ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

A CHARACTERIZATION OF MONOTONELY HOMOGENEOUS DENDRITES

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ABSTRACT. We develop tools for studying dendrites and monotone maps between them by examining the Cantor-Bendixson rank of certain subsets of arcs. Using these tools, we prove that the class of monotonely homogeneous dendrites is precisely the class of dendrites which contain a copy of the Omiljanowski dendrite L_0, answering a question originally asked by J.J. Charatonik.

1. Introduction. As in [**5**], we may put a quasi-order on the class of all dendrites by saying that $X \leq_{\mathbf{M}} Y$ if and only if there exists a monotone surjection from Y onto X. Two dendrites X and Y are said to be *monotonely equivalent* if $X \leq_{\mathbf{M}} Y$ and $Y \leq_{\mathbf{M}} X$.

The *standard universal dendrite* of order $n \in \{3, 4, \dots, \omega\}$, denoted D_n, is the unique dendrite with a dense set of ramification points, each of which has order n (see [**1**, (6), page 490]). It is known that each standard universal dendrite admits a monotone map onto D_ω, and that D_ω admits a monotone map onto every dendrite [**2**, Theorem 6.2, Corollary 6.4], so in particular, all standard universal dendrites are monotonely equivalent. In fact, the property of monotone equivalence to the standard universal dendrites is characterized by containment of the Omiljanowski dendrite L_0 (see [**2**, page 182] for the definition). This fact was originally shown as Theorem 6.12 in [**2**], but the proof contains an error; an alternative proof can be found in [**11**] as Theorem 5.7.

Given a class of mappings $\mathcal{M}$, a space X is said to be $\mathcal{M}$-homogeneous if for every $x, y \in X$, there is a mapping $f \in \mathcal{M}$ from X onto itself such that $f(x) = y$. In [**2**], J.J. Charatonik shows that each standard universal dendrite is monotonely homogeneous and asks for an intrinsic characterization of this property among dendrites. Since any dendrite which is monotonely equivalent to a monotonely homogeneous dendrite

2010 AMS *Mathematics subject classification.* Primary 54F50, 54C10, 54F65.
Keywords and phrases. Continuum, dendrite, monotone, Cantor-Bendixson.
Received by the editors on August 7, 2008, and in revised form on February 23, 2009.

DOI:10.1216/RMJ-2011-41-6-2057

is itself monotonely homogeneous, a natural question, which appears as Problem 3 of "Open problems on dendroids" in the book *Open Problems in Topology* II [**9**, page 319] (attributed to J.J. Charatonik and W.J. Charatonik), is whether monotone equivalence to the standard universal dendrites (equivalently, containment of L_0) characterizes monotone homogeneity among dendrites (see also [**2, 4**]). In Section 4, we answer this question in the affirmative.

2. Preliminaries. In this paper, all spaces are assumed to be metric. A *continuum* is a compact, connected space, and a *dendrite* is a locally connected continuum that contains no simple closed curve. It is known that every subcontinuum of a dendrite is a dendrite [**7**, Section 51, VI, Theorem 4, page 301]. Dendrites are uniquely arcwise connected, and for x, y in a dendrite X, we denote by xy the arc from x to y.

All mappings will be assumed continuous. A mapping $f : X \to Y$ between continua is *monotone* if the preimage of each point is connected.

For a dendrite X, the *order* of a subcontinuum Y of X is the number of components of $X \setminus Y$. We define the order of a point p in X to be the order of the singleton $\{p\}$. Points of order 1 are called *end points*, and points of order 3 or more are called *ramification points*. The set of end points is denoted by $E(X)$, and the set of ramification points is denoted $R(X)$.

Given a space E, for each ordinal α, the *Cantor-Bendixson derivative* of order α, denoted $E^{(\alpha)}$, is defined inductively as follows:

- $E^{(0)} = E$.
- $E^{(\beta+1)} = \{e \in E \mid e \text{ is a limit point in } E^{(\beta)}\}$.
- $E^{(\gamma)} = \cap_{\beta<\gamma} E^{(\beta)}$ for limit ordinals γ.

