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INEQUALITIES INVOLVING MULTIVARIATE CONVEX FUNCTIONS III

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ABSTRACT. Two inequalities involving multivariate convex functions are proven. They provide refinements of the first Hermite-Hadamard inequality. An optimization problem for smooth convex functions is also discussed.

1. Introduction. Convex functions play an important role in mathematical analysis, optimization theory and probability theory, to mention the most prominent areas of mathematics. Among numerous known inequalities for convex functions the following one is fundamental. Let f be an n-variate convex function defined on an open set U in $\mathbf{R}^n$ $(n \geq 1)$. Then

$$f\bigg(\sum_{i=0}^{n} w_i x^i\bigg) \leq \sum_{i=0}^{n} w_i f(x^i),$$

where $w_i > 0$ for all i, $w_0 + \cdots + w_n = 1$ and $x^0, \dots, x^n \in U$. This classical result is due to Jensen (see, e.g., [**3**]). A refinement of this inequality with equal weights

$$f\bigg(\frac{1}{n+1}\sum_{i=0}^{n} x^i\bigg) \leq \frac{1}{\mathrm{vol}_n(\sigma)}\int_\sigma f(x)\,dx \leq \frac{1}{n+1}\sum_{i=0}^{n} f(x^i), \tag{1.1}$$

where σ is the simplex in $\mathbf{R}^n$ with vertices $x^0, \dots, x^n$ and $\mathrm{vol}_n(\sigma)$ is the n-dimensional volume of σ, has been established in [**2**, Corollary 3.1]. Inequalities (1.1) are called the Hermite-Hadamard inequalities. Further generalizations of (1.1) are established in [**1**].

This paper is a continuation of our research whose results are reported in [**1, 2**] and is organized as follows. In Section 2 we provide notation

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and definitions which will be used in Section 3 which contains two new inequalities related to the first inequality in (1.1). Also, we will show that the first Hermite-Hadamard inequality is the sharpest one in the infinite family of inequalities defined in (3.1).

2. Notation and definitions. Let us introduce notation and definitions which will be used throughout the sequel. The dot product of $x, y \in \mathbf{R}^n$ $(n \geq 1)$ is defined in the usual way, i.e., $x \cdot y = x_1y_1 + \cdots + x_ny_n$. For a given set $S \subset \mathbf{R}^n$ the symbol $\mathrm{vol}_n(S)$ will stand for a positive Lebesgue measure of S.

The Euclidean simplex E_n in $\mathbf{R}^n$ is defined as follows

$$E_n = \{u : u = (u_1, \dots, u_n),\ u_i \geq 0,\ 1 \leq i \leq n,\ u_1 + \cdots + u_n \leq 1\}.$$

For later use we define $u_0 = 1 - u_1 - \cdots - u_n$, where $(u_1, \dots, u_n) \in E_n$.

Let $\{x^0, x^1, \dots, x^n\} \subset \mathbf{R}^n$. Simplex σ with vertices at $x^0, x^1, \dots, x^n$ is defined as

$$\sigma = \left\{x : x = \sum_{i=0}^{n} u_i x^i\right\},$$

where $(u_1, \dots, u_n) \in E_n$ and u_0 is defined above. In what follows we will always assume that the vectors $x^0, x^1, \dots, x^n$ form a nondegenerate simplex, i.e., that $\mathrm{vol}_n(\sigma) > 0$, where

$$\mathrm{vol}_n(\sigma) = \frac{|\det X|}{n!}, \tag{2.1}$$

and X is the matrix with columns $x^1 - x^0, x^2 - x^0, \dots, x^n - x^0$.

Let μ be a probability measure defined on E_n. The natural weights w_i $(0 \leq i \leq n)$ of μ are given by

$$w_i = \int_{E_n} u_i \mu(u)\, du, \tag{2.2}$$

where $du = du_1 \cdots du_n$. The weights w_i are nonnegative numbers and they sum up to 1.

The following formula

$$\int_{E_n} f\left(\sum_{i=0}^{n} u_i x^i\right) \mu(u)\, du = \frac{1}{n!\mathrm{vol}_n(\sigma)} \int_\sigma f(x) \mu(X^{-1}(x - x^0))\, dx \tag{2.3}$$

(see [**1**, Lemma 3.1]) will be utilized in the next section. In (2.3), $f \in C(\sigma)$ and $dx = dx_1 \cdots dx_n$.

For a function $f : \sigma \to \mathbf{R}$ having continuous partial derivatives of order two, the symbols ∇f and Hf will stand for the gradient and the Hessian of f, i.e.,

$$\nabla f(y) = \left(\frac{\partial f(y)}{\partial y_1}, \dots , \frac{\partial f(y)}{\partial y_n} \right)$$

and

$$Hf(y) = \left[\frac{\partial^2 f(y)}{\partial y_i \partial y_j} \right]_{i,j=1}^{n},$$

respectively.

