta}\theta_x\,|\varphi_x|\,;x\right)\Omega_N\left(f;\delta\right)\\ &= \Big\{T_{n,q_n}\left(\theta_x;x\right)\\ &\quad+\frac{1}{\delta}T_{n,q_n}\left(\theta_x\,|\varphi_x|\,;x\right)\Big\}\Omega_N\left(f;\delta\right)\\ &\le \Big\{T_{n,q_n}\left(\theta_x;x\right)\\ &\quad+\frac{1}{\delta}\sqrt{T_{n,q_n}\left(\theta_x^2;x\right)}\sqrt{T_{n,q_n}\left(\varphi_x^2;x\right)}\Big\}\Omega_N\left(f;\delta\right).\end{aligned}$$

The last increase is based upon the Cauchy-Schwarz inequality. Further, by using inequalities (21) and (31), we obtain

$$\begin{aligned}(36)\quad &|T_{n,q_n}(f;x)-f(x)|\\ &\le\left(B_{n,N,q_n}+\frac{1}{\delta}\sqrt{2B_{n,2N,q_n}c_{n,q_n}}\sqrt{\max\left\{x,x^2\right\}}\right)(1+x^N)\Omega_N\left(f;\delta\right).\end{aligned}$$

The following inequality $h/w_N \le 2/w_{N+1}$ holds on $\mathbf{R}_+$ both for $h=e_0$ and for $h=\sqrt{\max\{e_1,e_2\}}$. Applying it in (36) and choosing $\delta=\sqrt{c_{n,q_n}}$, the claimed result follows. □

Our concern is to determine the magnitude with respect to n of the constant appearing in (35), under the hypothesis $q_n\to 1^-$, $n\to\infty$. As a first step, keeping in view (15), we have $\beta_{n,k,i_k,q_n}(0)=\mathcal{O}([n]_{q_n}^{i_k+1})$, $0\le i_k\le k$. Examining (14), we deduce $\gamma_{n,r,q_n}=(-1)^r D_{q_n}^r\phi_n(0)=\mathcal{O}([n]_{q_n}^r)$, $r\in\mathbf{N}_0$. Next, (28) guarantees $A_{n,m,q_n}=\mathcal{O}([n]_{q_n}^m)$ and, consequently, (32) implies $B_{n,m,q_n}=\mathcal{O}(1)$. Thus, under the assumptions of Theorem 3, inequality (35) implies the following global estimate

$$\|T_{n,q_n}f-f\|_{N+1}^*\le K_N\Omega_N\left(f;\sqrt{c_{n,q_n}}\right),\quad f\in C_N\left(\mathbf{R}_+\right),$$

where K_N is a constant independent of f and n.

5. On some particular cases. Two special cases of the $(T_{n,q})_n$ sequence have been revealed in [**19**, Section 5], these representing q-analogues of some known discrete positive approximation processes.

5.1. We choose $\phi_n(x) := E_q(-[n]_q x)$, $x \geq 0$, $n \in \mathbf{N}$. Here E_q is the known expansion in the q-calculus of the exponential function defined as follows:

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}, \quad x \in \mathbf{R},$$

see, e.g., [**9**, page 31]. This way, $T_{n,q}$ operators turn into $S_n^*(\cdot;q;\cdot)$, a q-analogue of Szász-Mirakyan operators indicated in [**19**, equation (5.1)], where α_n was taken to be equal to 1.

We get

$$a_{n,q} = 0, \qquad b_{n,q} = \frac{1}{[n]_q}, \qquad c_{n,q} = \frac{1}{[n]_q}, \quad n \in \mathbf{N}, \tag{37}$$

the quantities introduced by (22) and (25).

5.2. Choosing $\phi_n(x) := (1 + q^n x)_q^{-n}$, $x \geq 0$, $n \in \mathbf{N}$, $T_{n,q}$ operators become a q-analogue of the ordinary Baskakov operators, a slight modification of $V_n^*(\cdot;q;\cdot)$ indicated in [**19**, equation (5.6)]. More precisely, in this case we have $T_{n,q}(f;qx) = V_n^*(f;q;x)$. This time, we get

$$a_{n,q} = \frac{1}{q[n]_q}, \qquad b_{n,q} = \frac{1}{[n]_q}, \qquad c_{n,q} = \frac{1+q}{q[n]_q}, \quad n \in \mathbf{N}. \tag{38}$$

Further on, in the above two special cases, we consider $q = q_n$, where $0 < q_n < 1$ and $\lim_n q_n = 1$. Examining (37) and (38) with $q = q_n$, on the basis of our results established in the previous section, we deduce the following.

Remark 2. The order of approximation of f by the q-analogues of Szász-Mirakjan and by ordinary Baskakov operators is $\mathcal{O}(1/\sqrt{[n]_{q_n}})$.

On the other hand, it is known that each of these original classical linear positive operators has the order of approximation $\mathcal{O}(1/\sqrt{n})$.

However, $1/\sqrt{n} \leq 1/\sqrt{[n]_{q_n}}$ for any $n \in \mathbf{N}$. To maximize the order of approximation by T_{n,q_n}, we are interested to have $[n]_{q_n}$ be of the same order as n. Thus, a question should still be raised: what additional assumption upon the sequence $(q_n)_n$ guarantees carrying out this requirement? The answer was given by Derriennic [**6**, Lemma 3.4]: it is necessary and sufficient that $n_0 \in \mathbf{N}$ and $c > 0$ exist such that, for any $n > n_0$, $q_n^n \geq c$ holds.

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