The *Cantor-Bendixson rank* of E, denoted $\operatorname{rank}(E)$, is defined to be the least ordinal α such that $E^{(\alpha)} = E^{(\alpha+1)}$.

For a compact metric space Y, it is known that $Y^{(\alpha)} = \varnothing$ for some α if and only if Y is countable. In this case, it is also true that $\operatorname{rank}(Y)$ is a successor ordinal, and that the last non-empty derivative of Y is a finite set.

3. Arc rank. The concept of *arc rank* will prove fundamental in the proof of the main theorem. We will begin with the relevant definitions and then prove the basic facts about arc rank that will be needed in the last section.

Let X be a dendrite. For an arc $A = pq \subset X$, we define $I(A)$ to be the set $(A \cap \overline{R(X)}) \cup \{p, q\}$ of *interesting* points of A. If $I(A)$ is countable, then the *arc rank* of A, denoted $\mathrm{arank}_X(A)$, is defined to be the Cantor-Bendixson rank of $I(A)$. Otherwise, we will say that $\mathrm{arank}_X(A)$ does not exist, and that A is *non-scattered*. The notation $I^{(\alpha)}(A)$ will be used as shorthand for $[I(A)]^{(\alpha)}$.

The arc rank of a point x in a dendrite X, denoted $\mathrm{arank}_X(x)$, is defined as follows. For an ordinal β, we say that $\mathrm{arank}_X(x) > \beta$ if there exists an arc $B \subset X$ such that $I^{(\beta)}(B)$ contains x. Then $\mathrm{arank}_X(x)$ is defined to be the least ordinal α such that $\mathrm{arank}_X(x)$ is not greater than α (if such an ordinal exists).

Finally, to investigate local properties of dendrites, we will need a third notion of arc rank. For an arc $A \subset X$ containing a point $x \in X$, we define the arc rank of x in A, denoted $\mathrm{arank}_X(x; A)$, to be the least ordinal α such that $I^{(\alpha)}(A)$ does not contain x.

The following theorem is the primary motivation for using arc rank to investigate monotone maps between dendrites.

Theorem 3.1. *Let $f : X \to Y$ be a monotone surjection between dendrites, and let $a_x b_x \subset X$ and $a_y b_y \subset Y$ be arcs such that $f[a_x b_x] = a_y b_y$. Then $I^{(\alpha)}(a_y b_y) \subset f[I^{(\alpha)}(a_x b_x)]$ for all ordinals α.*

Proof. First, we treat the case $\alpha = 0$. Let $y \in I(a_y b_y)$. If $y \in \{a_y, b_y\}$, then $y \in f[\{a_x, b_x\}]$, and so $y \in f[I(a_x b_x)]$. Otherwise, $y \in \overline{R(Y)}$, and so [**6**, Theorem II.1] implies that $y \in f[\overline{R(X)}]$. Let $x \in \overline{R(X)}$ be such that $f(x) = y$, and let p be the closest point of $a_x b_x$ to x. If $p = x$, then $y \in f[\overline{R(X)} \cap a_x b_x] \subset f[I(a_x b_x)]$. Otherwise, since $y \in a_y b_y = f[a_x b_x]$, we may choose $q \in a_x b_x$ such that $f(q) = y$, and so by monotonicity, we have $f[xq] = y$ and thus $f(p) = y$. If $p \in \{a_x, b_x\}$, then $y \in f[\{a_x, b_x\}] \subset f[I(a_x b_x)]$. If not, we claim that $p \in R(X)$. In this case, both pa_x and pb_x are nondegenerate arcs, and their intersection is the singleton $\{p\}$. By definition of p, the intersection of

px and $a_x b_x$ is also $\{p\}$, so $pa_x \cup pb_x \cup px$ is a triod centered at p, and we conclude that $p \in R(X)$. Therefore, $y \in f[R(X) \cap a_x b_x] \subset f[I(a_x b_x)]$.