3. Main results. We are in a position to prove the following.

Theorem 3.1. *Let $f : \sigma \to \mathbf{R}$ be a convex function with continuous partial derivatives of order one, and let*

$$A = \sum_{i=0}^{n} w_i x^i,$$

where the w's are defined in (2.2) *and the vectors $x^0, \dots , x^n$ span the proper simplex in $\mathbf{R}^n$. Then the following inequality*

$$\int_{E_n} f\left(\sum_{i=0}^{n} u_i x^i \right) \mu(u)\, du \geq f(y) + (A - y) \cdot \nabla f(y) \tag{3.1}$$

holds true for every $y \in \sigma$.

Proof. Convexity of f implies that the inequality

$$f(x) \geq f(y) + (x - y) \cdot \nabla f(y)$$

is valid for all $x, y \in \sigma$ (see, e.g., [**3**, page 11]). Letting above $x = \sum_{i=0}^{n} u_i x^i$ and next integrating both sides against a probability measure μ we obtain

$$\begin{aligned}\int_{E_n} f\left(\sum_{i=0}^{n} u_i x^i\right)\mu(u)\,du \\ &\geq f(y) + \int_{E_n}\left[\sum_{i=0}^{n} u_i(x^i - y)\right]\cdot\nabla f(y)\mu(u)\,du \\ &= f(y) + \sum_{i=0}^{n}\left(\int_{E_n} u_i\mu(u)\,du\right)(x^i - y)\cdot\nabla f(y).\end{aligned}$$

Application of (2.2) to the last integral completes the proof. □

Corollary 3.1. *Under the assumptions of Theorem* 3.1 *the following inequality*

$$\begin{aligned}0 &\leq \frac{1}{n+1}\sum_{i=0}^{n} f(x^i) - \frac{1}{\mathrm{vol}_n(\sigma)}\int_\sigma f(x)\,dx \\ &\leq \frac{1}{(n+1)^2}\sum_{0\leq i<j\leq n}(x^i - x^j)\cdot\left[\nabla f(x^i) - \nabla f(x^j)\right]\end{aligned} \tag{3.2}$$

is valid.

Proof. The first inequality in (3.2) is known (see, e.g., (1.1)). For the proof of the second one we let $\mu(u) = n!$ ($u \in E_n$). It is well known that in this case $w_i = 1/(n+1)$ ($0 \leq i \leq n$). This in conjunction with (3.1) and (2.3) gives

$$\frac{1}{\mathrm{vol}_n(\sigma)}\int_\sigma f(x)\,dx \geq f(y) + \frac{1}{n+1}\sum_{i=0}^{n}(x^i - y)\cdot\nabla f(y),$$

($y \in \sigma$). Letting above $y = x^0$, $y = x^1, \dots, y = x^n$ and next summing up the resulting inequalities, we obtain

$$\begin{aligned}\frac{1}{\mathrm{vol}_n(\sigma)}\int_\sigma f(x)\,dx \geq \frac{1}{n+1}\sum_{i=0}^{n} f(x^i) + \frac{1}{(n+1)^2}\sum_{0\leq i<j\leq n}(x^j - x^i) \\ \cdot\left[\nabla f(x^i) - \nabla f(x^j)\right].\end{aligned}$$

Hence the assertion follows. □

We close this section with the following.

Theorem 3.2. *Assume that the convex function $f : \sigma \to \mathbf{R}$ has continuous partial derivatives of order two. Then the right side of* (3.1) *attains the largest value if $y = \sum_{i=0}^{n} w_i x^i$, where the w's are defined in* (2.2).

Proof. Let $\phi(y)$ denote the right side of (3.1), i.e., let

$$\phi(y) = f(y) + (A - y) \cdot \nabla f(y), \tag{3.3}$$

where $A = \sum_{i=0}^{n} w_i x^i$. Let $a_1, \dots, a_l, \dots, a_n$ be the components of A, i.e., $a_l = \sum_{i=0}^{n} w_i x_l^i$. Then (3.3) can be written as

$$\phi(y) = f(y) + \sum_{l=1}^{n} (a_l - y_l) \frac{\partial f(y)}{\partial y_l}.$$

Hence

$$\nabla \phi(y) = \big[Hf(y)\big](A - y) \tag{3.4}$$

because

$$\frac{\partial \phi(y)}{\partial y_i} = \sum_{l=1}^{n} (a_l - y_l) \frac{\partial^2 f(y)}{\partial y_i \partial y_l} \tag{3.5}$$

$(1 \le i \le n)$. Convexity of f implies that the Hessian $Hf(y)$ is positive semidefinite. This in conjunction with (3.4) leads to the conclusion that the vector A is the critical point of the function ϕ. Differentiating (3.5) and next letting $y = A$ we obtain

$$H\phi(A) = -Hf(A).$$

Thus the Hessian of the function ϕ is seminegative definite at A. This in turn implies that the function $\phi(y)$ attains its global maximum at $y = A$. The proof is complete. □

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