Thus $I(a_y b_y) \subset f[I(a_x b_x)]$, so by (4.11) and (4.12) in [**5**], it follows that

$$I^{(\alpha)}(a_y b_y) \subset (f[I(a_x b_x)])^{(\alpha)} \subset f[I^{(\alpha)}(a_x b_x)]$$

for all ordinals α. □

Corollary 3.2. *Let X, Y be dendrites, and let $f : X \to Y$ be a monotone surjection. Then for each ordinal α, and for each $y \in Y$ such that* $\operatorname{arank}_Y(y) > \alpha$*, there is a point $x \in X$ such that* $\operatorname{arank}_X(x) > \alpha$ *and $f(x) = y$.*

Corollary 3.3. *Let A be an arc in a dendrite X, and let f be a monotone map from X onto a dendrite Y. Then the arc rank of $f[A]$ in Y is no larger than the arc rank of A in X.*

In addition to the three notions of arc rank defined above, we would like to define the arc rank of a dendrite globally. The obvious way to assign such a rank is to take the supremum over all arcs in the dendrite. The following lemma shows that such a construction gives a meaningful answer precisely when the dendrite contains no non-scattered arcs.

Lemma 3.4. *Given a nondegenerate dendrite X, if* $\operatorname{arank}_X(A)$ *exists for each $A \subset X$, then the arc ranks of arcs in X have a common, countable upper bound.*

Proof. First, we note that if each arc has arc rank, then each point has arc rank. Let $x \in X$, and for each pair of distinct components of $X \setminus \{x\}$, take an arc from an end point in one component to an end point in the other component. This forms a countable collection of arcs $\mathcal{A}$ such that for an arbitrary arc A containing x, some element of $\mathcal{A}$ contains a neighborhood of x in A. Thus, if x has arc rank α in any arc of X, then it has arc rank at least α in some element of $\mathcal{A}$. Therefore, since ω_1 has uncountable cofinality, the arc rank of x over all arcs is bounded above by a countable ordinal, so x has arc rank.

Now, choose a countable collection of arcs A_n such that every point of X that is not an end point is contained in the union. For example, if X is not an arc (in which case the theorem is obvious), we may take the collection of arcs whose end points are ramification points of X, and add to this collection all arcs from an isolated end point (of which there must be at most countably many) to the nearest point of $\overline{R(X)}$. If $\overline{R(X)} \setminus E(X)$ was uncountable, then the intersection with some arc of this collection would be uncountable, and that arc would have no arc rank, contrary to our assumption. Thus, arguing by cofinality as before, the arc ranks of points in $\overline{R(X)} \setminus E(X)$ are collectively bounded above by some countable ordinal α.

Let A be an arbitrary arc of X with $\mathrm{arank}_X(A) = \beta$, and let $x \in I^{(\beta-1)}(A)$. (Since β is the Cantor-Bendixson rank of a countable compact metric space, it is a successor ordinal). Because A contains at most two points of $E(X)$, for any sequence of distinct points of $I(A)$ converging to x, all but finitely many of them must be in $\overline{R(X)} \setminus E(X)$, and thus have rank no larger than α in $I(A)$. Therefore, the arc rank of x in A (and therefore the arc rank of A itself) is bounded above by $\alpha + 1$, and since A was arbitrary, all arcs share an upper bound of $\alpha + 1$. □

Finally, we show that the arc rank of a point $x \in X$ can be determined by looking only in one "direction" from x. Note that if the inequalities in the lemma are replaced by equality, the result is not true in general for points of infinite order.

Lemma 3.5. *Let X be a dendrite, let β be a countable ordinal, and let $x \in X$ be such that* $\mathrm{arank}_X(x) > \beta$. *Then there is some component C of $X \setminus \{x\}$ such that* $\mathrm{arank}_{\overline{C}}(x) > \beta$.

Proof. Let $A \subset X$ be an arc containing x such that $x \in I^{(\beta)}(A)$. If β is a successor ordinal, let β_n be the constant sequence $\beta - 1$, and if β is a limit ordinal, let β_n be a sequence of ordinals such that $\beta_n \uparrow \beta$. Choose a sequence of points $x_n \in I^{(\beta_n)}(A)$ such that $x_n \to x$. Since A intersects at most two components of $X \setminus \{x\}$, we may assume that each x_n lies in the same component, call it C. Let A' be the arc obtained as the intersection of A and the closure of C. Then each x_n is contained in A', so $x \in I^{(\beta)}(A')$, and we conclude that $\mathrm{arank}_{\overline{C}}(x) > \beta$. □

4. Main results. Based on the definition, we would intuitively expect that monotonely homogeneous dendrites display a high degree of self-similarity. The following lemma quantifies some of that intuition.

Lemma 4.1. *Let X be a nondegenerate monotonely homogeneous dendrite, and let x be a point of X. Then there is some component of $X \setminus \{x\}$ whose closure is monotonely equivalent to X.*

Proof. Choose e to be an arbitrary end point of X. By monotone homogeneity, there is a monotone map from X onto itself taking x to e. Since $X \setminus \{e\}$ is connected, the mapping m must shrink all but one component of $X \setminus \{x\}$ to a point, namely e. If C is the unique component with nondegenerate image, then the image of C under f is at least $X \setminus \{e\}$. Since $x \in \overline{C}$ and $f(x) = e$, the closure of C maps onto X, and since $\overline{C}$ is a subcontinuum of X, there is a monotone map from X onto $\overline{C}$, so we conclude that $\overline{C}$ and X are monotonely equivalent. □

The preceding theorem is the cornerstone of the proof of the main theorem, and the reason for using arc rank to examine monotonely homogeneous dendrites. We would like to show that monotonely homogeneous dendrites lie at the top of the monotone hierarchy, along with the universal dendrites. The following theorem shows that they must at least lie at the top of the arc rank hierarchy.

Theorem 4.2. *Let X be a nondegenerate monotonely homogeneous dendrite. Then there is an arc in X with no arc rank.*

Proof. Suppose not. Then by Lemma 3.4, the supremum of $\operatorname{arank}(A)$ taken over all arcs $A \subset X$ is a countable ordinal α.

If the supremum α is attained, then as the rank of a compact set, α is a successor ordinal, and we let β_n be the constant sequence $\alpha - 1$. If the supremum is not attained, then α is a limit ordinal, and we let β_n be a sequence of ordinals such that $\beta_n \uparrow \alpha$.

Let x_1 be an arbitrary point of X. By induction, we will construct an arc starting from x_1 and contradicting our definition of α.

Suppose that, for some n, we have defined a sequence of points $x_1, x_2, \dots, x_n$ such that:

(1) $x_1x_2 \subset x_1x_3 \subset \cdots \subset x_1x_n$ is an increasing sequence of arcs.

(2) For $k = 2, 3, \ldots, n$, the arc rank of $x_{k-1}x_k$ is greater than β_k.

(3) There is a component X_n of $X \setminus \{x_n\}$ such that $\overline{X_n}$ is monotonely equivalent to X and $X_n \cap x_1x_n = \varnothing$.

Let $P = \{x \in X : \operatorname{arank}_X(x) > \beta_{n+1}\}$. By Lemma 3.5 and Lemma 4.1, for each $p \in P$ we may find components C_p and D_p of $X \setminus \{p\}$ such that $\overline{C_p}$ is monotonely equivalent to X, and $\operatorname{arank}_{\overline{D_p}}(p) > \beta_{n+1}$.

If for some $p \in P$ we may choose $C_p \neq D_p$, then let e be an arbitrary point of $E(X) \cap D_p$, and let $q = p$.

If not, then we claim that for some $p \in P$, there is a subcontinuum M of C_p such that M is monotonely equivalent to X. Suppose not. Then every subcontinuum of X that is monotonely equivalent to X contains every point of P. If $|P| \geq 2$, then there is some point y that separates two of them. Then the closure of each component of $X \setminus \{y\}$ misses a point of P, so none is monotonely equivalent to X, contradicting Lemma 4.1. If $|P| = 1$, then Corollary 3.2 implies that every monotone self-map of X leaves the unique element of P fixed, contradicting monotone homogeneity. Thus, the claim is shown, and we may choose $p \in P$ and a subcontinuum M of C_p such that M is monotonely equivalent to X. Choose q to be the nearest point to p in M, and choose $e = p$.

In either case, since $\overline{X_n}$ is monotonely equivalent to X, which is monotonely homogeneous, we may find a monotone map $m : \overline{X_n} \to X$ such that $m(x_n) = e$. Since by construction the arc eq has arc rank greater than β_{n+1}, and since the arc from x_n to the nearest point of $m^{-1}(q)$, call it q', maps onto eq, by Corollary 3.3, this arc has arc rank greater than β_{n+1}. Moreover, by construction, there is some component of $\overline{X_n} \setminus m^{-1}(q)$ other than the one containing x_n whose closure is monotonely equivalent to X. Therefore, letting $x_{n+1} = q'$, we have

(1) $x_1x_2 \subset x_1x_3 \subset \cdots \subset x_1x_{n+1}$ is an increasing sequence of arcs.

(2) For $k = 2, 3, \ldots, n+1$, the arc rank of $x_{k-1}x_k$ is greater than β_k.

(3) There is some component X_{n+1} of $X \setminus \{x_{n+1}\}$ such that $\overline{X_{n+1}}$ is monotonely equivalent to X, and $X_{n+1} \cap x_1x_{n+1} = \varnothing$.

By induction, this results in a nested sequence of arcs $x_1x_2 \subset x_1x_3 \subset \cdots$, so the sequence $\{x_n\}$ converges to some point x of X. If the global (supremum) arc rank α was attained, then the arc rank of some point in each arc x_nx_{n+1} is greater than $\alpha - 1$, so the arc rank of x_1x is greater than α, a contradiction. If α was not attained, then the arc rank of x_1x is greater than each ordinal less than α, so it must be at least α, a contradiction. Therefore, our initial assumption was false, and we conclude that some arc of X has no arc rank. □

Corollary 4.3. *If X is a monotonely homogeneous nondegenerate dendrite, then the subcontinuum of X irreducible about the set of points of X which have no arc rank is nondegenerate.*

Since the irreducible subcontinuum in Corollary 4.3 may not be irreducible about *its own* set of points with no arc rank, we must iterate the construction. Our final lemma shows that this iteration always halts on a nondegenerate and monotonely homogeneous dendrite.

Lemma 4.4. *Let X be a nondegenerate monotonely homogeneous dendrite. Then there is a subcontinuum of X which retains these properties and which is, in addition, irreducible about its set of points with no arc rank.*

Proof. For each ordinal α (not necessarily countable), define X_α as follows:

- $X_0 = X$.
- $X_{\alpha+1}$ is the irreducible subcontinuum of X_α about the set of points x such that $\operatorname{arank}_{X_\alpha}(x)$ does not exist.
- $X_\gamma = \cap_{\beta<\gamma} X_\beta$ for limit ordinals γ.

We claim that if $f : X \to X$ is a monotone surjection, then $X_\alpha \subset f[X_\alpha]$ for all ordinals α. The case $\alpha = 0$ is trivial. If the claim is true for all $\alpha \leq \alpha_0$, then by considering $f|X_{\alpha_0} : X_{\alpha_0} \to f[X_{\alpha_0}] \supset X_{\alpha_0}$, Corollary 3.2 implies that $(f[X_{\alpha_0}])_1 \subset f[X_{\alpha_0+1}]$. But clearly $X_{\alpha_0+1} = (X_{\alpha_0})_1 \subset (f[X_{\alpha_0}])_1$, so the claim holds for $\alpha = \alpha_0 + 1$. If γ is a limit ordinal and the claim holds for all $\beta < \gamma$, then the fact that

$X_\gamma \subset f[X_\gamma]$ is an easy consequence of each X_β being compact, and f being continuous. By induction, this demonstrates the claim for all α.

This implies that each nonempty X_α is monotonely homogeneous. For $x, y \in X_\alpha$, take a monotone self-map f of X which takes x to y. Then by the claim above, $f[X_\alpha]$ is a subcontinuum of X which contains X_α. Composing f with a monotone retraction of $f[X_\alpha]$ onto X_α, we have a monotone self-map of X_α taking x to y.

This means that if α is the least ordinal such that X_α is empty, then $X_{\alpha-1}$ must be a single point by Corollary 4.3 (α must be a successor ordinal because nested intersections of nonempty compact spaces are nonempty). However, this would contradict monotone homogeneity of X, since by the first claim above, each monotone self-map of X would be forced to leave that point fixed. Therefore, each X_α is nonempty, and thus nondegenerate and monotonely homogeneous. Moreover, there must exist some α' such that $X_{\alpha'} = X_{\alpha'+1}$ (take α' with cardinality larger than that of X, for example), and then $X_{\alpha'}$ is the required subcontinuum. □

Finally, with all of these tools in hand, we are prepared to prove the main theorem.

Theorem 4.5. *Let X be a nondegenerate monotonely homogeneous dendrite. Then there exists a monotone map from X onto a dendrite with a dense set of ramification points.*

Proof. By Lemma 4.4, X contains a nondegenerate monotonely homogeneous subcontinuum that is irreducible about its set of points which have no arc rank. By taking a monotone retraction of X onto this subcontinuum, we may suppose without loss of generality that X has this property.

Define a relation on X by $x \sim y$ if and only if xy has arc rank. This is clearly an equivalence relation, since xy has arc rank exactly when $I(xy)$ is countable. Also clear is that equivalence classes of $\sim$ are arcwise connected. We claim that they are closed as well. Let $\{x_n\}$ be a convergent sequence of points which lie in a single equivalence class, and let p be its limit. For each k, let c_k be the nearest point of x_1p to x_k. Since equivalence classes are connected, either p is in the same equivalence class as the x_n, or each x_n lies in the same component of $X \setminus \{p\}$. In the latter case, x_1p and x_kp intersect on a final segment for each k, so $c_k \in x_kp$ for each k, and $\lim c_k = p$. Moreover, since by construction $x_kc_k \cup c_kx_1 = x_kx_1$, each c_k is in the same equivalence class as x_1. Therefore, up to reordering, x_1c_n is an increasing sequence of arcs such that $I(x_1c_n)$ is countable for each n, and $x_1p \setminus \{p\} = \cup_n x_1c_n$, so $x_1 \sim p$, and the claim is shown.

The equivalence relation thus gives a decomposition of X into subcontinua. Since every sequence of disjoint subcontinua in a dendrite forms a null-sequence [**10**, Chapter 5, (2.6), page 92], the decomposition is automatically upper semicontinuous, and therefore defines a monotone map f on X. We claim that the image contains no free arcs. First, note that the image of each ramification point is a ramification point. Indeed, since X is irreducible about its set of points which have no arc rank, for any point $r \in R(X)$, each component of $X \setminus \{r\}$ contains a point (and therefore an arc) with no arc rank. The image of an arc with no arc rank under f is nondegenerate by definition, so the image of each component of $X \setminus \{r\}$ is nondegenerate, and the image of r is a ramification point. Thus, the preimage of any nondegenerate free arc $a'b' \subset f[X]$ would contain a nondegenerate free arc $ab \subset X$ which maps onto it. If ab was free, then a would be equivalent to b, and then the image of ab would be a single point, contradicting our assumption. Therefore, X admits a monotone map onto a nondegenerate dendrite that contains no nondegenerate free arcs and which therefore has a dense set of ramification points. $\square$

Since every dendrite with a dense set of ramification points is monotonely equivalent to each of the standard universal dendrites [**2**, Theorem 6.7], and since monotone equivalence to the standard universal dendrites is characterized by containment of the Omiljanowski dendrite L_0 (see [**2**, Theorem 6.12] and [**11**, Theorem 5.7]), Theorem 4.5 implies that the class of monotonely homogeneous dendrites is precisely the class of those dendrites which contain a homeomorphic copy of L_0.

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